Introduction to Riemann Surfaces, exercise sheet no. 5

1. Prove that any holomorphic map $f : S^2 \to S^2$ is a rational function.

2. In freshman calculus class, we were told how to compute $\int R(x, \sqrt{x^2 + bx + c})dx$ for some rational function R by using the Euler substitution: $\sqrt{x^2 + bx + c} = \pm x + t$. Explain where this magic substitution comes from, by studying the normalization and parameterization of the algebraic curve $y^2 = x^2 + bx + c$.

3. Find the genus of the normalization X^* for the affine algebraic curve

$$w^k = (z - z_1)(z - z_2)\dots(z - z_m)$$

and the map $\pi(z, w) = z$, as a function of k and of the distinct points $z_1, \dots, z_m \in \mathbb{C}$.

4. Let $U \subseteq \mathbb{C}$ be open, $\psi_0 : U \to \mathbb{C}$ holomorphic, $y_0 \in U$. Look at all curves $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(0) = y_0$ such that ψ_0 has an analytic continuation $\{(U_t, \psi_t)\}_{t\in[0,1]}$ along γ. We say that $\gamma \sim \tilde{\gamma}$ if when

$$\gamma(1) = \tilde{\gamma}(1), \psi_1(\gamma(1)) = \psi_1(\tilde{\gamma}(1)).$$

Let X be the collection of all equivalence maps, and consider on X two maps

$$F([\gamma]) = \gamma(1), \Psi([\gamma]) = \psi_1(\gamma(1))$$

Prove that there is a unique complex structure on X, such that F and Ψ are holomorphic, and F is a local homeomorphism.

5. The Riemann surface X constructed in the previous exercise is called "the Riemann surface associated with a holomorphic function ψ_0". Explain what is X in the case where $U = D(1, \varepsilon)$ and $\psi_0(z)$ is a branch of $\sqrt{z}$ in U. What is the normalization of $F : X \to \mathbb{C}$? and of $\Psi : X \to \mathbb{C}$?

6. Describe the Riemann surface of the inverse function to $f(z) = z^3 - z$.

7. Prove that in a simply-connected domain $D \subseteq \mathbb{C}$, for any $z \notin D$ there is an analytic branch of $\log(z - z_0)$ in D.

8. Define a harmonic function on a Riemann surface. Define a conjugate harmonic function. Prove that in a simply-connected Riemann surface, any harmonic function has a conjugate.