Introduction to Riemann Surfaces, exercise sheet no. 2

1. Prove the inverse function theorem: Let $U \subseteq \mathbb{C}$ be open, let $f : U \to \mathbb{C}$ be holomorphic and one-to-one. Then $V = f(U)$ is open, $g = f^{-1}$ is holomoprhic on V, and $f'(z) \cdot g'(f(z)) \equiv 1$.

2. Let $F(z, w)$ be holomorphic in z and w, and let $g(z)$ and $h(z)$ be holomorphic functions of a complex variable z. Prove that $F(g(z), h(z))$ is holomorphic.

3. Let $\mathcal{A} = \{(U_\alpha, \tilde{U}_\alpha, \psi_\alpha)\}_{\alpha \in I}$ and $\mathcal{B} = \{(V_\beta, \tilde{V}_\beta, \psi_\beta)\}_{\beta \in I}$ be two complex structures ("atlas of complex charts") on a Hausdorff topological space X. Assume that $\mathcal{A} \subseteq \mathcal{B}$. Prove that the two resulting Riemann surfaces are equivalent.

4. Let X be a Riemann surface. Prove that $f : X \to \mathbb{C}_\infty$ which is not identically ∞ is holomoprhic if and only if it is meromorphic ("holomorphic except for poles") in any local coordinate.

5. Prove that all biholomorphisms of $\mathbb{C}_\infty$ are Möbius maps $z \mapsto (az + b)/(cz + d)$. Moreover, this group of biholomorphisms is isomorphic to $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\pm I$. Finally, each Möbius transformation may be obtained by applying a stereographic projection from the plane to a sphere in $\mathbb{R}^3$, then applying a translation and a rotation in space to the sphere, and then a stereographic projection back from the sphere to the plane.

6. The Schwartz derivative of a holomorphic function f on $\mathbb{C}$ is
$$(Sf)(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 .$$
Show that for any Möbius transformation g we have that $S(g \circ f) = S(f)$.

7. Let $P \subseteq \mathbb{R}^3$ be a connected, oriented polyhedron. Prove that $\mathbb{R}^3 \setminus P$ has two connected components.

8. Let $D = \{z \in \mathbb{C}\,;\, 0 < |z| < 1\}$ and let Γ be the group generated by $z \mapsto e^{2\pi i/n} z$. Prove that the Riemann surface D/Γ is equivalent to the cylinder $\mathbb{C}/Z$.

9. Let $D = \mathbb{C} \setminus \{0\}$ and let Γ be the group generated by the map $z \mapsto 2z$. Prove that D/Γ is equivalent as a Riemann surface to the torus $\mathbb{C}/L$ where L is some lattice in $\mathbb{C}$.

10. Let $U \subseteq \mathbb{C}^2$ be open, let $F : U \to \mathbb{C}$ be holomorphic with $|F_z|^2 + |F_w|^2 > 0$ throughout U. Prove that the function $\pi_2(z, w) = w$ is holomorphic on the Riemann surface $X = \{(z, w) \in U\,;\, F(z, w) = 0\}$. [Also at points with $F_w = 0$].