Introduction to Riemann Surfaces, exercise sheet no. 6

1. Explain how to define the integral of a holomorphic 1-form α on a Riemann surface X with respect to a curve $\gamma : [0,1] \to X$ which is assumed merely continuous. (By the way, this is possible also for a non-holomorphic 1-form).

2. Let $X \subseteq \mathbb{C}^2$ be a Riemann surface defined by the equation $F(z,w) = 0$ for a holomorphic $F : \mathbb{C}^2 \to \mathbb{C}$. Prove that on X we have the relation
$$\frac{\partial F}{\partial z} \cdot dz + \frac{\partial F}{\partial w} \cdot dw = 0.$$

3. For $z \in D = D(0,1) \subseteq \mathbb{C}$ and $t \in [0, 2\pi]$ we define $\theta^*(t)$ via the requirement that $e^{it}, e^{i\theta^*(t)}$ and z are collinear. Prove that for any harmonic function u on D that is continuous up to the boundary,
$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta^*(t))dt \qquad\qquad (z \in D)$$
Deduce that the Poisson integral of a continuous function on ∂D is continuous up to the boundary.

4. Consider the Riemann surface $X = \{w^2 = P(z)\}$, where $P(z) = (z - z_1)(z - z_2)(z - z_3)$ with $z_1, z_2, z_3 \in \mathbb{C}$ distinct, and set $\pi(z,w) = z$. Let $R = \max_j |z_j|$. Find an explicit isomorphism between $(\pi^{-1}(\mathbb{C} \setminus RD), \pi)$ and $(\mathbb{C} \setminus \sqrt{R}D, z \mapsto z^2)$.

5. Show that a Riemann surface which is a quotient of $\mathbb{C}$ is either the cylinder $\mathbb{C} \setminus \{0\}$ or a torus $\mathbb{C}/\Lambda$ for a certain two-dimensional lattice $\Lambda \subseteq \mathbb{C}$. (Well, or $\mathbb{C}$ itself). Find all holomorphic maps from a torus to itself.

6. Let $\Lambda_1, \Lambda_2 \subseteq \mathbb{C}$ be two-dimensional lattices, and let $P_{\Lambda_1}, P_{\Lambda_2}$ be the associated Weierstrass functions. Assume that there is a linear transformation mapping the critical values of P_{Λ_1} to those of P_{Λ_2}. Deduce from Riemann's existence and uniqueness theorem that $\mathbb{C}/\Lambda_1$ is equivalent to $\mathbb{C}/\Lambda_2$.

7. Prove that if there is no linear transformation, then $\mathbb{C}/\Lambda_1$ is not equivalent to $\mathbb{C}/\Lambda_2$.

8. Consider the Riemann surface $X = \{w^2 = (E+z)(1-z^2)\}$ for some $E \in \mathbb{R} \setminus \{\pm 1\}$. In class we showed that X is equivalent to $(\mathbb{C}/\Lambda) \setminus \{0\}$ for a certain lattice $\Lambda \subseteq \mathbb{C}$, via the Weierstrass function and its derivative. Prove that Λ is a rectangular lattice, perhaps by integrating dz/w over loops in X.

9. Let $P(x)$ be a real polynomial of degree 4 with two local minima. For a parameter $E \in \mathbb{R}$, consider the equation $(dx/dt)^2 + P(x(t)) = E$, the Newtonian motion of a particle in a line under potential energy $P(x)$. Pick E such that there are two distinct periodic solutions, each oscillating near a local minimum. Prove that they have the same period.