Introduction to Riemann Surfaces, exercise sheet no. 3

1. Verify that if $f : D \to \mathbb{C} \cup \{\infty\}$ has a pole at zero, then its multiplicity at zero is indeed the order of the pole (i.e., the number N such that $f(z) = \sum_{k=-N}^{\infty} a_k z^k$ with $a_{-N} \neq 0$).

2. Verify that if $X = \{F(z, w) = 0\}$ with F holomorphic, and $\partial F/\partial w \neq 0$ at a point $p \in X$, then $\pi(z, w) = z$ has multiplicity one at the point p.

3. Let $f : X \to \mathbb{C}$ be holomorphic, $U, V \subseteq X$ connected open sets such that $f|_U$ and $f|_V$ are biholomorphisms. Assume that $U \cap V \neq \emptyset$ and $f(U) = f(V)$. Prove that $U = V$.

4. Prove Hadamard's theorem: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a proper, continuous map. Assume that f is a local homeomorphism (any $p \in \mathbb{R}^n$ has an open nbhd U such that $f(U)$ is open and $f : U \to f(U)$ is a homeomorphism). Prove that f is onto. If you already know what a covering map is, prove also that f is one-to-one.

5. Let $P(z, w)$ be a polynomial. Prove that by applying a random, invertible linear transformation T, we almost-surely obtain a polynomial $Q = P \circ T$ of the form

$$Q(z, w) = cw^d + P_{d-1}(z)w^{d-1} + P_{d-2}(z)w^{d-2} + \ldots + P_0(z)$$

where $P_0, \ldots, P_{d-1}$ are polynomials and $c \neq 0$.

6. (a) Prove the Gauss lemma: If $f(y)$ is an irreducible polynomial such that

$$f(y)|P(x, y)Q(x, y),$$

then either $f(y)|P(x, y)$ or $f(y)|Q(x, y)$. [Hint: $f(y)|P(x, y)$ if and only if $P(x, y) = \sum P_i(y)x^i$ and $f|P_i$ for all i].

   (b) Let k be a field (e.g., the field of rational functions in $\mathbb{C}$), and let $k[z]$ be the ring of polynomials in z with coefficients in k. Prove that if $P \in k[z]$ is irreducible and Q is not a multiple of P, then there exist $\alpha, \beta \in k[z]$ such that $\alpha P + \beta Q \equiv 1$.

   (c) Let $P(z, w)$ be an irreducible polynomial. We may view P as a polynomial in $k[z]$ where $k = \mathbb{C}(w)$ is the field of rational functions in $\mathbb{C}$. Deduce from the Gauss lemma that P is irreducible in $k[z]$.

   (d) Let $P(z, w), Q(z, w)$ be polynomials, P irreducible and Q is not a multiple of P. Prove that there exist polynomials $\alpha(z, w), \beta(z, w)$ and $g(w)$ such that

$$\alpha(z, w)P(z, w) + \beta(z, w)Q(z, w) \equiv g(w).$$