Convex Localization and Optimal Transport

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Please solve at least 10 of the exercises below (weighted count) according to your choice. Submit your solution by August 15 at the link:

https://www.dropbox.com/request/u8D9KVLDY1ddaGjW1W7P

To be more precise, most of the exercises below are worth 1 point, but few towards the end are marked as being worth 2, 3 or 5 points. You are requested to accumulate 10 points.

If you find a mistake or an inaccuracy in any of the questions, please send me an e-mail as soon as you find them. You can also suggest additional exercises that are related to the course material.

1. **Heuristic explanation of Neumann boundary conditions**: Let $K \subset \mathbb{R}^n$ be a bounded, open set with a smooth boundary. For $i \geq 0$ set

$$\lambda_i = \inf_{0 \not\equiv f \perp \varphi_0, \ldots, \varphi_{i-1}} \frac{\int_K |\nabla f|^2}{\int_K f^2}, \tag{1}$$

where the infimum runs over all functions $f \in L^2(K)$, and when $i \geq 1$ we require that f be orthogonal to $\varphi_0, \ldots, \varphi_{i-1}$ in $L^2(K)$.

You may assume that for any i the infimum in (1) is attained and that the minimizer φ_i may be chosen to be a smooth function up to the boundary (this is actually true as you learn in PDE class). Prove that for any $i \geq 0$

$$\begin{cases} \Delta \varphi_i(x) = -\lambda_i \varphi_i(x) & \forall x \in K \\ \nabla \varphi_i(x) \perp \nu(x) & \forall x \in \partial K \end{cases}$$

where $\nu(x)$ is the normal to ∂K at the point x. [Hint: look at the Rayleigh quotient of $\varphi_i + \varepsilon g$ as $\varepsilon \to 0$, first for g compactly-supported and then without this assumption]

<u>Poincaré constants</u>

2. Show that the Poincaré constant of the uniform measure on the cube $[0,1]^n$ is $1/\pi^2$, the same value as the Poincaré constant of the interval $[0,1]$. In general, suppose that $\mu_1, \ldots, \mu_k$ are Borel probability measures on $\mathbb{R}^n$, and consider the product probability measure $\mu_1 \otimes \ldots \otimes \mu_k$ on $(\mathbb{R}^n)^k$. Prove that

$$C_P(\mu_1 \otimes \ldots \otimes \mu_k) = \max_{i=1,\ldots,k} C_P(\mu_i).$$

3. In this guided exercise we study the Poincaré constant of the standard Gaussian measure γ_n in $\mathbb{R}^n$, and we estimate it using "round needles". [The Poincaré constant actually equals one. This exercise is a variant of a theorem by Maurey and Pisier]

   (a) Why is the Poincaré constant at least one?

   (b) Let X, Y be two independent standard Gaussian random vectors in $\mathbb{R}^n$. For $\theta \in [0, \pi/2]$ set
   $$X_\theta = (\cos\theta)X + (\sin\theta)Y.$$
   Prove that the two random variables X_θ and $\frac{\partial}{\partial\theta}X_\theta$ are independent, standard Gaussian random vectors in $\mathbb{R}^n$. [Hint: what is the joint distribution of (X, Y)?]

   (c) Prove that for any smooth function f with $\mathbb{E}|\nabla f(X)|^2 < \infty$ and any $\theta \in [0, \pi/2]$,
   $$\mathbb{E}\left|\frac{\partial}{\partial\theta} f(X_\theta)\right|^2 = \mathbb{E}|\nabla f(X) \cdot Y|^2 = \mathbb{E}|\nabla f(X)|^2.$$

   (d) Under the same assumptions on f, prove that
   $$\mathbb{E}|f(X) - f(Y)|^2 = \mathbb{E}\left(\int_0^{\pi/2} \frac{\partial}{\partial\theta} f(X_\theta) d\theta\right)^2 \leq \frac{\pi}{2}\int_0^{\pi/2} \mathbb{E}\left(\frac{\partial}{\partial\theta} f(X_\theta)\right)^2 d\theta.$$

   (e) Show that the Poincaré constant of the standard Gaussian is at most $\pi^2/8$.

   (f) Think how come this proof yields a constant which does not depend on the dimension, while being similar in spirit to the original proof by Poincaré we studied in class.

4. Here we show that the Poincaré constant of the unit sphere S^{n-1} is at most $C_P(X)/(n - C_P(X))$, where X is a standard Gaussian random vector in $\mathbb{R}^n$ (it is actually an equality and $C_P(X) = 1$, by the way). Write $\Theta = X/|X|$.

   (a) Why is Θ distributed uniformly on S^{n-1}? Prove that $|X|$ and Θ are independent.

   (b) Let $f : S^{n-1} \to \mathbb{R}$ be a smooth function. Denote $F(x) = |x| f(x/|x|)$ for any $0 \neq x \in \mathbb{R}^n$. Prove that
   $$\mathbb{E}|f(\Theta)|^2 = \frac{1}{n}\mathbb{E}|F(X)|^2.$$

   (c) Prove that for any $0 \neq x \in \mathbb{R}^n$,
   $$\nabla F(x) = \nabla_{S^{n-1}} f\left(\frac{x}{|x|}\right) + f\left(\frac{x}{|x|}\right)\frac{x}{|x|}.$$

   (d) Show that
   $$\mathbb{E}|\nabla F(X)|^2 = \mathbb{E}|\nabla_{S^{n-1}} f(\Theta)|^2 + \mathbb{E}|f(\Theta)|^2.$$

   (e) Prove the relation above about the Poincaré constant of the unit sphere and of the standard Gaussian.

Around Cheeger's inequality

5. Let $p \geq 1$ and let X be a random variable with $\mathbb{E}|X|^p < \infty$. Let $\tilde{m} \in \mathbb{R}$ be a median of X and write $m = \mathbb{E}X$. Prove that

$$\frac{1}{2}\|X - m\|_p \leq \|X - \tilde{m}\|_p \leq 3\|X - m\|_p.$$

6. Let (X, d, μ) be a measure-metric space, $p \geq 1$, $A > 0$, and assume that for any $f \in L^p(\mu)$, setting $m = \int f d\mu$,

$$\|f - m\|_{L^p(\mu)} \leq A \left(\int_X |\nabla f|^p d\mu \right)^{1/p}.$$

Prove that for any $q > p$,

$$\|f - m\|_{L^q(\mu)} \leq 10 \frac{q}{p} A \left(\int_X |\nabla f|^q d\mu \right)^{1/q}$$

[hint: switch from mean to median, and use the argument for $p = 1, q = 2$ we saw in class].

7. Let (X, d, μ) be a measure-metric space satisfying a Poincaré inequality with constant A, i.e., for any $f \in L^2(\mu)$,

$$Var_\mu(f) \leq A \cdot \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

Prove that for any 1-Lipschitz function $f : X \to \mathbb{R}$, setting $m = \int f d\mu$,

$$\mu\left\{x \in X \,;\, |f(x) - m| \geq t\right\} \leq Ce^{-ct/\sqrt{A}} \qquad \text{for all } t \in \mathbb{R},$$

for some universal constants $c, C > 0$. This was proven by Gromov and Milman in the 1980s. [hint: use the relation between exponential tail and linear growth of moments we saw in class.]

Log-concavity

Here we prove the Prékopa-Leindler inequality in three steps:

8. Let $0 < \lambda < 1$ and let $f, g, h : \mathbb{R} \to (0, \infty)$ be three integrable functions such that for all $x, y \in \mathbb{R}$,

$$h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda. \tag{2}$$

Assume that $\|f\|_\infty = \|g\|_\infty$. Prove that

$$\int_{\mathbb{R}} h \geq (1 - \lambda) \int_{\mathbb{R}} f + \lambda \int_{\mathbb{R}} g$$

[hint: use layer cake representation, show that $\{h \geq t\} \supseteq (1 - \lambda)\{f \geq t\} + \lambda\{g \geq t\}$ and use one-dimensional Brunn-Minkowski, for non-empty sets only!]

9. Let $0 < \lambda < 1$ and let $f, g, h : \mathbb{R} \to (0, \infty)$ be three integrable functions such that for all $x, y \in \mathbb{R}$, condition (2) hold true. Prove that

$$\int_{\mathbb{R}} h \geq \left(\int_{\mathbb{R}} f \right)^{1-\lambda} \cdot \left(\int_{\mathbb{R}} g \right)^{\lambda}. \tag{3}$$

[hint: approximate by bounded functions, normalize and use the previous exercise]

10. Let $0 < \lambda < 1$ and let $f, g, h : \mathbb{R}^n \to (0, \infty)$ be three integrable functions such that for all $x, y \in \mathbb{R}^n$, condition (2) hold true. Prove (3) by induction on the dimension.

[hint: set $F(y) = \int_{\mathbb{R}} f(t, y) dt$ for $y \in \mathbb{R}^{n-1}$, and similarly for G and H, and prove that $H((1-\lambda)x + \lambda y) \geq F(x)^{1-\lambda} G(y)^{\lambda}$].

11. Prove that log-concavity is preserved under convolutions, pointwise products, and that if X is a log-concave random vector then so is $T(X)$ for linear map T.

Isoperimetry in one dimension

The next four exercises are based on arguments of Bobkov.

12. Let $\rho : \mathbb{R} \to (0, \infty)$ be a smooth, positive, log-concave probability density. Fix $p \in (0, 1)$ and $h > 0$ and write $N_\delta(A) = \{x \in \mathbb{R}\,; \inf_{y \in A} |x - y| < \delta\}$ for the δ-neighborhood.

For $a \in \mathbb{R}$ with $\mu([a, \infty)) > p$ write $I_p(a) \subseteq \mathbb{R}$ for the interval $[a, b] \subseteq \mathbb{R}$ with $\mu([a, b]) = p$. Prove that the function $\varphi(a) = \mu(N_h(I_p(a))$ is unimodal: increasing till some point, and decreasing afterwards.

[hint: show that $b'(a) = \rho(a)/\rho(b)$ and that φ'/ρ is decreasing).

13. Write $\mathcal{F}$ for the collection of all subsets of $\mathbb{R}$ with finitely many connected components, and write $\mathcal{F}_0$ for the collection of all half-lines (i.e., sets of the form $[a, \infty)$ or $(-\infty, a]$). Fix $p \in (0, 1), h > 0$ and use the previous exercise to show that

$$\inf_{\substack{A \in \mathcal{F} \\ \mu(A) = p}} \mu(N_h(A)) = \inf_{\substack{A \in \mathcal{F}_0 \\ \mu(A) = p}} \mu(N_h(A)).$$

[hint: induction on the number of intervals in the set, and move them around]

14. Consider the measure-metric space $(\mathbb{R}, |\cdot|, \mu)$ where μ is a log-concave probability measure on $\mathbb{R}$, with a smooth, positive density $\rho : \mathbb{R} \to (0, \infty)$. Use the previous exercise and prove that it suffices to look at half-lines in order to determine the isoperimetric (Cheeger) constant.

15. Let μ be a log-concave probability measure on $\mathbb{R}$ with a smooth, positive, density ρ. Set $\Phi(x) = \mu((-\infty, x))$ and prove that $\rho \circ \Phi^{-1} : (0, 1) \to \mathbb{R}$ is concave. Conclude that the Cheeger constant is attained for a set which is a half-line whose μ-measure is exactly $1/2$.

Recalling polar coordinates

16. (a) Recall the Jacobian of polar coordinates in $\mathbb{R}^n$, i.e., the function $J(r)$ such that

$$\int_{\mathbb{R}^n} \varphi = \int_{S^{n-1}} \int_0^\infty \varphi(r\theta) J(r) dr d\sigma_{n-1}(\theta)$$

for any integrable function φ. Here σ_{n-1} is the uniform probability measure on the sphere S^{n-1}. Is the function $J(r)$ log-concave?

(b) Polar coordinates on the sphere: find the function $J(r)$ such that for any continuous $\varphi : S^{n-1} \to \mathbb{R}$,

$$\int_{S^{n-1}} \varphi d\sigma_{n-1} = \int_{S^{n-2}} \int_0^\pi \varphi(\cos(r) e_1 + \sin(r)\theta) J(r) dr d\sigma_{n-2}(\theta)$$

where $S^{n-2} = \{x \in S^{n-1} ; x_1 = 0\}$, where $e_1 = (1, 0, \dots, 0)$ and $S^{n-1} = \{x \in \mathbb{R}^n ; \sum_i x_i^2 = 1\}$. Is J log-concave?

Tails and moments

17. Let X be a random variable and let $\alpha > 0$. Prove that the following are equivalent:

(a) There exists $A > 0$ such that $\mathbb{P}(|X| \geq t) \leq 2e^{-t^\alpha / A}$ for any $t > 0$.

(b) There exists $A > 0$ such that $\|X\|_p \leq A p^{1/\alpha}$ for all $p \geq 1$, where $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$.

(Such random variables are called a ψ_α-random variables)

18. Complete the proof of the Proposition we discussed in class: Let X be a real-valued, log-concave random variable. Then for any $p > 0$,

$$\|X\|_p \leq C(p+1)\|X\|_0$$

where $C > 0$ is a universal constant. (in class we proved it under the additional assumption $\mathbb{E}X = 0$).

19. Let $K \subseteq \mathbb{R}^n$ be a convex body and let X be a random vector distributed uniformly in K. Assume that $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i X_j = \delta_{ij}$ for $i, j = 1, \dots, n$. Prove that for any $\theta \in S^{n-1}$, the random vector

$$\langle X, \theta \rangle$$

is a log-concave random variable of mean zero and variance one, whose support contains an interval of the form $[-c, c]$ where $c > 0$ is a universal constant.

20. Under the assumptions of the previous exercise, prove that $cB^n \subseteq K$ where $c > 0$ is a universal constant and $B^n = \{x \in \mathbb{R}^n ; |x| \leq 1\}$.

21. Under the assumptions of the previous exercise, prove that $K \subseteq cnB^n$. [This exercise is worth 2 points, not just 1 like the others]

Grünbaum's theorem

22. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, log-concave probability density. Assume that $\int_{-\infty}^{\infty} x f(x) dx = 0$ and that f is supported in some interval $[-R, R]$. Denote $F(t) = \int_{-R}^{t} f(x) dx$.

    (a) Prove that there exists $\alpha > 0$ such that $F(t) \leq F(0) e^{\alpha t}$ for all $t \in \mathbb{R}$.

    (b) Prove that $F(0) \geq 1/e$ by verifying that

$$R = \int_{-R}^{R} F(t) dt \leq \int_{-R}^{1/\alpha} F(0) e^{\alpha t} dt + (R - 1/\alpha) \leq R + (eF(0) - 1)/\alpha.$$

23. Let $K \subseteq \mathbb{R}^n$ be a convex body whose barycenter lies at the origin. Let $H \subseteq \mathbb{R}^n$ be half-space with the origin in its boundary. Prove that

$$Vol_n(K \cap H) \geq \frac{1}{e} \cdot Vol_n(K).$$

Rockafellar's theorem

24. Prove the following variant of Rockafellar's theorem: A subset $A \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is d-cyclically-monotone if for any $N \geq 1$ and $(x_1, y_1), \ldots, (x_N, y_N) \in A$,

$$\sum_{i=1}^{N} d(x_i, y_i) \leq \sum_{i=1}^{N} d(x_i, y_{i+1})$$

with $y_{N+1} = y_1$. Prove that A is d-cyclically-monotone if and only if there exists a 1-Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$ such that for all $(x, y) \in A$,

$$u(x) - u(y) = d(x, y).$$

Is the analogous statement true in any metric space?

Applications of convex localization

25. Use convex localization and prove the Gaussian isoperimetric inequality: Let $A \subseteq \mathbb{R}^n$ be a measurable set and let $H \subseteq \mathbb{R}^n$ be a half-space with $\gamma_n(A) = \gamma_n(H)$, where γ_n is the standard Gaussian measure in $\mathbb{R}^n$. Prove that for any $r > 0$,

$$\gamma_n(A + rB^n) \geq \gamma_n(H + rB^n).$$

[This question is worth 3 points]

Alexandrov's second-differentiability theorem

26. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and denote $g(x) = f(x) + |x|^2/2$. Show that the set $\{(y, x) \, ; \, x \in \mathbb{R}^n, y \in \partial g(x)\}$ is the graph of a surjective 1-Lipschitz function on $\mathbb{R}^n$.

    Then use Rademacher's theorem on the differentiability of Lipschitz functions and appropriate versions of the inverse function theorem to conclude Alexandrov's theorem: For almost any $x_0 \in \mathbb{R}^n$, there exists a polynomial of second degree P_{x_0} such that

    $$|f(x) - P_{x_0}(x)| = o(|x - x_0|^2)$$

    as $x \to x_0$.

    [This question is worth 5 points]

    Gaussian waist inequality

27. Prove the version of Borsuk-Ulam's theorem that was needed in the proof.

    [This question is worth 5 points]