Introduction to Riemann Surfaces, exercise sheet no. 4

1. Find the critical values of the map $f(z) = z/(1-z)^2$ from $\mathbb{C}$ to $\mathbb{C} \cup \{+\infty\}$. By the way, what is the image of the open unit disc?

2. Let $X = \mathbb{C} \setminus \{\pm 1\}$ and $Y = \mathbb{C} \setminus \{(\pi/2) + \pi\mathbb{Z}\}$. Prove that $\sin : Y \to X$ is a holomorphic cover map and find all deck transformations.

3. Let X be a compact Riemann surface, and let $Y = X \setminus F$ for a finite set F. Show that any non-constant holomorphic map $f : Y \to \mathbb{C}$ has a dense image.

4. Assume that X is a simply-connected Riemann surface, G a group of holomorphic biholomorphisms satisfying the free action condition. Set $Y = X/G$. Prove that $\pi_1(Y) \cong G$.

5. Let $X = \{(z, w) \in \mathbb{C}^2 \,;\, z^2 = \sin(w)\}$. Prove that X is a Riemann surface, and that the map $f(z, w) = w$ is a proper, holomorphic map on X with infinitely many critical values. This shows that the set of critical values is discrete, but not necessarily finite. [Thanks to Lev Buhovksy for this example].

6. Explain why the fundamental group of X from the previous question is not finitely generated. [One possibility is to make cuts from $2\pi k$ to $\pi(2k + 1)$].

7. Prove the uniqueness of the universal cover. Let X_1, X_2, Y be connected Riemann surfaces, let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be holomorphic covering maps with X_1 and X_2 simply-connected. Prove that there exists a biholomorphism $g : X_1 \to X_2$ such that $f_1 = f_2 \circ g$.

8. Show that $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a universal covering map, as well as the map $\exp : H \to D \setminus \{0\}$ where H is the left half-plane and D the unit disc. What are the deck transformations in the last example?

9. Let $f : X \to D \setminus \{0\}$ be a holomorphic covering map, where X is a connected Riemann surface. In class we saw that if f is proper with $\deg(f) = k$, then (X, f) is equivalent to $(D \setminus \{0\}, z \mapsto z^k)$. Use the previous question in order to show that otherwise, (X, f) is equivalent to $(H, \exp)$.