

# Needle decompositions in Riemannian geometry

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## Contents

Chapter 1. Introduction	1
Chapter 2. Regularity of geodesic foliations	7
2.1. Transport rays	7
2.2. Whitney's extension theorem for $C^{1,1}$	12
2.3. Riemann normal coordinates	16
2.4. Proof of the regularity theorem	21
Chapter 3. Conditioning a measure with respect to a geodesic foliation	27
3.1. Geodesics emanating from a $C^{1,1}$ -hypersurface	27
3.2. Decomposition into ray clusters	37
3.3. Needles and Ricci curvature	44
Chapter 4. The Monge-Kantorovich problem	54
Chapter 5. Some applications	62
5.1. The inequalities of Buser, Ledoux and E. Milman	62
5.2. A Poincaré inequality for geodesically-convex domains	64
5.3. The isoperimetric inequality and its relatives	65
Chapter 6. Further research	68
Appendix. The Feldman-McCann proof of Lemma 2.4.1	72
Bibliography	75



## Abstract

The localization technique from convex geometry is generalized to the setting of Riemannian manifolds whose Ricci curvature is bounded from below. In a nutshell, our method is based on the following observation: When the Ricci curvature is non-negative, log-concave measures are obtained when conditioning the Riemannian volume measure with respect to a geodesic foliation that is orthogonal to the level sets of a Lipschitz function. The Monge mass transfer problem plays an important role in our analysis.

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## CHAPTER 1

# Introduction

The *localization technique* in convex geometry is a method for reducing  $n$ -dimensional problems to one-dimensional problems, that was developed by Gromov and Milman [29], Lovász and Simonovits [36] and Kannan, Lovász and Simonovits [31]. Its earliest appearance seems to be found in the work of Payne and Weinberger [41], where the following inequality is stated: For any bounded, open, convex set  $K \subset \mathbb{R}^n$  and an integrable,  $C^1$ -function  $f : K \rightarrow \mathbb{R}$ ,

$$(1) \quad \int_K f = 0 \quad \implies \quad \int_K f^2 \leq \frac{\text{Diam}^2(K)}{\pi^2} \int_K |\nabla f|^2,$$

where  $\text{Diam}(K) = \sup_{x,y \in K} |x - y|$  is the diameter of  $K$ , and  $|\cdot|$  is the standard Euclidean norm in  $\mathbb{R}^n$ . The localization proof of (1) goes roughly as follows: Given  $f$  with  $\int_K f = 0$ , one finds a hyperplane  $H \subset \mathbb{R}^n$  passing through the barycenter of  $K$  such that

$$\int_{K \cap H^+} f = \int_{K \cap H^-} f = 0,$$

where  $H^-, H^+ \subset \mathbb{R}^n$  are the two half-spaces determined by the hyperplane  $H$ . The problem of proving (1) is reduced to proving the two inequalities:

$$\int_{K \cap H^\pm} f^2 \leq \frac{\text{Diam}^2(K \cap H^\pm)}{\pi^2} \int_{K \cap H^\pm} |\nabla f|^2.$$

The next step is to again bisect each of the two half-spaces separately, retaining the requirement that the integral of  $f$  is zero. Thus one recursively obtains finer and finer partitions of  $\mathbb{R}^n$  into convex cells. At the  $k^{\text{th}}$  step, the proof of (1) is reduced to  $2^k$  “smaller” problems of a similar nature. At the limit, the original problem is reduced to a lower-dimensional problem, and eventually even to a one-dimensional problem. This one-dimensional problem has turned out to be relatively simple to solve.

This bisection technique has no clear analog in the context of an abstract Riemannian manifold. The purpose of this article is to try and bridge this gap between convex geometry and Riemannian geometry.

There are only two parameters of a given Riemannian manifold that play a role in our analysis: the dimension of the manifold, and a uniform lower bound  $\kappa$  for its Ricci curvature. We say that an  $n$ -dimensional Riemannian manifold  $\mathcal{M}$  satisfies the curvature-dimension condition  $CD(\kappa, n)$  with  $\kappa \in \mathbb{R}$  if

$$\text{Ric}_{\mathcal{M}}(v, v) \geq \kappa \cdot g(v, v) \quad \text{for all } p \in \mathcal{M}, v \in T_p \mathcal{M},$$

where  $g$  is the Riemannian metric tensor and  $\text{Ric}_{\mathcal{M}}$  is the Ricci tensor of  $\mathcal{M}$ . The contribution of Bakry and Émery [2] has made it clear that *weighted* Riemannian manifolds are convenient for the study of curvature-dimension conditions. A

weighted Riemannian manifold is a triplet  $(\mathcal{M}, d, \mu)$ , where  $\mathcal{M}$  is an  $n$ -dimensional Riemannian manifold with Riemannian distance function  $d$ , and where the measure  $\mu$  has a smooth, positive density  $e^{-\rho}$  with respect to the Riemannian volume measure on  $\mathcal{M}$ . The *generalized Ricci tensor* of the weighted Riemannian manifold  $(\mathcal{M}, d, \mu)$  is defined via

$$Ric_{\mu} := Ric_{\mathcal{M}} + Hess_{\rho},$$

where  $Hess_{\rho}$  is the Hessian form associated with the smooth function  $\rho : \mathcal{M} \rightarrow \mathbb{R}$ . The *generalized Ricci tensor with parameter  $N$* , for  $N \in (-\infty, 1) \cup [n, +\infty]$  is defined for  $p \in \mathcal{M}$  and  $v \in T_p\mathcal{M}$  as follows:

$$Ric_{\mu, N}(v, v) := \begin{cases} Ric_{\mu}(v, v) - (\partial_v \rho)^2 / (N - n) & N \neq n, +\infty \\ Ric_{\mu}(v, v) & N = +\infty \\ Ric_{\mathcal{M}}(v, v) & N = n, \rho \equiv Const \end{cases}$$

The standard agreement is that  $Ric_{\mu, n}$  is undefined unless  $\rho$  is a constant function. For  $\kappa \in \mathbb{R}$  and  $N \in (-\infty, 1) \cup [n, +\infty]$  we say that  $(\mathcal{M}, d, \mu)$  satisfies the curvature-dimension condition  $CD(\kappa, N)$  when

$$Ric_{\mu, N}(v, v) \geq \kappa \cdot g(v, v) \quad \text{for all } p \in \mathcal{M}, v \in T_p\mathcal{M}.$$

For instance, the  $CD(0, \infty)$ -condition is equivalent to the requirement that the generalized Ricci tensor be non-negative. We refer the reader to Bakry, Gentil and Ledoux [4] for background on weighted Riemannian manifolds of class  $CD(\kappa, N)$ . In this article, a *minimizing geodesic* is a curve  $\gamma : A \rightarrow \mathcal{M}$ , where  $A \subseteq \mathbb{R}$  is connected (i.e., an interval), such that

$$d(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in A.$$

It is also customary to refer to the set  $\gamma(A) = \{\gamma(t); t \in A\}$  itself as a minimizing geodesic.

**DEFINITION 1.1.** *Let  $\kappa \in \mathbb{R}, 1 \neq N \in \mathbb{R} \cup \{\infty\}$  and let  $\nu$  be a measure supported on the Riemannian manifold  $\mathcal{M}$ . We say that  $\nu$  is a “ $CD(\kappa, N)$ -needle” if there exist a non-empty, open interval  $A \subseteq \mathbb{R}$ , a smooth function  $\Psi : A \rightarrow \mathbb{R}$  and a minimizing geodesic  $\gamma : A \rightarrow \mathcal{M}$  such that:*

- (i) *Denote by  $\theta$  the measure on  $A \subseteq \mathbb{R}$  whose density with respect to the Lebesgue measure is  $e^{-\Psi}$ . Then  $\nu$  is the push-forward of  $\theta$  under the map  $\gamma$ .*
- (ii) *The following inequality holds in the entire set  $A$ :*

$$(2) \quad \Psi'' \geq \kappa + \frac{(\Psi')^2}{N - 1},$$

where in the case  $N = \infty$ , we interpret the term  $(\Psi')^2 / (N - 1)$  as zero.

Condition (2) is equivalent to condition  $CD(\kappa, N)$  for the weighted Riemannian manifold  $(A, d, \theta)$  with  $d(x, y) = |x - y|$ . Examples of needles include:

1. *Log-concave needles* which are defined to be  $CD(0, \infty)$ -needles. In this case,  $\Psi$  is a convex function. Log-concave needles are valuable when studying the uniform measure on convex sets in  $\mathbb{R}^n$  for large  $n$ .
2. A  *$\sin^n$ -concave needle* is a  $CD(n, n + 1)$ -needle. These are relevant to the sphere  $S^{n+1}$ , since the  $(n + 1)$ -dimensional unit sphere is of class  $CD(n, n + 1)$ .



3. The  $N$ -concave needles are  $CD(0, N + 1)$ -needles with  $N > 0$ . Here,  $f^{1/N}$  is a concave function, where  $f = e^{-\Psi}$  is the density of the measure  $\theta$ . For  $N < 0$ , the  $CD(0, N + 1)$ -condition is equivalent to the convexity of  $f^{-1/|N|}$ .
4. A  $\kappa$ -log-concave needle is a  $CD(\kappa, \infty)$ -needle.

These examples are discussed by Gromov [28, Section 4]. We say that the Riemannian manifold  $\mathcal{M}$  is *geodesically-convex* if any two points in  $\mathcal{M}$  may be connected by a minimizing geodesic. By the Hopf-Rinow theorem, any complete, connected Riemannian manifold is geodesically-convex. In this article there is no need to assume that the Riemannian manifold  $\mathcal{M}$  is complete, it suffices to require geodesic convexity. A partition of  $\mathcal{M}$  is a collection of non-empty disjoint subsets of  $\mathcal{M}$  whose union equals  $\mathcal{M}$ .

**THEOREM 1.2** (“Localization theorem”). *Let  $n \geq 2, \kappa \in \mathbb{R}$  and  $N \in (-\infty, 1) \cup [n, +\infty]$ . Assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold of class  $CD(\kappa, N)$  which is geodesically-convex. Let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a  $\mu$ -integrable function with  $\int_{\mathcal{M}} f d\mu = 0$ . Assume that there exists a point  $x_0 \in \mathcal{M}$  with  $\int_{\mathcal{M}} |f(x)| \cdot d(x_0, x) d\mu(x) < \infty$ .*

*Then there exist a partition  $\Omega$  of  $\mathcal{M}$ , a measure  $\nu$  on  $\Omega$  and a family  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in \Omega}$  of measures on  $\mathcal{M}$  such that:*

- (i) *For any Lebesgue-measurable set  $A \subseteq \mathcal{M}$ ,*

$$\mu(A) = \int_{\Omega} \mu_{\mathcal{I}}(A) d\nu(\mathcal{I})$$

*(In particular, the map  $\mathcal{I} \mapsto \mu_{\mathcal{I}}(A)$  is well-defined  $\nu$ -almost everywhere and it is a  $\nu$ -measurable map). In other words, we have a “disintegration of the measure  $\mu$ ”.*

- (ii) *For  $\nu$ -almost any  $\mathcal{I} \in \Omega$ , the set  $\mathcal{I} \subseteq \mathcal{M}$  is a minimizing geodesic, the measure  $\mu_{\mathcal{I}}$  is supported on  $\mathcal{I}$ , and either  $\mathcal{I}$  is a singleton or else  $\mu_{\mathcal{I}}$  is a  $CD(\kappa, N)$ -needle.*
- (iii) *For  $\nu$ -almost any  $\mathcal{I} \in \Omega$  we have  $\int_{\mathcal{I}} f d\mu_{\mathcal{I}} = 0$ .*

In Chapter 5 we demonstrate that Theorem 1.2 leads to alternative proofs of some familiar inequalities from convex and Riemannian geometry. These include the isoperimetric inequality, the Poincaré and log-Sobolev inequalities, the Payne-Weiberger/Yang-Zhong inequality, the Brunn-Minkowski type inequality of Cordero-Erausquin, McCann and Schmuckenschläger, among others. Some of these inequalities are consequences of the following Riemannian analog of the four functions theorem of Kannan, Lovász and Simonovits [31]:

**THEOREM 1.3** (“The four functions theorem”). *Let  $n \geq 2, \alpha, \beta > 0, \kappa \in \mathbb{R}, N \in (-\infty, 1) \cup [n, +\infty]$ . Let  $(\mathcal{M}, d, \mu)$  be an  $n$ -dimensional weighted Riemannian manifold of class  $CD(\kappa, N)$  which is geodesically-convex. Let  $f_1, f_2, f_3, f_4 : \mathcal{M} \rightarrow [0, +\infty)$  be measurable functions such that there exists  $x_0 \in \mathcal{M}$  with*

$$\int_{\mathcal{M}} (f_1(x) + f_2(x) + f_3(x) + f_4(x)) \cdot (1 + d(x_0, x)) d\mu(x) < \infty.$$

Assume that  $f_1^\alpha f_2^\beta \leq f_3^\alpha f_4^\beta$  almost-everywhere in  $\mathcal{M}$  and that for any probability measure  $\eta$  on  $\mathcal{M}$  which is a  $CD(\kappa, N)$ -needle,

$$\left( \int_{\mathcal{M}} f_1 d\eta \right)^\alpha \left( \int_{\mathcal{M}} f_2 d\eta \right)^\beta \leq \left( \int_{\mathcal{M}} f_3 d\eta \right)^\alpha \left( \int_{\mathcal{M}} f_4 d\eta \right)^\beta$$

whenever  $f_1, f_2, f_3, f_4$  are  $\eta$ -integrable. Then,

$$\left( \int_{\mathcal{M}} f_1 d\mu \right)^\alpha \left( \int_{\mathcal{M}} f_2 d\mu \right)^\beta \leq \left( \int_{\mathcal{M}} f_3 d\mu \right)^\alpha \left( \int_{\mathcal{M}} f_4 d\mu \right)^\beta.$$

Theorem 1.2 was certainly known in the case where  $\mathcal{M} = \mathbb{R}^n$  or  $\mathcal{M} = S^{n-1}$ . However, even in these symmetric spaces, our proof of Theorem 1.2 is very different from the traditional bisection proofs given in Gromov and Milman [29] or Lovász and Simonovits [36]. The geodesic foliations that we construct in Theorem 1.2 are *integrable*, meaning that there is a function  $u : \mathcal{M} \rightarrow \mathbb{R}$  such that the geodesics appearing in the partition are integral curves of  $\nabla u$ . This integrability property makes the construction of the partition somewhat more “canonical”. In contrast, there are many arbitrary choices that one makes during the bisection process, as there could be many hyperplanes that bisect a domain in  $\mathbb{R}^n$  into two subsets of equal volumes. For a function  $u : \mathcal{M} \rightarrow \mathbb{R}$  we define its Lipschitz seminorm by

$$\|u\|_{Lip} = \sup_{x \neq y \in \mathcal{M}} \frac{|u(x) - u(y)|}{d(x, y)}.$$

Given a 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  and a point  $y \in \mathcal{M}$ , we say that  $y$  is a *strain point* of  $u$  if there exist  $x, z \in \mathcal{M}$  for which

$$u(y) - u(x) = d(x, y) > 0, \quad u(z) - u(y) = d(y, z) > 0, \quad d(x, z) = d(x, y) + d(y, z).$$

Write  $Strain[u] \subseteq \mathcal{M}$  for the collection of all strain points of  $u$ . The set  $Strain[u]$  resembles the *transport set* defined at the beginning of Chapter 3 in Evans and Gangbo [21]. It is explained below that  $Strain[u]$  is a measurable subset of  $\mathcal{M}$ , and that the relation

$$x \sim y \quad \iff \quad |u(x) - u(y)| = d(x, y)$$

is an equivalence relation on  $Strain[u]$ . Write  $T^\circ[u]$  for the collection of all equivalence classes. (The rationale behind the small circle in the notation  $T^\circ[u]$  should become clearer in Section 2.1, once  $T[u]$  is defined). It is proven below that for any  $\mathcal{I} \in T^\circ[u]$  there exists a minimizing geodesic  $\gamma : A \rightarrow \mathcal{M}$  with  $\gamma(A) = \mathcal{I}$  and

$$(3) \quad u(\gamma(t)) = t \quad \text{for all } t \in A.$$

Let  $\pi : Strain[u] \rightarrow T^\circ[u]$  be the partition map, i.e.,  $x \in \pi(x) \in T^\circ[u]$  for all  $x \in Strain[u]$ . The conditioning of  $\mu$  with respect to the geodesic foliation  $T^\circ[u]$  is described in the following theorem:

**THEOREM 1.4.** *Let  $n \geq 2, \kappa \in \mathbb{R}$  and  $N \in (-\infty, 1) \cup [n, +\infty)$ . Assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold of class  $CD(\kappa, N)$  which is geodesically-convex. Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Then there exist a measure  $\nu$  on the set  $T^\circ[u]$  and a family  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$  of measures on  $\mathcal{M}$  such that:*

- (i) *For any Lebesgue-measurable set  $A \subseteq \mathcal{M}$ , the map  $\mathcal{I} \mapsto \mu_{\mathcal{I}}(A)$  is well-defined  $\nu$ -almost everywhere and is a  $\nu$ -measurable map. If a subset  $S \subseteq T^\circ[u]$  is  $\nu$ -measurable, then  $\pi^{-1}(S) \subseteq Strain[u]$  is a measurable subset of  $\mathcal{M}$ .*

(ii) For any Lebesgue-measurable set  $A \subseteq \mathcal{M}$ ,

$$\mu(A \cap \text{Strain}[u]) = \int_{T^\circ[u]} \mu_{\mathcal{I}}(A) d\nu(\mathcal{I}).$$

(iii) For  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ , the measure  $\mu_{\mathcal{I}}$  is a  $CD(\kappa, N)$ -needle supported on  $\mathcal{I} \subseteq \mathcal{M}$ . Furthermore, the set  $A \subseteq \mathbb{R}$  and the minimizing geodesic  $\gamma : A \rightarrow \mathcal{M}$  from Definition 1.1 may be selected so that  $\mathcal{I} = \gamma(A)$  and so that (3) holds true.

We call the 1-Lipschitz function  $u$  from Theorem 1.4 the *guiding function* of the needle-decomposition. In the case where the function  $u$  is the distance function from a smooth hypersurface, the conclusion of Theorem 1.4 is essentially a classical computation in Riemannian geometry which may be found in Bayle [5], Gromov [26, 27], Heintze and Karcher [30] and Morgan [39]. That computation is related to Paul Levy's proof of the isoperimetric inequality. It is beneficial to analyze arbitrary Lipschitz functions in Theorem 1.4, because of the relation to the dual Monge-Kantorovich problem presented in the following:

**THEOREM 1.5** (“Localization theorem with a guiding function”). *Let  $n \geq 2$ ,  $\kappa \in \mathbb{R}$  and  $N \in (-\infty, 1) \cup [n, +\infty]$ . Assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold of class  $CD(\kappa, N)$  which is geodesically-convex. Let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a  $\mu$ -integrable function with  $\int_{\mathcal{M}} f d\mu = 0$ . Assume that there exists a point  $x_0 \in \mathcal{M}$  with  $\int_{\mathcal{M}} |f(x)| \cdot d(x_0, x) d\mu(x) < \infty$ . Then,*

(A) *There exists a 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  such that*

$$(4) \quad \int_{\mathcal{M}} u f d\mu = \sup_{\|v\|_{L^1} \leq 1} \int_{\mathcal{M}} v f d\mu.$$

(B) *For any such function  $u$ , the function  $f$  vanishes  $\mu$ -almost everywhere in  $\mathcal{M} \setminus \text{Strain}[u]$ . Furthermore, let  $\nu$  and  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$  be measures on  $T^\circ[u]$  and  $\mathcal{M}$ , respectively, satisfying conclusions (i), (ii) and (iii) of Theorem 1.4. Then for  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ ,*

$$(5) \quad \int_{\mathcal{I}} f d\mu_{\mathcal{I}} = 0.$$

(C) *For any such function  $u$ , there exist  $\Omega, \nu, \{\mu_{\mathcal{I}}\}_{\mathcal{I} \in \Omega}$  satisfying the conclusions of Theorem 1.2, which also satisfy the following property: For  $\nu$ -almost any  $\mathcal{I} \in \Omega$ , there exist an interval  $A \subseteq \mathbb{R}$  and a minimizing geodesic  $\gamma : A \rightarrow \mathcal{M}$  with  $\gamma(A) = \mathcal{I}$  such that (3) holds true.*

Our article owes much to previous investigations of the Monge-Kantorovich problem. An integrable foliation by straight lines satisfying an analog of (5) was mentioned already by Monge in 1781, albeit on a heuristic level (see, e.g., Cayley's review of Monge's work [14]). The optimization problem (4) entered the arena with the work of Kantorovich [32, Section VIII.4].

An analytic resolution of the Monge-Kantorovich problem which is satisfactory for our needs is provided by Evans and Gangbo [21], with subsequent developments by Ambrosio [1], Caffarelli, Feldman and McCann [13], Feldman and McCann [22] and Trudinger and Wang [44]. Ideas from these papers have helped us in dealing with the following difficulty: We are obliged to work with the second fundamental form of the level set  $\{u = t_0\}$  in order to use the Ricci curvature and conclude

that  $\mu_{\mathcal{I}}$  is a  $CD(\kappa, N)$ -needle. However, the function  $u$  is an arbitrary Lipschitz function, and it is not entirely clear how to interpret its Hessian. Chapter 2 is devoted to overcoming this difficulty, by showing that inside the set  $Strain[u]$  the function  $u$  behaves as if it were a  $C^{1,1}$ -function. The conditioning of  $\mu$  with respect to the partition  $T^\circ[u]$  is discussed in Chapter 3, in which we prove Theorem 1.4. Chapter 4 is dedicated to the proofs of Theorem 1.2 and Theorem 1.5.

Throughout this article, by a smooth function or manifold we always mean  $C^\infty$ -smooth. All differentiable manifolds are assumed smooth and all of our Riemannian manifolds have smooth metric tensors. We do not consider Riemannian manifolds with a boundary. When we mention a measure  $\nu$  on a set  $X$  we implicitly consider a  $\sigma$ -algebra of  $\nu$ -measurable subsets of  $X$ . All of our measures in this paper are *complete*, meaning that if  $\nu(A) = 0$  and  $B \subseteq A$ , then  $B$  is  $\nu$ -measurable. When we push-forward the measure  $\nu$ , we implicitly also push-forward its  $\sigma$ -algebra. Note that the concept of a Lebesgue-measurable subset of a differentiable manifold is well-defined (e.g., Section 3.1 below). When we write “a measurable set”, without any reference to a specific measure, we simply mean Lebesgue-measurable. An interval  $A \subseteq \mathbb{R}$  is an arbitrary connected set, which may be bounded or unbounded. We write  $\log$  for the natural logarithm.

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## Regularity of geodesic foliations

### 2.1. Transport rays

Let  $\mathcal{M}$  be an  $n$ -dimensional Riemannian manifold which is geodesically-convex and let  $d$  be the Riemannian distance function on  $\mathcal{M}$ . As before, a curve  $\gamma : I \rightarrow \mathcal{M}$  is a minimizing geodesic if  $I \subseteq \mathbb{R}$  is an interval and

$$d(\gamma(s), \gamma(t)) = d(s, t) \quad \text{for all } s, t \in I.$$

A curve  $\gamma : J \rightarrow \mathcal{M}$  is a *geodesic* if  $J \subseteq \mathbb{R}$  is an interval, and for any  $x \in J$  there exists a relatively-open interval  $I \subseteq J$  containing  $x$  such that  $\gamma|_I$  is a minimizing geodesic. Thus, we only discuss geodesic curves of unit velocity, and not of arbitrary velocity as is customary. For the basic concepts in Riemannian geometry that we use here we refer the reader, e.g., to the first ten pages of Cheeger and Ebin [16]. In particular, it is well-known that all geodesic curves are smooth, and that for  $p \in \mathcal{M}$  and a unit vector  $v \in T_p\mathcal{M}$  there is a unique geodesic curve  $\gamma_{p,v}$  with  $\gamma_{p,v}(0) = p$  and  $\dot{\gamma}_{p,v}(0) = v$ . Let  $I_{p,v} \subseteq \mathbb{R}$  be the maximal set on which  $\gamma_{p,v}$  is well-defined, which is an open interval containing zero. Denote

$$\exp_p(tv) = \gamma_{p,v}(t) \quad \text{for } t \in I_{p,v}.$$

The exponential map  $\exp_p : T_p\mathcal{M} \rightarrow \mathcal{M}$  is a partially-defined function, which is well-defined and smooth on an open subset of  $T_p\mathcal{M}$  containing the origin.

LEMMA 2.1.1. *Let  $A \subseteq \mathbb{R}$  be an arbitrary subset, and let  $\gamma : A \rightarrow \mathcal{M}$  satisfy*

$$(1) \quad d(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in A.$$

*Denote  $\text{conv}(A) = \{\lambda t + (1 - \lambda)s; s, t \in A, 0 \leq \lambda \leq 1\}$ . Then there exists a minimizing geodesic  $\tilde{\gamma} : \text{conv}(A) \rightarrow \mathcal{M}$  with  $\tilde{\gamma}|_A = \gamma$ .*

PROOF. We may assume that  $\#(A) \geq 3$ , because if  $A$  contains only two points  $x$  and  $y$ , then we may connect  $\gamma(x)$  to  $\gamma(y)$  by a minimizing geodesic. Fix  $s \in A$  with  $\inf A < s < \sup A$ . According to (1), for any  $r, t \in A$  with  $r < s < t$ ,

$$(2) \quad d(\gamma(r), \gamma(s)) + d(\gamma(s), \gamma(t)) = d(\gamma(r), \gamma(t)).$$

Denote  $a = \gamma(r), b = \gamma(s), c = \gamma(t)$ . Select any minimizing geodesic  $\gamma_1$  from  $a$  to  $b$ , and any minimizing geodesic  $\gamma_2$  from  $b$  to  $c$ . We claim that  $\gamma_1$  and  $\gamma_2$  make a zero angle at the point  $b$ . Indeed by (2), the concatenation of the curves  $\gamma_1$  and  $\gamma_2$  forms a minimizing geodesic from  $a$  to  $c$ , which is necessarily smooth, hence the curves  $\gamma_1$  and  $\gamma_2$  must fit together at the point  $b$ . We conclude that there exists a unit vector  $v \in T_{\gamma(s)}\mathcal{M}$ , such that for any  $x \in A \setminus \{s\}$ , the vector  $\text{sgn}(x - s)v$  is tangent to any minimizing geodesic from  $\gamma(s)$  to  $\gamma(x)$ . Here,  $\text{sgn}(x)$  is the sign of  $x \in \mathbb{R} \setminus \{0\}$ . Denote

$$\tilde{\gamma}(x) = \exp_{\gamma(s)}((x - s)v).$$

Then  $\tilde{\gamma}$  is the geodesic emanating from  $\gamma(s)$  in the direction of  $v$ , and it satisfies  $\tilde{\gamma}(x) = \gamma(x)$  for any  $x \in A$ . The geodesic curve  $\tilde{\gamma}$  is thus well-defined on the interval  $\text{conv}(A)$ , with  $\tilde{\gamma}|_A = \gamma$ . Furthermore, it follows from (1) that the geodesic  $\tilde{\gamma} : \text{conv}(A) \rightarrow \mathcal{M}$  is a minimizing geodesic, and the lemma is proven.  $\square$

The following definition was proposed by Evans and Gangbo [20] who worked under the assumption that  $\mathcal{M}$  is a Euclidean space, see Feldman and McCann [22] for the generalization to complete Riemannian manifolds.

DEFINITION 2.1.2. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  be a function with  $\|u\|_{Lip} \leq 1$ . A subset  $\mathcal{I} \subseteq \mathcal{M}$  is a “transport ray” associated with  $u$  if*

$$(3) \quad |u(x) - u(y)| = d(x, y) \quad \text{for all } x, y \in \mathcal{I}$$

and if for any  $\mathcal{J} \supsetneq \mathcal{I}$  there exist  $x, y \in \mathcal{J}$  with  $|u(x) - u(y)| \neq d(x, y)$ . In other words,  $\mathcal{I}$  is a maximal set that satisfies condition (3). We write  $T[u]$  for the collection of all transport rays associated with  $u$ .

By continuity, the closure of a transport ray is also a transport ray, and by maximality any transport ray is a closed set. By Zorn’s lemma, any subset  $\mathcal{I} \subseteq \mathcal{M}$  satisfying (3) is contained in a certain transport ray. For the rest of this section, we fix a function  $u : \mathcal{M} \rightarrow \mathbb{R}$  with  $\|u\|_{Lip} \leq 1$ . The following lemma shows that transport rays are geodesic arcs in  $\mathcal{M}$  on which  $u$  grows at speed one. For a map  $F$  defined on a set  $A$  we write  $F(A) = \{F(x); x \in A\}$ .

LEMMA 2.1.3. *Let  $\mathcal{J} \in T[u]$  and denote  $A = u(\mathcal{J}) \subseteq \mathbb{R}$ . Then  $A$  is an interval, and there exists a minimizing geodesic  $\gamma : A \rightarrow \mathcal{M}$  with  $\gamma(A) = \mathcal{J}$  which satisfies*

$$(4) \quad u(\gamma(t)) = t \quad \text{for } t \in A.$$

PROOF. The map  $u : \mathcal{J} \rightarrow A$  is invertible, according to (3). By defining  $\gamma(u(x)) = x$  for  $x \in \mathcal{J}$ , we see from (3) that

$$d(\gamma(s), \gamma(t)) = |s - t| \quad \text{for any } s, t \in A.$$

We may thus apply Lemma 2.1.1, and conclude that  $\gamma$  may be extended to a curve  $\tilde{\gamma} : \text{conv}(A) \rightarrow \mathcal{M}$  which is a minimizing geodesic. Furthermore, since  $\|u\|_{Lip} \leq 1$  with  $u(\gamma(t)) = t$  for  $t \in A$ , then necessarily

$$(5) \quad u(\tilde{\gamma}(t)) = t \quad \text{for } t \in \text{conv}(A).$$

The curve  $\tilde{\gamma}$  is a minimizing geodesic, and its image  $\mathcal{I} = \tilde{\gamma}(\text{conv}(A))$  satisfies (3), thanks to (5). However,  $\mathcal{I} \supseteq \mathcal{J} = \gamma(A)$ , while  $\mathcal{J}$  is a transport ray. The maximality property of  $\mathcal{J}$  entails that  $\mathcal{I} = \mathcal{J}$  and consequently  $A = \text{conv}(A)$  and  $\gamma \equiv \tilde{\gamma}$ . This shows that  $A$  is an interval and that  $\gamma : A \rightarrow \mathcal{M}$  is a minimizing geodesic with  $\gamma(A) = \mathcal{J}$ . The relation (4) follows from (5).  $\square$

Lemma 2.1.3 shows that any transport ray  $\mathcal{I} \subseteq \mathcal{M}$  is a minimizing geodesic with a canonical parametrization  $\gamma : A \rightarrow \mathcal{I}$  satisfying (4). When we write that a unit vector  $v \in T\mathcal{M}$  is tangent to  $\mathcal{I}$  we mean that  $v = \dot{\gamma}(t)$  for some  $t \in A$ . We say that

$$\{\gamma(t); t \in \text{int}(A)\}$$

is the *relative interior* of the transport ray  $\mathcal{I}$ , where  $\text{int}(A) \subseteq \mathbb{R}$  is the interior of the interval  $A \subseteq \mathbb{R}$ . Note that a transport ray  $\mathcal{I}$  could be a singleton, and then its relative interior turns out to be empty. The set

$$\{\gamma(t); t \in A \setminus \text{int}(A)\}$$

is defined to be the *relative boundary* of the transport ray  $\mathcal{I}$ . Since  $A \subseteq \mathbb{R}$  is an interval, then the relative boundary of any transport ray contains at most two points. Recall the following definition from Chapter 1:

DEFINITION 2.1.4. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  be a 1-Lipschitz function. Define  $\text{Strain}[u]$  as the collection of all points  $y \in \mathcal{M}$  for which there exist  $x, z \in \mathcal{M}$  with*

$$(6) \quad d(x, y) = u(y) - u(x) > 0, \quad d(y, z) = u(z) - u(y) > 0, \quad d(x, y) + d(y, z) = d(x, z).$$

LEMMA 2.1.5. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  be a 1-Lipschitz function. Then  $\text{Strain}[u] \subseteq \mathcal{M}$  equals the union of all relative interiors of transport rays associated with  $u$ .*

PROOF. Suppose first that  $y \in \mathcal{M}$  belongs to the relative interior of a transport ray  $\mathcal{I}$ . Let  $\gamma : A \rightarrow \mathcal{I}$  be the parametrization of the minimizing geodesic  $\mathcal{I}$  given by Lemma 2.1.3. Since  $y$  is in the relative interior of  $\mathcal{I}$ , there exists  $r, s, t \in A$  with  $r < s < t$  such that  $\gamma(s) = y$ . Setting

$$x = \gamma(r), \quad z = \gamma(t),$$

we see that (6) holds true in view of (4). Therefore  $y \in \text{Strain}[u]$ . To prove the other direction, let us pick a point  $y \in \text{Strain}[u]$ , and let  $x, z \in \mathcal{M}$  be points such that (6) holds true. The triplet  $\mathcal{I} = \{x, y, z\}$  satisfies (3). By Zorn's lemma,  $\mathcal{I}$  is contained in a transport ray  $\mathcal{J}$ , and the point  $y$  must belong to the relative interior of  $\mathcal{J}$  as  $u(x) < u(y) < u(z)$ .  $\square$

The short proof of the following lemma appears in Feldman and McCann [22, Lemma 10]:

LEMMA 2.1.6. *For any transport ray  $\mathcal{I} \in T[u]$  and a point  $x$  in the relative interior of  $\mathcal{I}$ , the function  $u$  is differentiable at  $x$ , and  $\nabla u(x)$  is a unit vector tangent to  $\mathcal{I}$ .*

LEMMA 2.1.7. *For any  $x \in \text{Strain}[u]$  there exists a unique  $\mathcal{I} \in T[u]$  such that  $x \in \mathcal{I}$ . Furthermore,  $x$  belongs to the relative interior of  $\mathcal{I}$ .*

PROOF. From Lemma 2.1.5 we know that  $x$  belongs to the relative interior of a certain transport ray. Lemma 2.1.6 implies that  $u$  is differentiable at  $x$  and that  $\nabla u(x)$  is a unit vector. Consider the geodesic

$$(7) \quad \gamma(t) = \exp_x(t\nabla u(x))$$

which is well-defined in a maximal interval  $(a, b) \subseteq \mathbb{R}$  containing zero. Define

$$(8) \quad A = \{t \in (a, b); u(\gamma(t)) = u(x) + t\}.$$

Note that  $0 \in A$ . Since  $\gamma$  is a geodesic and  $\|\nabla u\|_{Lip} \leq 1$ , then  $A$  is necessarily an interval and  $\gamma : A \rightarrow \mathcal{M}$  is a minimizing geodesic. In fact, by (8) the set  $\gamma(A)$  is contained in a certain transport ray.

We will show that  $\gamma(A)$  is the unique transport ray containing  $x$ . Indeed, let  $\mathcal{I} \in T[u]$  be a transport ray such that  $x \in \mathcal{I}$ . Since  $x$  is contained in the relative interior of some transport ray, then  $\mathcal{I}$  is not a singleton by the maximality property of transport rays. Note that  $\nabla u(x)$  is necessarily tangent to  $\mathcal{I}$ : this follows from

equation (4) of Lemma 2.1.3 and from the fact that  $\nabla u(x)$  is a unit vector. We conclude from (7), (8) and Lemma 2.1.3 that  $\mathcal{I} \subseteq \gamma(A)$ . However, we said earlier that  $\gamma(A)$  is contained in a certain transport ray, hence by maximality  $\mathcal{I} = \gamma(A)$ . Therefore  $\gamma(A)$  is the unique transport ray containing  $x$ . The point  $x \in \text{Strain}[u]$  belongs to the relative interior of the transport ray  $\gamma(A)$ , according to Lemma 2.1.5.  $\square$

DEFINITION 2.1.8. For a point  $y \in \mathcal{M}$  define  $\alpha_u(y)$  to be the supremum over all  $\varepsilon > 0$  for which there exist  $x, z \in \mathcal{M}$  and  $\delta > 0$  with

$$(9) \quad d(x, y) = u(y) - u(x) \geq \varepsilon, \quad d(y, z) = u(z) - u(y) \geq \delta, \quad d(x, y) + d(y, z) = d(x, z).$$

The supremum over an empty set is defined to be  $-\infty$ . Similarly, for  $y \in \mathcal{M}$  define  $\beta_u(y)$  as the supremum over all  $\delta > 0$  for which there exist  $x, z \in \mathcal{M}$  and  $\varepsilon > 0$  such that (9) holds true.

By comparing (6) and (9), we see that

$$\text{Strain}[u] = \{y \in \mathcal{M}; \alpha_u(y) > 0\} = \{y \in \mathcal{M}; \beta_u(y) > 0\}.$$

LEMMA 2.1.9. Let  $y \in \text{Strain}[u]$  and let  $\mathcal{J} \in T[u]$  be the unique transport ray containing  $y$ . Then,

$$(10) \quad \alpha_u(y) = u(y) - \inf_{z \in \mathcal{J}} u(z), \quad \beta_u(y) = \left[ \sup_{z \in \mathcal{J}} u(z) \right] - u(y).$$

PROOF. If  $x, z \in \mathcal{M}$  and  $\varepsilon, \delta > 0$  satisfy (9), then the triplet  $\mathcal{I} = \{x, y, z\}$  satisfies (3). By Zorn's lemma,  $\mathcal{I}$  is contained in a transport ray, which is necessarily  $\mathcal{J}$ , since  $\mathcal{J}$  is the unique transport ray containing  $y$ . We thus see that in the definition above of  $\alpha_u(y)$  and  $\beta_u(y)$ , it suffices to restrict attention to  $x, z \in \mathcal{J}$ . Now (10) follows from Lemma 2.1.3.  $\square$

The conclusions of Lemma 2.1.3, Lemma 2.1.6 and Lemma 2.1.9 may be summarized as follows:

PROPOSITION 2.1.10. Let  $y \in \text{Strain}[u]$ . Set  $A = (-\alpha_u(y), \beta_u(y)) \subseteq \mathbb{R}$ . Then there exists a minimizing geodesic  $\gamma : A \rightarrow \mathcal{M}$  with  $\gamma(0) = y$ , such that  $\gamma(A)$  is the relative interior of a transport ray, and for all  $t \in A$ ,

$$u(\gamma(t)) = u(y) + t, \quad \dot{\gamma}(t) = \nabla u(\gamma(t)).$$

A transport ray which is a singleton is called a *degenerate* transport ray. According to Lemma 2.1.3, a transport ray  $\mathcal{I} \in T[u]$  is non-degenerate if and only if its relative interior is non-empty.

LEMMA 2.1.11. The following relation is an equivalence relation on  $\text{Strain}[u]$ :

$$(11) \quad x \sim y \quad \iff \quad |u(x) - u(y)| = d(x, y).$$

As in Chapter 1, we write  $T^\circ[u]$  for the collection of all equivalence classes. Then  $T^\circ[u]$  is the collection of all relative interiors of non-degenerate transport rays.



PROOF. According to Lemma 2.1.5 and Lemma 2.1.7, the collection of all relative interiors of non-degenerate transport rays is a partition of  $Strain[u]$ . Let  $x, y \in Strain[u]$ . We need to show that  $x \sim y$  if and only if  $x$  and  $y$  belong to the relative interior of the same transport ray.

Assume first that  $x \sim y$ . Then  $\mathcal{I} = \{x, y\}$  satisfies (3), and hence there exists a transport ray  $\mathcal{J} \in T[u]$  such that  $x, y \in \mathcal{J}$ . However,  $x, y \in Strain[u]$  and  $\mathcal{J}$  is a transport ray containing  $x$  and  $y$ . From Lemma 2.1.7 we conclude that  $x$  and  $y$  belong to the relative interior of  $\mathcal{J}$ . Conversely, suppose that  $x, y \in Strain[u]$  belong to the relative interior of a certain transport ray  $\mathcal{J} \in T[u]$ . By (11) and Definition 2.1.2, we have  $x \sim y$ . The proof is complete.  $\square$

A  $\sigma$ -compact set is a countable union of compact sets. A topological space is second-countable if its topology has a countable basis of open sets. Note that any geodesically-convex, Riemannian manifold  $\mathcal{M}$  is second-countable: Indeed, since  $\mathcal{M}$  is a metric space, it suffices to find a countable, dense subset. Fix  $a \in \mathcal{M}$  and a countable, dense subset of  $T_a\mathcal{M}$ . Since  $\mathcal{M}$  is geodesically-convex, the image of the latter subset under  $\exp_a$  is a countable, dense subset of  $\mathcal{M}$ . Therefore  $\mathcal{M}$  is second-countable, and any open cover of any subset  $S \subseteq \mathcal{M}$  has a countable subcover. Since  $\mathcal{M}$  is locally-compact and second-countable, it is  $\sigma$ -compact.

Define  $\ell_u(y) = \min\{\alpha_u(y), \beta_u(y)\}$  for  $y \in \mathcal{M}$ . Then  $\ell_u(y) > 0$  for  $y \in Strain[u]$  while  $\ell_u(y) = -\infty$  for  $y \notin Strain[u]$ .

LEMMA 2.1.12. *The functions  $\alpha_u, \beta_u, \ell_u : \mathcal{M} \rightarrow \mathbb{R} \cup \{\pm\infty\}$  are Borel-measurable.*

PROOF. We will only prove that  $\alpha_u$  is Borel-measurable. The argument for  $\beta_u$  is similar, while  $\ell_u$  is Borel-measurable as  $\ell_u = \min\{\alpha_u, \beta_u\}$ . For  $\varepsilon, \delta > 0$  we define  $A_{\varepsilon, \delta}$  to be the collection of all triplets  $(x, y, z) \in \mathcal{M}^3$  with

$$d(x, y) = u(y) - u(x) \geq \varepsilon, \quad d(y, z) = u(z) - u(y) \geq \delta, \quad d(x, y) + d(y, z) = d(x, z).$$

Then  $A_{\varepsilon, \delta}$  is a closed set, by the continuity of  $u$  and of the distance function. The Riemannian manifold  $\mathcal{M}$  is  $\sigma$ -compact, hence there exist compacts  $K_1 \subseteq K_2 \subseteq \dots$  such that  $\mathcal{M} = \cup_i K_i$ . Define

$$A_{i, \varepsilon, \delta} = A_{\varepsilon, \delta} \cap (K_i \times K_i \times K_i) \quad (i \geq 1, \varepsilon > 0, \delta > 0).$$

Note that  $A_{i, \varepsilon, \delta}$  is compact and hence  $\pi(A_{i, \varepsilon, \delta})$  is also compact, where  $\pi(x, y, z) = y$ . Clearly,  $A_{\varepsilon, \delta} = \cup_i A_{i, \varepsilon, \delta}$ . Let  $\alpha_{i, \varepsilon, \delta} : \mathcal{M} \rightarrow \mathbb{R} \cup \{-\infty\}$  be the function that equals  $\varepsilon$  on the compact set  $\pi(A_{i, \varepsilon, \delta})$  and equals  $-\infty$  otherwise. Then  $\alpha_{i, \varepsilon, \delta}$  is a Borel-measurable function and by Definition 2.1.8, for any  $y \in \mathcal{M}$ ,

$$\alpha_u(y) = \sup \{\varepsilon > 0; \exists \delta > 0, y \in \pi(A_{\varepsilon, \delta})\} = \sup \{\alpha_{i, \varepsilon, \delta}(y); \varepsilon, \delta \in \mathbb{Q} \cap (0, \infty), i \geq 1\}.$$

Hence  $\alpha_u$  is the supremum of countably many Borel-measurable functions, and is thus necessarily Borel-measurable.  $\square$

For  $\varepsilon > 0$  denote  $Strain_\varepsilon[u] = \{x \in \mathcal{M}; \ell_u(x) > \varepsilon\}$ . Thus,

$$Strain[u] = \bigcup_{\varepsilon > 0} Strain_\varepsilon[u] = \{x \in \mathcal{M}; \ell_u(x) > 0\}.$$

The function  $u$  is basically an arbitrary Lipschitz function, yet the following theorem asserts higher regularity of  $u$  inside the set  $Strain[u]$ . Denote  $B_{\mathcal{M}}(p, \delta) = \{x \in \mathcal{M}; d(x, p) < \delta\}$ .

**THEOREM 2.1.13.** *Let  $\mathcal{M}$  be a geodesically-convex Riemannian manifold. Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  be a function with  $\|u\|_{Lip} \leq 1$ . Let  $p \in \mathcal{M}, \varepsilon_0 > 0$ . Then there exist  $\delta > 0$  and a  $C^{1,1}$ -function  $\tilde{u} : B_{\mathcal{M}}(p, \delta) \rightarrow \mathbb{R}$  such that for any  $x \in \mathcal{M}$ ,*

$$x \in B_{\mathcal{M}}(p, \delta) \cap \text{Strain}_{\varepsilon_0}[u] \quad \implies \quad \tilde{u}(x) = u(x), \quad \nabla \tilde{u}(x) = \nabla u(x).$$

The remainder of this chapter is organized as follows: Section 2.2 contains the standard background on  $C^{1,1}$ -functions. In Section 2.3 we discuss the Riemann normal coordinates, and in Section 2.4 we complete the proof of Theorem 2.1.13. Our proof of Theorem 2.1.13 is related to the arguments of Evans and Gangbo [21] and to the contributions by Ambrosio [1], Caffarelli, Feldman and McCann [13], Feldman and McCann [22] and Trudinger and Wang [44]. The new ingredient in our analysis is the use of Whitney's extension theorem.

## 2.2. Whitney's extension theorem for $C^{1,1}$

Given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  we write  $\partial_i f = \partial f / \partial x_i$  for its  $i^{\text{th}}$  partial derivative, so that  $\nabla f = (\partial_1 f, \dots, \partial_n f)$ . Denote by  $|\cdot|$  the standard Euclidean norm in  $\mathbb{R}^n$ , and  $x \cdot y$  is the usual scalar product of  $x, y \in \mathbb{R}^n$ . For an open, convex set  $K \subseteq \mathbb{R}^n$  and a  $C^1$ -function  $\varphi = (\varphi_1, \dots, \varphi_m) : K \rightarrow \mathbb{R}^m$  we set

$$(1) \quad \|\varphi\|_{C^{1,1}} = \sup_{x \in K} (|\varphi(x)| + \|\varphi'(x)\|_{op}) + \sup_{x \neq y \in K} \frac{\|\varphi'(x) - \varphi'(y)\|_{op}}{|x - y|},$$

where the derivative  $\varphi'(x)$  is an  $m \times n$  matrix whose  $(i, j)$ -entry is  $\partial_j \varphi_i(x)$ , and

$$\|A\|_{op} = \sup_{0 \neq v \in \mathbb{R}^n} |Av|/|v|$$

is the operator norm. Similarly, we may define the  $C^{1,1}$ -norm of a function  $\varphi : K \rightarrow Y$ , where  $X$  and  $Y$  are finite-dimensional linear spaces possessing inner products and where  $K \subseteq X$  is an open, convex set. In fact, formula (1) remains valid in the latter scenario, yet in this case we need to interpret  $\varphi'(x)$  as a linear map from  $X$  to  $Y$  and not as a matrix. For an open set  $U \subseteq \mathbb{R}^n$ , we say that  $f : U \rightarrow \mathbb{R}^m$  is a  $C^{1,1}$ -function if for any  $x \in U$  there exists  $\delta > 0$  such that

$$\left\| f|_{B(x, \delta)} \right\|_{C^{1,1}} < \infty$$

where  $f|_{B(x, \delta)}$  is the restriction of  $f$  to the open ball  $B(x, \delta) = \{y \in \mathbb{R}^n; |y - x| < \delta\}$ . In other words, a  $C^1$ -function  $f : U \rightarrow \mathbb{R}^m$  is a  $C^{1,1}$ -function if and only if the derivative  $f'$  is a locally-Lipschitz map into the space of  $m \times n$  matrices. Any  $C^2$ -function  $f : U \rightarrow \mathbb{R}^m$  is automatically a  $C^{1,1}$ -function. A map  $\varphi : U \rightarrow V$  is a  $C^{1,1}$ -diffeomorphism, for open sets  $U, V \subseteq \mathbb{R}^n$ , if  $\varphi$  is an invertible  $C^{1,1}$ -map and the inverse map  $\varphi^{-1} : V \rightarrow U$  is also  $C^{1,1}$ . The  $C^1$ -version of the following lemma may be found in any textbook on multivariate calculus.

**LEMMA 2.2.1.** *(i) Let  $U_1 \subseteq \mathbb{R}^n$  and  $U_2 \subseteq \mathbb{R}^m$  be open sets. Let  $f_2 : U_2 \rightarrow \mathbb{R}^k$  and  $f_1 : U_1 \rightarrow U_2$  be  $C^{1,1}$ -functions. Then  $f_2 \circ f_1$  is also a  $C^{1,1}$ -function.*

*(ii) Let  $U \subseteq \mathbb{R}^n$  be an open set and let  $f : U \rightarrow \mathbb{R}^n$  be a  $C^{1,1}$ -function. Assume that  $x_0 \in U$  is such that  $\det f'(x_0) \neq 0$ . Then there exists  $\delta > 0$  such that  $f|_{B(x_0, \delta)}$  is a  $C^{1,1}$ -diffeomorphism onto some open set  $V \subseteq \mathbb{R}^n$ .*

- (iii) Let  $U \subseteq \mathbb{R}^n$  be an open set and let  $f : U \rightarrow \mathbb{R}$  be a  $C^{1,1}$ -function. Assume that  $x_0 \in U$  is such that  $\nabla f(x_0) \neq 0$ . Then there exists an open set  $V \subseteq U$  containing the point  $x_0$ , an open set  $\Omega \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$  of the form  $\Omega = \Omega_0 \times (a, b) \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$  and a  $C^{1,1}$ -diffeomorphism  $G : \Omega \rightarrow V$  such that for any  $(y, t) \in \Omega$ ,

$$f(G(y, t)) = t.$$

PROOF. (i) We know that  $h = f_2 \circ f_1$  is a  $C^1$ -function. The map  $x \mapsto f_2'(f_1(x))$  is locally-Lipschitz, since it is the composition of two locally-Lipschitz maps. Since  $f_1'$  is locally-Lipschitz, the product

$$h'(x) = f_2'(f_1(x)) \cdot f_1'(x)$$

is also locally-Lipschitz. Hence  $h$  is a  $C^{1,1}$ -function.

- (ii) The usual inverse function theorem for  $C^1$  guarantees the existence of  $\delta > 0$  and an open set  $V \subseteq \mathbb{R}^n$  such that  $f : B(x_0, \delta) \rightarrow V$  is a  $C^1$ -diffeomorphism. Let  $g : V \rightarrow B(x_0, \delta)$  be the inverse map. The map  $g'(x) = (f'(g(x)))^{-1}$  is the composition of three locally-Lipschitz maps, which are  $x \mapsto g(x)$ ,  $y \mapsto f'(y)$  and  $A \mapsto A^{-1}$ . Hence  $g'$  is locally-Lipschitz and  $g$  is  $C^{1,1}$ .
- (iii) This follows from (ii) in exactly the same way that the implicit function theorem follows from the inverse function theorem in the  $C^1$  case, see e.g. Edwards [19, Chapter III.3].  $\square$

Lemma 2.2.1(i) shows that the concept of a  $C^{1,1}$ -function on a differentiable manifold is well-defined:

DEFINITION 2.2.2. Let  $\mathcal{M}$  and  $\mathcal{N}$  be differentiable manifolds. A function  $f : \mathcal{M} \rightarrow \mathcal{N}$  is a  $C^{1,1}$ -function if  $f$  is  $C^{1,1}$  in any local chart. A  $C^{1,1}$ -function  $f : \mathcal{M} \rightarrow \mathcal{N}$  is a  $C^{1,1}$ -diffeomorphism if it is invertible and the inverse function  $f^{-1} : \mathcal{N} \rightarrow \mathcal{M}$  is also  $C^{1,1}$ .

Let  $K \subseteq \mathbb{R}^n$  be an open, convex set and let  $f : K \rightarrow \mathbb{R}$  satisfy  $M := \|f\|_{C^{1,1}} < \infty$ . It follows from the definition (1) that for  $x, y \in K$ ,

$$(2) \quad |\nabla f(x) - \nabla f(y)| \leq M|x - y|.$$

For  $x, y \in K$  we also have, denoting  $x_t = (1 - t)x + ty$ ,

$$(3) \quad |f(x) + \nabla f(x) \cdot (y - x) - f(y)| = \left| \int_0^1 [\nabla f(x) - \nabla f(x_t)] \cdot (y - x) dt \right| \leq \frac{M}{2}|x - y|^2.$$

Conditions (2) and (3), which are basically Taylor's theorem for  $C^{1,1}$ -functions, capture the essence of the concept of a  $C^{1,1}$ -function, as is demonstrated in Theorem 2.2.3 below. For points  $x, y \in \mathbb{R}^n$  and for  $f : \{x, y\} \rightarrow \mathbb{R}$  and  $V : \{x, y\} \rightarrow \mathbb{R}^n$  we define  $\|(f, V)\|_{x,y}$  to be the infimum over all  $M \geq 0$  for which the following three conditions hold:

- (i)  $|f(x)| \leq M$ ,  $|V(x)| \leq M$ ,
- (ii)  $|V(y) - V(x)| \leq M|y - x|$ ,
- (iii)  $|f(x) + V(x) \cdot (y - x) - f(y)| \leq M|y - x|^2$ .

This infimum is in fact a minimum. Note that  $\|(f, V)\|_{x,y}$  is not necessarily the same as  $\|(f, V)\|_{y,x}$ .

THEOREM 2.2.3 (Whitney's extension theorem for  $C^{1,1}$ ). *Let  $A \subseteq \mathbb{R}^n$  be an arbitrary set, let  $f : A \rightarrow \mathbb{R}$  and  $V : A \rightarrow \mathbb{R}^n$ . Assume that*

$$(4) \quad \sup_{x,y \in A} \|(f, V)\|_{x,y} < \infty.$$

*Then there exists a  $C^{1,1}$ -function  $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$  such that for any  $x \in A$ ,*

$$\tilde{f}(x) = f(x), \quad \nabla \tilde{f}(x) = V(x).$$

For a proof of Theorem 2.2.3 see Stein [42, Chapter VI.2.3] or the original paper by Whitney [45]. Whitney's theorem is usually stated under the additional assumption that  $A \subseteq \mathbb{R}^n$  is a closed set, but it is straightforward to extend  $f$  and  $V$  from  $A$  to the closure  $\bar{A}$  by continuity, preserving the validity of assumption (4).

Given a differentiable manifold  $\mathcal{M}$  and a subset  $A \subseteq \mathcal{M}$ , a 1-form on  $A$  is a map  $\omega : A \rightarrow T^*\mathcal{M}$  with  $\omega(x) \in T_x^*\mathcal{M}$  for  $x \in A$ . Let  $\mathcal{M}, \mathcal{N}$  be differentiable manifolds and let  $\varphi : \mathcal{M} \rightarrow \mathcal{N}$  be a  $C^1$ -map. For a 1-form  $\omega$  on  $A \subseteq \mathcal{N}$  we write  $\varphi^*\omega$  for the pull-back of  $\omega$  under the map  $\varphi$ . Thus  $\varphi^*\omega$  is a 1-form on  $\varphi^{-1}(A)$ . Write  $(\mathbb{R}^n)^*$  for the space of all linear functionals from  $\mathbb{R}^n$  to  $\mathbb{R}$ . With any  $\ell \in (\mathbb{R}^n)^*$  we associate the vector  $V_\ell \in \mathbb{R}^n$  which satisfies

$$\ell(x) = x \cdot V_\ell \quad \text{for any } x \in \mathbb{R}^n.$$

Since  $T_x^*(\mathbb{R}^n)$  is canonically isomorphic to  $(\mathbb{R}^n)^*$ , any 1-form  $\omega$  on a subset  $A \subseteq \mathbb{R}^n$  may be identified with a map  $\omega : A \rightarrow (\mathbb{R}^n)^*$ . Defining  $V_\omega(x) := V_{\omega(x)} \in \mathbb{R}^n$  we recall the formula

$$(5) \quad V_{\varphi^*\omega}(x) = \varphi'(x)^* \cdot V_\omega(\varphi(x)),$$

where  $B^*$  is the transpose of the matrix  $B$ . Here,  $\omega$  is a 1-form on a subset  $A \subseteq \mathbb{R}^m$ , the function  $\varphi$  is a  $C^1$ -map from an open set  $U \subseteq \mathbb{R}^n$  to  $\mathbb{R}^m$ , and the formula (5) is valid for any  $x \in U \cap \varphi^{-1}(A)$ . For  $x, y \in \mathbb{R}^n$  and for  $f : \{x, y\} \rightarrow \mathbb{R}$  and  $\omega : \{x, y\} \rightarrow (\mathbb{R}^n)^*$  we define

$$\|(f, \omega)\|_{x,y} := \|(f, V_\omega)\|_{x,y}.$$

LEMMA 2.2.4. *Let  $K_1, K_2 \subseteq \mathbb{R}^n$  be open, convex sets. Let  $R \geq 1$  and let  $\varphi : K_1 \rightarrow K_2$  be a  $C^1$ -diffeomorphism with*

$$(6) \quad \|\varphi^{-1}\|_{C^{1,1}} \leq R.$$

*Let  $x, y \in K_2$ , denote  $A = \{x, y\}$ , let  $f : A \rightarrow \mathbb{R}$ , and let  $\omega : A \rightarrow (\mathbb{R}^n)^*$  be a 1-form on  $A$ . Denote  $\tilde{A} = \varphi^{-1}(A)$ ,  $\tilde{\omega} = \varphi^*\omega$ ,  $\tilde{f} = f \circ \varphi$ , and  $\tilde{x} = \varphi^{-1}(x)$ ,  $\tilde{y} = \varphi^{-1}(y)$ . Then,*

$$\|(f, \omega)\|_{x,y} \leq C_{n,R} \left\| (\tilde{f}, \tilde{\omega}) \right\|_{\tilde{x}, \tilde{y}},$$

*where  $C_{n,R} > 0$  is a constant depending solely on  $n$  and  $R$ .*

PROOF. It follows from (1), (6) and the convexity of  $K_2$  that the map  $\psi := \varphi^{-1}$  is  $R$ -Lipschitz. Thus,

$$(7) \quad |\tilde{y} - \tilde{x}| = |\psi(y) - \psi(x)| \leq R|y - x|.$$

Set  $V = V_\omega : A \rightarrow \mathbb{R}^n$  and  $\tilde{V} = V_{\tilde{\omega}} : \tilde{A} \rightarrow \mathbb{R}^n$ . Since  $\tilde{\omega} = \varphi^*\omega$  then  $\omega = \psi^*\tilde{\omega}$  and from (5),

$$V(x) = \psi'(x)^* \cdot \tilde{V}(\tilde{x}).$$

Denote  $M = \|(\tilde{f}, \tilde{\omega})\|_{\tilde{x}, \tilde{y}} = \|(\tilde{f}, \tilde{V})\|_{\tilde{x}, \tilde{y}}$ . It suffices to show that  $f$  and  $V$  satisfy conditions (i), (ii) and (iii) from the definition of  $\|(f, V)\|_{x, y}$  with  $M$  replaced by  $2M(R^2 + nR + 1)$ . To that end, observe that

$$|f(x)| = |\tilde{f}(\tilde{x})| \leq M, \quad |V(x)| = |\psi'(x)^* \cdot \tilde{V}(\tilde{x})| \leq MR.$$

Thus condition (i) is satisfied. To prove condition (ii), we compute that

$$(8) \quad |V(y) - V(x)| = \left| \psi'(y)^* \tilde{V}(\tilde{y}) - \psi'(x)^* \tilde{V}(\tilde{x}) \right| \\ \leq \left| \psi'(y)^* (\tilde{V}(\tilde{y}) - \tilde{V}(\tilde{x})) \right| + \left| (\psi'(y)^* - \psi'(x)^*) \tilde{V}(\tilde{x}) \right| \leq RM (|\tilde{y} - \tilde{x}| + |y - x|).$$

Condition (ii) holds in view of (7) and (8). Denote  $\psi = (\psi_1, \dots, \psi_n)$ . From (3) and (7) we obtain

$$|f(x) + V(x) \cdot (y - x) - f(y)| = |\tilde{f}(\tilde{x}) + \psi'(x)^* \tilde{V}(\tilde{x}) \cdot (y - x) - \tilde{f}(\tilde{y})| \\ \leq |\tilde{f}(\tilde{x}) + \tilde{V}(\tilde{x}) \cdot (\tilde{y} - \tilde{x}) - \tilde{f}(\tilde{y})| + |\tilde{V}(\tilde{x})| \cdot |\psi'(x)(y - x) - (\psi(y) - \psi(x))| \\ \leq M|\tilde{x} - \tilde{y}|^2 + M \sum_{i=1}^n |\nabla \psi_i(x) \cdot (y - x) - (\psi_i(y) - \psi_i(x))| \leq M(R^2 + nR)|y - x|^2.$$

Condition (iii) is thus satisfied and the lemma is proven.  $\square$

**COROLLARY 2.2.5.** *Let  $\mathcal{M}$  be an  $n$ -dimensional differentiable manifold, let  $R \geq 1$  and let  $U \subseteq \mathcal{M}$  be an open set. Assume that for any  $a \in U$  we are given a convex, open set  $U_a \subseteq \mathbb{R}^n$  and a  $C^{1,1}$ -diffeomorphism  $\varphi_a : U_a \rightarrow U$ . Suppose that for any  $a, b \in U$ ,*

$$(9) \quad \|\varphi_b^{-1} \circ \varphi_a\|_{C^{1,1}} \leq R.$$

*Let  $A \subseteq U$ . Let  $f : A \rightarrow \mathbb{R}$  and let  $\omega$  be a 1-form on  $A$ . For  $a \in U$  set  $f_a = f \circ \varphi_a$  and  $\omega_a = \varphi_a^* \omega$ . Suppose that for any  $x, y \in A$  there exists  $a \in U$  for which*

$$(10) \quad \|(f_a, \omega_a)\|_{\varphi_a^{-1}(x), \varphi_a^{-1}(y)} \leq R.$$

*Then there exists a  $C^{1,1}$ -function  $\tilde{f} : U \rightarrow \mathbb{R}$  with*

$$(11) \quad \tilde{f}|_A = f, \quad d\tilde{f}|_A = \omega,$$

*where  $d\tilde{f}$  is the differential of the function  $\tilde{f}$ .*

**PROOF.** Fix  $b \in U$  and denote  $A_b = \varphi_b^{-1}(A) \subseteq U_b \subseteq \mathbb{R}^n$ . Abbreviate  $\varphi_{b,a} = \varphi_a^{-1} \circ \varphi_b$ . Let  $x, y \in A_b \subseteq \mathbb{R}^n$ . According to (10) there exists  $a \in U$  for which

$$(12) \quad \|(f_a, \omega_a)\|_{\varphi_{b,a}(x), \varphi_{b,a}(y)} \leq R.$$

We may apply Lemma 2.2.4, thanks to (9) and (12), and conclude that for any  $x, y \in A_b$ ,

$$(13) \quad \|(f_b, \omega_b)\|_{x, y} \leq C_{n, R},$$

for some  $C_{n, R} > 0$  depending only on  $n$  and  $R$ . Recall that for any linear functional  $\ell \in (\mathbb{R}^n)^*$  there corresponds a vector  $V_\ell \in \mathbb{R}^n$  defined via

$$\ell(z) = V_\ell \cdot z \quad (z \in \mathbb{R}^n).$$

In particular, for  $x \in A_b$  we have  $\omega_b(x) \in (\mathbb{R}^n)^*$  and let us set  $V_b(x) := V_{\omega_b(x)} \in \mathbb{R}^n$ . According to (13), the function  $f_b : A_b \rightarrow \mathbb{R}$  and the vector field  $V_b : A_b \rightarrow \mathbb{R}^n$  satisfy

$$\sup_{x,y \in A_b} \|(f_b, V_b)\|_{x,y} \leq C_{n,R} < \infty.$$

Theorem 2.2.3 thus produces a  $C^{1,1}$ -function  $\tilde{f}_b : U_b \rightarrow \mathbb{R}$  with

$$\tilde{f}_b(x) = f_b(x), \quad \nabla \tilde{f}_b(x) = V_b(x) \quad (x \in A_b).$$

In particular  $d\tilde{f}_b|_{A_b} = \omega_b$ . Setting  $\tilde{f}(x) = \tilde{f}_b(\varphi_b^{-1}(x))$  for  $x \in U$ , we obtain a function  $\tilde{f} : U \rightarrow \mathbb{R}$  satisfying (11). The function  $\tilde{f}$  is a  $C^{1,1}$ -function since it is the composition of two  $C^{1,1}$ -functions.  $\square$

REMARK 2.2.6. Corollary 2.2.5 admits the following formal generalization: Rather than stipulating that  $U_a$  is a subset of  $\mathbb{R}^n$  for any  $a \in U$ , we may assume that  $U_a \subseteq X_a$ , where  $X_a$  is an  $n$ -dimensional linear space with an inner product. This generalization is completely straightforward, and it does not involve any substantial modifications to neither the formulation nor the proof of Corollary 2.2.5.

### 2.3. Riemann normal coordinates

Let  $\mathcal{M}$  be an  $n$ -dimensional Riemannian manifold with Riemannian distance function  $d$ . For  $a \in \mathcal{M}$  we write  $\langle \cdot, \cdot \rangle$  for the Riemannian scalar product in  $T_a\mathcal{M}$ , and  $|\cdot|$  is the norm induced by this scalar product. Given a  $C^2$ -function  $h : T_a\mathcal{M} \rightarrow \mathbb{R}$  and a point  $X \in T_a\mathcal{M}$  we may speak of the gradient  $\nabla h(X) \in T_a\mathcal{M}$  and of the Hessian operator  $\nabla^2 h(X) : T_a\mathcal{M} \rightarrow T_a\mathcal{M}$ , which is a symmetric operator such that

$$(1) \quad h(Y) = h(X) + \langle \nabla h(X), Y - X \rangle + \frac{1}{2} \langle \nabla^2 h(X)(Y - X), Y - X \rangle + o(|Y - X|^2).$$

On a formal level, since  $T_a\mathcal{M}$  is a linear space, we canonically identify  $T_X(T_a\mathcal{M}) \cong T_a\mathcal{M}$  for any  $X \in T_a\mathcal{M}$ . Therefore the gradient  $\nabla h(X)$  belongs to  $T_a\mathcal{M} \cong T_X(T_a\mathcal{M})$ . A subset  $U \subseteq \mathcal{M}$  is *strongly convex* if for any two points  $x, y \in U$  there exists a *unique* minimizing geodesic in  $\mathcal{M}$  that connects  $x$  and  $y$ , and furthermore this minimizing geodesic is contained in  $U$ , while there are no other geodesic curves contained in  $U$  that join  $x$  and  $y$ . See, e.g., Chavel [15, Section IX.6] for more information. The following standard lemma expresses the fact that a Riemannian manifold is “locally-Euclidean”.

LEMMA 2.3.1. *Let  $\mathcal{M}$  be a Riemannian manifold and let  $p \in \mathcal{M}$ . Then there exists  $\delta_0 = \delta_0(p) > 0$  such that the following hold:*

- (i) *The ball  $B_{\mathcal{M}}(p, 2\delta_0)$  is strongly convex and its closure is compact.*
- (ii) *Denote  $U = B_{\mathcal{M}}(p, \delta_0/2)$  and for  $a \in U$  set*

$$U_a = \{X \in T_a\mathcal{M}; \exp_a(tX) \in U \text{ for all } t \in [0, 1]\}.$$

*Then  $U_a \subseteq T_a\mathcal{M}$  is a bounded, open set and  $\exp_a$  is a smooth diffeomorphism between  $U_a$  and  $U$ .*

- (iii) *Define  $f_{a,X}(Y) = \frac{1}{2} \cdot d^2(\exp_a X, \exp_a Y)$  for  $a \in U, X, Y \in U_a$ . Then  $f_{a,X} : U_a \rightarrow \mathbb{R}$  is a smooth function, and its Hessian operator  $\nabla^2 f_{a,X}$  satisfies*

$$(2) \quad \frac{1}{2} \cdot Id \leq \nabla^2 f_{a,X}(Y) \leq 2 \cdot Id \quad (a \in U, X, Y \in U_a),$$

in the sense of symmetric operators, where  $Id$  is the identity operator.

(iv) For any  $a, x \in U$  and  $0 < \delta \leq \delta_0$ , the set  $U_a \cap \exp_a^{-1}(B_{\mathcal{M}}(x, \delta))$  is a convex subset of  $T_a\mathcal{M}$ . In particular,  $U_a$  is convex.

(v) For any  $a \in U$  and  $X, Y \in U_a$ ,

$$\frac{1}{2} \cdot |X - Y| \leq d(\exp_a X, \exp_a Y) \leq 2 \cdot |X - Y|.$$

(vi) For  $a, b \in U$  consider the transition map  $\varphi_{a,b} : U_a \rightarrow U_b$  defined by  $\varphi_{a,b} = \exp_b^{-1} \circ \exp_a$ . Then,

$$(3) \quad \sup_{a,b \in U} \|\varphi_{a,b}\|_{C^{1,1}} < \infty.$$

PROOF. We will see that the conclusions of the lemma hold for any sufficiently small  $\delta_0$ , i.e., there exists  $\tilde{\delta}_0 > 0$  such that the conclusions of the lemma hold for any  $0 < \delta_0 < \tilde{\delta}_0$ . For  $a \in \mathcal{M}, \delta > 0$  and  $X \in T_a\mathcal{M}$  we define  $B_{T_a\mathcal{M}}(X, \delta) = \{Y \in T_a\mathcal{M}; |X - Y| < \delta\}$ .

Item (i) is the content of Whitehead's theorem, see [16, Theorem 5.14] or [15, Theorem IX.6.1]. Regarding (ii), note that  $U_a \subseteq B_{T_a\mathcal{M}}(0, \delta_0)$ , and hence  $U_a$  is bounded. The openness of  $U_a$  and the fact that  $\exp_a : U_a \rightarrow U$  is a smooth diffeomorphism are standard, see [16, Chapter I]. Thus (ii) holds true. We move to item (iii). By strong convexity, the function

$$f_{a,X}(Y) = d^2(\exp_a X, \exp_a Y)/2$$

is a smooth function of  $Y \in U_a$ , which depends smoothly also on  $a \in U$  and  $X \in U_a$ . The Hessian operator of  $f_{p,0}$  at the point  $0 \in T_p\mathcal{M}$  is precisely the identity, as follows from (1) and [16, Corollary 1.9]. By smoothness, the Hessian operator of  $f_{a,X}$  at the point  $Y \in T_a\mathcal{M}$  is at least  $\frac{1}{2} \cdot Id$  and at most  $2Id$ , whenever  $a$  is sufficiently close to  $p$  and  $X, Y$  are sufficiently close to zero. In other words, assuming that  $\delta_0$  is at most a certain positive constant determined by  $p$ , we know that for  $a \in B_{\mathcal{M}}(p, 2\delta_0)$  and  $X, Y \in B_{T_a\mathcal{M}}(0, 2\delta_0)$ ,

$$(4) \quad \frac{1}{2} \cdot Id \leq \nabla^2 f_{a,X}(Y) \leq 2 \cdot Id.$$

Thus (iii) is proven. It follows from (4) that the function  $f_{a,X}$  is convex in the Euclidean ball  $B_{T_a\mathcal{M}}(0, 2\delta_0)$ . Let  $a, x \in U$  and  $0 < \delta \leq \delta_0$ . Then  $B_{\mathcal{M}}(x, \delta) \subseteq B_{\mathcal{M}}(a, 2\delta_0)$ . Let  $X \in U_a$  be the unique vector such that  $x = \exp_a(X)$  and observe that

$$(5) \quad \{Y \in U_a; f_{a,X}(Y) \leq \delta^2/2\} = U_a \cap \exp_a^{-1}(B_{\mathcal{M}}(x, \delta)) \subseteq B_{T_a\mathcal{M}}(0, 2\delta_0).$$

Since  $f_{a,X}$  is convex in  $B_{T_a\mathcal{M}}(0, 2\delta_0)$ , then (5) implies that the set

$$U_a \cap \exp_a^{-1}(B_{\mathcal{M}}(x, \delta))$$

is convex. Therefore (iv) is proven. Thanks to the convexity of  $U_a$  we may use Taylor's theorem, and conclude from (2) that for  $a \in U, X, Y \in U_a$ ,

$$(6) \quad \frac{1}{4} \cdot |X - Y|^2 \leq |f_{a,X}(Y) - (f_{a,X}(X) + \nabla f_{a,X}(X) \cdot (Y - X))| \leq |X - Y|^2.$$

However  $f_{a,X}(X) = 0$ , and also  $\nabla f_{a,X}(X) = 0$  since  $Y \mapsto f_{a,X}(Y)$  attains its minimum at the point  $X$ . Therefore (v) follows from (6). Finally, the smooth map  $\varphi_{a,b} = \exp_b^{-1} \circ \exp_a : U_a \rightarrow U_b$  smoothly depends also on  $a, b \in U$ . Since the closure

of  $U$  is compact, the continuous function  $\|\varphi_{a,b}\|_{C^{1,1}}$  is bounded over  $a, b \in U$ , and (3) follows.  $\square$

For the rest of this section, we fix a point  $p \in \mathcal{M}$ , and let  $\delta_0 = \delta_0(p) > 0$  be the radius whose existence is guaranteed by Lemma 2.3.1. Set  $U = B_{\mathcal{M}}(p, \delta_0/2)$  and for  $a \in U$  let  $U_a \subseteq T_a\mathcal{M}$  be defined as in Lemma 2.3.1(ii), so that  $\exp_a : U_a \rightarrow U$  is a smooth diffeomorphism. When we say that a constant  $C$  depends on  $p$ , we implicitly allow this constant to depend also on the Riemannian structure of  $\mathcal{M}$  and on the dimension  $n$ .

Since  $T_X(T_a\mathcal{M}) \cong T_a\mathcal{M}$  for any  $a \in \mathcal{M}$  and  $X \in T_a\mathcal{M}$ , we may view the differential of the map  $\exp_a$  at the point  $X \in T_a\mathcal{M}$  as a map

$$\text{dexp}_X : T_a\mathcal{M} \rightarrow T_x\mathcal{M},$$

where  $x = \exp_a(X)$ . We define  $\Pi_{x,a} : T_x\mathcal{M} \rightarrow T_a\mathcal{M}$  to be the adjoint map, where we identify  $T_x\mathcal{M} \cong T_x^*\mathcal{M}$  and  $T_a\mathcal{M} \cong T_a^*\mathcal{M}$  by using the Riemannian scalar products. In other words, for  $V \in T_x\mathcal{M}$  we define  $\Pi_{x,a}(V) \in T_a\mathcal{M}$  via

$$(7) \quad \langle \Pi_{x,a}(V), W \rangle_a = \langle V, \text{dexp}_X(W) \rangle_x \quad \text{for all } W \in T_a\mathcal{M}.$$

Here,  $\langle \cdot, \cdot \rangle_a$  is the Riemannian scalar product in  $T_a\mathcal{M}$ , and  $\langle \cdot, \cdot \rangle_x$  is the Riemannian scalar product in  $T_x\mathcal{M}$ . Following Feldman and McCann [22], for  $a \in U$  and  $X, Y \in U_a$  we denote  $x = \exp_a(X), y = \exp_a(Y)$  and define

$$F_a(X, Y)$$

to be the unique vector  $V \in U_x \subseteq T_x\mathcal{M}$  for which  $\exp_x(V) = y$ . It follows from Lemma 2.3.1 that the vector  $F_a(X, Y) \in U_x$  is well-defined by strong convexity, as  $x, y \in U$  and  $\exp_x : U_x \rightarrow U$  is a diffeomorphism. Note that for any  $a \in U$  and  $X, Y \in U_a$ ,

$$|F_a(X, Y)| = d(\exp_a X, \exp_a Y).$$

Given  $a \in U$  and  $X, Y \in U_a$  we define

$$(8) \quad \overrightarrow{XY} = \Pi_{x,a}(F_a(X, Y)) \in T_a\mathcal{M}.$$

Intuitively, we think of  $\overrightarrow{XY}$  as a vector in  $T_a\mathcal{M}$  which represents ‘‘how  $\exp_a(Y)$  is viewed from  $\exp_a(X)$ ’’.

LEMMA 2.3.2. *Let  $h : U \rightarrow \mathbb{R}, t \in \mathbb{R}, a \in U$ . Suppose that  $X, Y \in U_a$  and set  $x = \exp_a(X), y = \exp_a(Y)$  and  $h_a = h \circ \exp_a$ . Assume that  $h$  is differentiable at the point  $x$  with  $\nabla h(x) = t \cdot F_a(X, Y)$ . Then  $\nabla h_a(X) = t \cdot \overrightarrow{XY}$ .*

PROOF. Let us pass to 1-forms. Then  $dh_a = \exp_a^*(dh)$ , and for any  $W \in T_a\mathcal{M}$ ,

$$(9) \quad \begin{aligned} \langle \nabla h_a(X), W \rangle_a &= (dh_a)_X(W) = (dh)_x(\text{dexp}_X(W)) \\ &= \langle \nabla h(x), \text{dexp}_X(W) \rangle_x = \langle tF_a(X, Y), \text{dexp}_X(W) \rangle_x. \end{aligned}$$

From (7) and (9) we obtain that  $\nabla h_a(X) = \Pi_{x,a}(tF_a(X, Y)) = t\Pi_{x,a}(F_a(X, Y))$ . The lemma thus follows from (8).  $\square$

LEMMA 2.3.3. *Let  $a \in U, X, Y \in U_a$ . Assume that there exists  $\alpha \in \mathbb{R}$  such that  $X = \alpha Y$ . Then,*

$$(10) \quad \overrightarrow{XY} = Y - X,$$



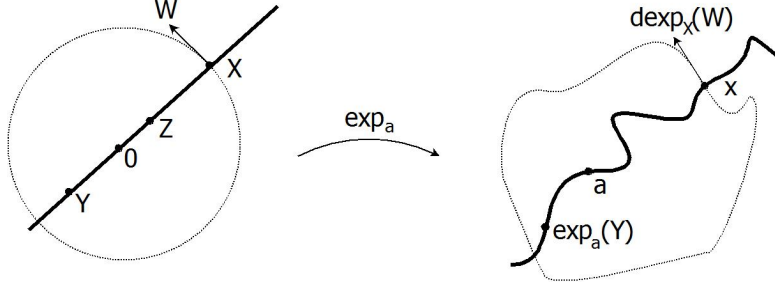


FIGURE 1. Proof of Lemma 2.3.3

and

$$(11) \quad |\overrightarrow{XY}| = d(\exp_a X, \exp_a Y).$$

PROOF. Let  $Z \in T_a \mathcal{M}$  be a unit vector such that  $X$  and  $Y$  are proportional to  $Z$ . Write  $\gamma(t) = \exp_a(tZ)$  for the geodesic leaving  $a$  in direction  $Z$ . Then  $\exp_a(X)$  and  $\exp_a(Y)$  lie on this geodesic and by the strong convexity of  $U$ ,

$$(12) \quad d(\exp_a(X), \exp_a(Y)) = |X - Y|.$$

Let  $\beta \in \mathbb{R}$  be such that  $Y - X = \beta Z$ . Denote  $x = \exp_a(X)$ . The vector  $\text{dexp}_X(Z) \in T_x \mathcal{M}$  is a unit tangent to  $\gamma$ . Additionally,  $F_a(X, Y) \in T_a \mathcal{M}$  is a vector of length  $d(\exp_a X, \exp_a Y) = |Y - X|$  which is tangential to the curve  $\gamma$ . It follows that

$$(13) \quad F_a(X, Y) = \beta \cdot \text{dexp}_X(Z).$$

Let us now use the Gauss lemma [16, Lemma 1.8], which states that for any  $W \in T_a \mathcal{M}$ ,

$$(14) \quad \langle Z, W \rangle_a = 0 \quad \implies \quad \langle \text{dexp}_X(Z), \text{dexp}_X(W) \rangle_x = 0.$$

Since  $Z$  and  $\text{dexp}_X(Z)$  are unit vectors, it follows from (14) that for any  $W \in T_a \mathcal{M}$ ,

$$(15) \quad \langle \text{dexp}_X(Z), \text{dexp}_X(W) \rangle_x = \langle Z, W \rangle_a.$$

From (7), (13) and (15) we conclude that  $\Pi_{x,a}(F_a(X, Y)) = \beta Z$ . Recalling that  $\overrightarrow{XY} = \Pi_{x,a}(F_a(X, Y))$  and that  $\beta Z = Y - X$ , we establish (10). Now (11) follows from (10) and (12).  $\square$

LEMMA 2.3.4. *Let  $a \in U$  and  $X, X_1, X_2, Y, Y_1, Y_2 \in U_a$ . Then,*

$$(16) \quad \left| \overrightarrow{XY_2} - \overrightarrow{XY_1} - (Y_2 - Y_1) \right| \leq C_p \cdot |X| \cdot |Y_2 - Y_1|,$$

and

$$(17) \quad \left| \overrightarrow{X_1Y} - \overrightarrow{X_2Y} - (X_2 - X_1) \right| \leq C_p \cdot |Y| \cdot |X_2 - X_1|.$$

Here,  $C_p > 0$  is a constant depending on  $p$ .

PROOF. For  $a \in U$  and  $X, Y \in U_a$  denote

$$(18) \quad H_{a,X}(Y) = \overrightarrow{XY} - Y.$$

Then  $H_{a,X} : U_a \rightarrow T_a\mathcal{M}$  is a smooth function. Since  $T_a\mathcal{M}$  is a linear space, then at the point  $Y \in U_a$  the derivative  $H'_{a,X}(Y)$  is a linear operator from the space  $T_a\mathcal{M}$  to itself. We claim that there exists a constant  $C_p > 0$  depending on  $p$  such that

$$(19) \quad \|H'_{a,X_2}(Y) - H'_{a,X_1}(Y)\|_{op} \leq C_p \cdot |X_2 - X_1| \quad \text{for } a \in U, X_1, X_2, Y \in U_a,$$

where  $\|S\|_{op} = \sup_{0 \neq V} |S(V)|/|V|$  is the operator norm. Write  $\mathcal{L}(T_a\mathcal{M})$  for the space of linear operators on  $T_a\mathcal{M}$ , equipped with the operator norm. For  $a \in U$  and  $Y \in U_a$  the map

$$(20) \quad U_a \ni X \mapsto H'_{a,X}(Y) \in \mathcal{L}(T_a\mathcal{M})$$

is a smooth map. In fact, the map in (20) may be extended smoothly to the larger domain  $a \in B_{\mathcal{M}}(p, 2\delta_0/3)$ ,  $X, Y \in B_{T_a\mathcal{M}}(0, 4\delta_0/3)$ . Since  $U_a$  is convex with a compact closure, the smooth map in (20) is necessarily a Lipschitz map, and the Lipschitz constant of this map depends continuously on  $a \in U$  and  $Y \in U_a$ . Since the closure of  $U$  is compact, the Lipschitz constant of the map in (20) is bounded over  $a \in U$  and  $Y \in U_a$ . This completes the proof of (19). From (18) and Lemma 2.3.3,

$$(21) \quad H_{a,0}(Y) = 0 \quad \text{for any } Y \in U_a.$$

From (21) we see that  $H'_{a,0}(Y) = 0$  for any  $Y \in U_a$ . The set  $U_a$  is convex, and by applying (19) with  $X_2 = X$  and  $X_1 = 0$  we obtain

$$\sup_{\substack{Y_1, Y_2 \in U_a \\ Y_1 \neq Y_2}} \frac{|H_{a,X}(Y_2) - H_{a,X}(Y_1)|}{|Y_2 - Y_1|} = \sup_{Y \in U_a} \|H'_{a,X}(Y)\|_{op} \leq C_p \cdot |X| \quad \text{for all } X \in U_a,$$

and (16) is proven. In order to prove (17), one needs to analyze  $\tilde{H}_{a,Y}(X) = \overrightarrow{XY} + X$ . According to Lemma 2.3.3 we know that  $\tilde{H}_{a,0}(X) = 0$  for any  $X \in U_a$ . The latter equality replaces (21), and the rest of the proof of (17) is entirely parallel to the analysis of  $H_{a,X}$  presented above.  $\square$

LEMMA 2.3.5. *Let  $a \in U$  and  $t_0 \in \mathbb{R}$ . Assume that  $V, Z \in U_a$  are such that  $t_0V \in U_a$ . Then, in the notation of Lemma 2.3.1(iii),*

$$(22) \quad f_{a,t_0V}(Z) \leq f_{a,t_0V}(V) + \langle (1-t_0)V, Z-V \rangle + |Z-V|^2.$$

PROOF. Fix  $X_0, Y_0 \in U_a$  and define  $x_0 = \exp_a(X_0) \in U$ ,  $y_0 = \exp_a(Y_0) \in U$ . Consider the function  $g_{y_0}(x) = \frac{1}{2} \cdot d(x, y_0)^2$ , defined for  $x \in U$ . Then  $\nabla g_{y_0}(x_0)$  equals the vector  $V \in U_{x_0} \subseteq T_{x_0}\mathcal{M}$  for which  $y_0 = \exp_{x_0}(-V)$ . Consequently,

$$(23) \quad \nabla g_{y_0}(x_0) = -F_a(X_0, Y_0).$$

Since  $f_{a,Y_0} = g_{y_0} \circ \exp_a$ , then from (23) and Lemma 2.3.2,

$$(24) \quad \nabla f_{a,Y_0}(X_0) = -\overrightarrow{X_0Y_0}.$$

According to (24) and Lemma 2.3.3, if  $X, Y \in U_a$  lie on the same line through the origin, then

$$\nabla f_{a,Y}(X) = -\overrightarrow{XY} = -(Y - X) = X - Y.$$

In particular,

$$(25) \quad \nabla f_{a,t_0V}(V) = V - t_0V = (1-t_0)V.$$

We may use Taylor's theorem in the convex set  $U_a \subseteq T_a\mathcal{M}$ , and deduce from the bound (2) in Lemma 2.3.1(iii) that

$$(26) \quad |f_{a,t_0V}(Z) - (f_{a,t_0V}(V) + \langle \nabla f_{a,t_0V}(V), Z - V \rangle)| \leq \frac{1}{2} \cdot 2 \cdot |Z - V|^2.$$

Now (22) follows from (25) and (26).  $\square$

## 2.4. Proof of the regularity theorem

In this section we prove Theorem 2.1.13. We begin with a geometric lemma:

LEMMA 2.4.1 (Feldman and McCann [22]). *Let  $\mathcal{M}$  be a Riemannian manifold with distance function  $d$ , and let  $p \in \mathcal{M}$ . Then there exists  $\delta_1 = \delta_1(p) > 0$  with the following property: Let  $x_0, x_1, x_2, y_0, y_1, y_2 \in B_{\mathcal{M}}(p, \delta_1)$ . Assume that there exists  $\sigma > 0$  such that*

$$(1) \quad d(x_i, x_j) = d(y_i, y_j) = \sigma|i - j| \leq d(x_i, y_j) \quad \text{for } i, j \in \{0, 1, 2\}.$$

Then,

$$\max\{d(x_0, y_0), d(x_2, y_2)\} \leq 10 \cdot d(x_1, y_1).$$

Together with Whitney's extension theorem, Lemma 2.4.1 is the central ingredient in our proof of Theorem 2.1.13. The proof of Lemma 2.4.1 provided by Feldman and McCann in [22, Lemma 16] is very clear and detailed, yet the notation is a bit different from ours. For the convenience of the reader, their proof is reproduced in the Appendix below.

Let us recall the assumptions of Theorem 2.1.13. The Riemannian manifold  $\mathcal{M}$  is geodesically-convex and the function  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfies  $\|u\|_{Lip} \leq 1$ . We are given a point  $p \in \mathcal{M}$  and a number  $\varepsilon_0 > 0$ . Define

$$(2) \quad \delta_2 = \min\left\{\frac{1}{10C_p}, \frac{\delta_0}{2}, \delta_1\right\} > 0$$

where  $C_p$  is the constant from Lemma 2.3.4, the constant  $\delta_0 = \delta_0(p)$  is provided by Lemma 2.3.1, and  $\delta_1 = \delta_1(p)$  is the constant from Lemma 2.4.1. Set

$$(3) \quad \sigma = \min\{\varepsilon_0/2, \delta_2/3\}, \quad M = 1 + \max\left\{\sup_{x \in U} |u(x)|, \frac{10^4}{\sigma}\right\},$$

where as before, we denote

$$U = B_{\mathcal{M}}(p, \delta_0/2),$$

which is a strongly convex set in  $\mathcal{M}$ . For  $a \in U$  we define  $U_a \subseteq T_a\mathcal{M}$  as in Lemma 2.3.1(ii), so that  $\exp_a : U_a \rightarrow U$  is a diffeomorphism.

PROPOSITION 2.4.2. *Let  $x, x_0, x_1, x_2, y_0, y_1, y_2 \in B_{\mathcal{M}}(p, \delta_2) \subseteq U$ . Assume that  $x$  lies on the geodesic arc between  $x_0$  and  $x_2$  while  $x \notin \{x_0, x_2\}$ . Suppose that*

$$(4) \quad u(x_i) = u(y_i) = u(x_0) + i\sigma \quad \text{for } i = 0, 1, 2,$$

and that

$$(5) \quad d(x_i, x_j) = d(y_i, y_j) = \sigma|i - j| \leq d(x_i, y_j) \quad \text{for } i, j \in \{0, 1, 2\}.$$

Denote  $a = x_0$  and let  $X, X_0, X_1, X_2, Y_0, Y_1, Y_2 \in U_a$  be such that  $x = \exp_a(X)$  and  $x_i = \exp_a(X_i), y_i = \exp_a(Y_i)$  for  $i = 0, 1, 2$ .

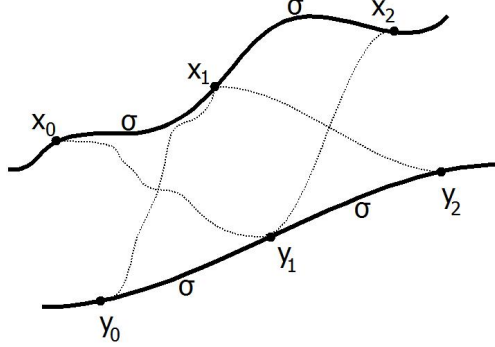


FIGURE 2. The dotted geodesics are of length at least  $\sigma$ .

Denote  $u_a = u \circ \exp_a$ . Then the function  $u_a : U_a \rightarrow \mathbb{R}$  is differentiable at the points  $X$  and  $Y_1$  and the following hold:

- (i)  $|u_a(X)| \leq M$  and  $|\nabla u_a(X)| \leq M$ ,
- (ii)  $|\nabla u_a(X) - \nabla u_a(Y_1)| \leq M|X - Y_1|$ ,
- (iii)  $|u_a(X) + \langle \nabla u_a(X), Y_1 - X \rangle - u_a(Y_1)| \leq M|X - Y_1|^2$ .

Before proving Proposition 2.4.2 let us see how it implies Theorem 2.1.13.

PROOF OF THEOREM 2.1.13. We will prove the theorem with

$$(6) \quad \delta = \sigma/2.$$

We would like to apply Whitney's extension theorem, in the form of Corollary 2.2.5 and Remark 2.2.6. Denote  $\varphi_a = \exp_a : U_a \rightarrow U$  for any  $a \in U$ . Then  $\varphi_a$  is a smooth diffeomorphism between the convex, open set  $U_a \subseteq T_a\mathcal{M}$  and the open set  $U \subseteq \mathcal{M}$ . Thanks to Lemma 2.3.1(vi), there exists a constant  $R = R_p > 0$  depending on  $p$  with the following property: For any  $a, b \in U$ , condition (9) from Corollary 2.2.5 holds true. Denote

$$(7) \quad A = \{x \in B_{\mathcal{M}}(p, \delta); \ell_u(x) > \varepsilon_0\} = B_{\mathcal{M}}(p, \delta) \cap \text{Strain}_{\varepsilon_0}[u].$$

Then  $A \subseteq U = B_{\mathcal{M}}(p, \delta_0/2)$  according to (2), (3) and (6). The function  $u$  is differentiable on the entire set  $A$ , according to Lemma 2.1.6. Define a 1-form  $\omega$  on  $A$  by setting  $\omega = du|_A$ . We will verify that the scalar function  $u : A \rightarrow \mathbb{R}$  and the 1-form  $\omega$  on the set  $A$  satisfy condition (10) from Corollary 2.2.5. In fact, for any  $x, y \in A$  we will show that there exists  $a \in U$  for which

$$(8) \quad \|(u_a, \omega_a)\|_{\varphi_a^{-1}(x), \varphi_a^{-1}(y)} \leq M,$$

where  $M$  was defined in (3) while  $u_a = u \circ \varphi_a$  and  $\omega_a = \varphi_a^*\omega$ . Once we prove (8), we have verified all of the requirements of Corollary 2.2.5 and Remark 2.2.6, and the theorem easily follows: From the conclusion of Corollary 2.2.5, there exists a  $C^{1,1}$ -function  $\tilde{u} : U \rightarrow \mathbb{R}$  with

$$(9) \quad \tilde{u}|_A = u|_A, \quad d\tilde{u}|_A = \omega = du|_A.$$

Since  $U \supseteq B_{\mathcal{M}}(p, \delta)$ , the theorem follows from (7) and (9). Therefore, all that remains is to show that for any  $x, y \in A$  there exists  $a \in U$  for which (8) holds true.

Let us fix  $x, y \in A$ . Since  $\ell_u(x) > \varepsilon_0 \geq 2\sigma$  and also  $\ell_u(y) > 2\sigma$  then by Proposition 2.1.10 there exist minimizing geodesics  $\gamma_x, \gamma_y : (-2\sigma, 2\sigma) \rightarrow \mathcal{M}$  with  $\gamma_x(0) = x, \gamma_y(0) = y$  such that

$$(10) \quad u(\gamma_x(t)) = u(x) + t, \quad u(\gamma_y(t)) = u(y) + t, \quad \text{for } t \in (-2\sigma, 2\sigma).$$

Recall that  $x, y \in A \subseteq B_{\mathcal{M}}(p, \delta)$ . Set  $t_0 = u(y) - u(x)$ . Since  $u$  is 1-Lipschitz, then (6) implies that  $|t_0| < \sigma$ . We now define

$$(11) \quad x_i = \gamma_x(t_0 + (i-1)\sigma), \quad y_i = \gamma_y((i-1)\sigma) \quad \text{for } i = 0, 1, 2.$$

Since  $|t_0| < \sigma$  then  $t_0 + (i-1)\sigma \in (-2\sigma, 2\sigma)$  and the points  $x_0, x_1, x_2, y_0, y_1, y_2$  are well-defined. Since  $t_0 = u(y) - u(x)$  then (10) and (11) yield

$$(12) \quad u(x_i) = u(y_i) = u(x_0) + i\sigma \quad \text{for } i = 0, 1, 2.$$

Recall that  $\|u\|_{Lip} \leq 1$  and that  $\gamma_x, \gamma_y$  are minimizing geodesics. We deduce from (11) and (12) that for  $i, j \in \{0, 1, 2\}$ ,

$$(13) \quad d(x_i, x_j) = d(y_i, y_j) = \sigma|i-j| = |u(x_i) - u(y_j)| \leq d(x_i, y_j).$$

Since  $\gamma_x(0) = x$  and  $|t_0| < \sigma$ , then by (11) the points  $x_0, x_1, x_2$  are of distance at most  $2\sigma$  from  $x$ . Similarly, the points  $y_0, y_1, y_2$  are of distance at most  $\sigma$  from  $y = y_1$ . Since  $x, y \in B_{\mathcal{M}}(p, \delta)$  we obtain

$$(14) \quad x, x_0, x_1, x_2, y_0, y_1, y_2 \in B(p, \delta_2) \subseteq U,$$

as  $\delta \leq \sigma/2 \leq \delta_2/6$ . Recall from (11) that  $x_0 = \gamma_x(t_0 - \sigma)$  and  $x_2 = \gamma_x(t_0 + \sigma)$ . Since  $\gamma_x(0) = x$  and  $|t_0| < \sigma$ , the point  $x$  lies on the geodesic arc from  $x_0$  to  $x_2$ . Furthermore,  $x \notin \{x_0, x_2\}$ . Thus all of the requirements of Proposition 2.4.2 are satisfied: This follows from (12), (13) and (14), as  $y = y_1$ . We are therefore permitted to use the conclusions of Proposition 2.4.2. Denote

$$a = x_0.$$

As in Proposition 2.4.2 we define  $X, X_0, X_1, X_2, Y_0, Y_1, Y_2 \in U_a$  via  $x = \exp_a(X)$  and  $x_i = \exp_a(X_i), y_i = \exp_a(Y_i)$  for  $i = 0, 1, 2$ . Recall that  $\varphi_a = \exp_a$ , that  $u_a = u \circ \varphi_a$  and that  $\omega_a = \varphi_a^* \omega = \varphi_a^*(du|_A) = du_a|_{\varphi_a^{-1}(A)}$ . Finally, the three conclusions of Proposition 2.4.2 mean precisely that

$$\|(u_a, \omega_a)\|_{\varphi_a^{-1}(x), \varphi_a^{-1}(y)} = \|(u_a, \omega_a)\|_{X, Y_1} = \|(u_a, \nabla u_a)\|_{X, Y_1} \leq M.$$

To summarize, given the arbitrary points  $x, y \in A$ , we found  $a \in U$  for which (8) holds true. The proof is thus complete.  $\square$

It still remains to prove Proposition 2.4.2. Recall from the previous section that for  $a \in U$  and  $X, Y \in U_a$  we defined a certain vector  $\overrightarrow{XY} \in T_a \mathcal{M}$ .

LEMMA 2.4.3. *Under the notation and assumptions of Proposition 2.4.2, we have*

$$(15) \quad \left| \overrightarrow{Y_1 Y_2} - \overrightarrow{X_1 X_2} \right| \leq 200 \cdot |X - Y_1|,$$

and

$$(16) \quad |\langle X_1, Y_1 - X_1 \rangle| \leq 7000 \cdot |X - Y_1|^2.$$

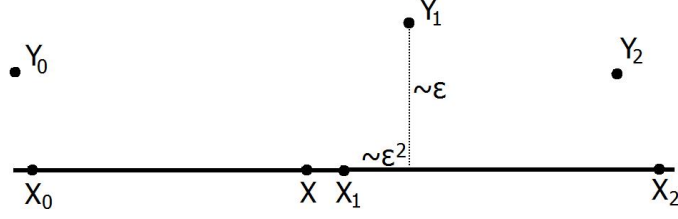


FIGURE 3. The projection of  $Y_1$  to the line is very close to  $X_1$ .

PROOF. Denote  $\varepsilon = d(x, y_1)$ . According to Lemma 2.3.1(v),

$$(17) \quad \frac{\varepsilon}{2} \leq |X - Y_1| \leq 2\varepsilon.$$

From (5), the point  $x_1$  is the midpoint of the geodesic arc between  $x_0$  and  $x_2$ . The point  $x$  also lies on the geodesic between  $x_0$  and  $x_2$ . Let  $K \in \{0, 2\}$  be such that  $x$  lies on the geodesic from  $x_1$  to  $x_K$ . According to (5),

$$(18) \quad d(x_1, x) + d(x, x_K) = d(x_1, x_K) = \sigma.$$

From (5) and (18),

$$(19) \quad \sigma \leq d(x_K, y_1) \leq d(x_K, x) + d(x, y_1) = (\sigma - d(x, x_1)) + d(x, y_1).$$

By using (19) and the fact that  $d(x, y_1) = \varepsilon$  we obtain

$$(20) \quad d(x_1, y_1) \leq d(x_1, x) + d(x, y_1) \leq 2d(x, y_1) = 2\varepsilon.$$

We would like to apply Lemma 2.4.1. Recall from (2) that  $\delta_2 \leq \delta_1$ , where  $\delta_1 = \delta_1(p)$  is the constant from Lemma 2.4.1. Therefore  $x_0, x_1, x_2, y_0, y_1, y_2 \in B_{\mathcal{M}}(p, \delta_1)$ . Moreover, assumption (1) holds in view of (5). We may therefore apply Lemma 2.4.1, and according to its conclusion,

$$(21) \quad d(x_i, y_i) \leq 10 \cdot d(x_1, y_1) \leq 20\varepsilon \quad (i = 0, 1, 2),$$

where we used (20) in the last passage. By Lemma 2.3.1(v), the inequality (21) yields

$$(22) \quad |X_i - Y_i| \leq 40\varepsilon \quad (i = 0, 1, 2).$$

Since  $a = x_0$  and  $\exp_a(X_0) = x_0$ , then  $X_0 = 0$ . According to Lemma 2.3.3, for  $i = 0, 1, 2$ ,

$$(23) \quad |Y_i| = |\overrightarrow{X_0 Y_i}| = d(x_0, y_i) \leq 2\delta_2, \quad |X_i| = |\overrightarrow{X_0 X_i}| = d(x_0, x_i) \leq 2\delta_2,$$

as  $x_0, x_1, x_2, y_0, y_1, y_2 \in B_{\mathcal{M}}(p, \delta_2)$ . From Lemma 2.3.4 combined with (22) and (23),

$$(24) \quad \left| \overrightarrow{Y_1 Y_2} - \overrightarrow{X_1 Y_2} - (X_1 - Y_1) \right| \leq C_p \cdot |Y_2| \cdot |Y_1 - X_1| \leq C_p \cdot 2\delta_2 \cdot 40\varepsilon \leq 10\varepsilon,$$

where we used the fact that  $\delta_2 C_p \leq 1/10$  in the last passage, as follows from (2). Similarly, according to Lemma 2.3.4 and to the inequalities (22) and (23),

$$(25) \quad \left| \overrightarrow{X_1 Y_2} - \overrightarrow{X_1 X_2} - (Y_2 - X_2) \right| \leq C_p \cdot |X_1| \cdot |Y_2 - X_2| \leq C_p \cdot 2\delta_2 \cdot 40\varepsilon \leq 10\varepsilon.$$

Finally, by using (22), (24) and (25),

$$\begin{aligned} |\overrightarrow{Y_1 Y_2} - \overrightarrow{X_1 X_2}| &= |(\overrightarrow{Y_1 Y_2} - \overrightarrow{X_1 Y_2}) + (\overrightarrow{X_1 Y_2} - \overrightarrow{X_1 X_2})| \\ &\leq 20\varepsilon + |(X_1 - Y_1) + (Y_2 - X_2)| \leq 20\varepsilon + |X_1 - Y_1| + |Y_2 - X_2| \leq 100\varepsilon, \end{aligned}$$

and (15) is proven, thanks to (17). We move on to the proof of (16). For  $a \in U$  and  $W, Z \in U_a$  define

$$(26) \quad d_a(W, Z) := d(\exp_a W, \exp_a Z).$$

Then  $d_a^2(W, Z) = 2f_{a,W}(Z)$ , in the notation of Lemma 2.3.1(iii). Using Lemma 2.3.5 with  $V = X_1, t_0 = 0$  and  $Z = Y_1$ ,

$$(27) \quad d_a^2(X_0, Y_1) \leq d_a^2(X_0, X_1) + \langle 2X_1, Y_1 - X_1 \rangle + 2|Y_1 - X_1|^2.$$

From (5) and (26),

$$d_a(X_0, Y_1) = d(x_0, y_1) \geq d(x_0, x_1) = d_a(X_0, X_1).$$

Therefore (27) entails

$$(28) \quad \langle X_1, Y_1 - X_1 \rangle \geq -|Y_1 - X_1|^2.$$

Since  $x_1$  is the midpoint of the geodesic between  $a = x_0$  and  $x_2$ , then  $x_2 = \exp_a(X_2) = \exp_a(2X_1)$ . Hence  $X_2 = 2X_1$ . By using Lemma 2.3.5 with  $V = X_1, t_0 = 2$  and  $Z = Y_1$  we obtain

$$(29) \quad d_a^2(X_2, Y_1) \leq d_a^2(X_2, X_1) + \langle -2X_1, Y_1 - X_1 \rangle + 2|Y_1 - X_1|^2.$$

As before, from (5) and (26) we deduce that  $d_a(X_2, Y_1) \geq d_a(X_2, X_1)$ . Therefore (29) leads to

$$(30) \quad \langle X_1, Y_1 - X_1 \rangle \leq |Y_1 - X_1|^2.$$

The desired conclusion (16) follows from (17), (22), (28) and (30).  $\square$

**PROOF OF PROPOSITION 2.4.2.** It follows from (4) and (5) that for  $i, j \in \{0, 1, 2\}$ ,

$$(31) \quad |u(x_i) - u(x_j)| = |u(y_i) - u(y_j)| = \sigma|i - j| = d(x_i, x_j) = d(y_i, y_j).$$

Recalling Definition 2.1.2, we see that the triplet  $\{x_0, x_1, x_2\}$  is contained in a certain transport ray  $\mathcal{I} \in T[u]$ . By Lemma 2.1.3 and strong convexity, the entire geodesic arc between  $x_0$  and  $x_2$  belongs to  $\mathcal{I}$ . In particular  $x$  belongs to the relative interior of  $\mathcal{I}$ . Since  $x \notin \{x_0, x_2\}$ , it follows from (4) that  $u(x) < u(x_2)$ . Lemma 2.1.6 shows that the function  $u$  is differentiable at  $x$  and  $\nabla u(x)$  is a unit vector tangent to  $\mathcal{I}$ . Hence

$$(32) \quad \nabla u(x) = \frac{F_a(X, X_2)}{d(x, x_2)},$$

according to the definition of  $F_a(X, X_2)$  from Section 2.3. Since  $X_0 = 0$ , we conclude from (5) and Lemma 2.3.3 that  $|X_1| = |\overrightarrow{X_0 X_1}| = d(x_0, x_1) = \sigma$ . Applying Lemma 2.1.3 with the transport ray  $\mathcal{I}$  that contains the three points  $a = x_0, x_1$  and  $x_2$ , we obtain

$$(33) \quad u\left(\exp_a\left(t \cdot \frac{X_1}{\sigma}\right)\right) = u(x_0) + t \quad \text{for } t \in [0, 2].$$

Next, from (31) we also deduce that the triplet  $\{y_0, y_1, y_2\}$  is contained in a certain transport ray  $\mathcal{J}$  and that  $y_1$  belongs to the relative interior of the transport ray  $\mathcal{J}$ .

Additionally,  $u(y_1) < u(y_2)$  according to (4). We deduce from Lemma 2.1.6 that the function  $u$  is differentiable at  $y_1$  with

$$(34) \quad \nabla u(y_1) = \frac{F_a(Y_1, Y_2)}{d(y_1, y_2)} = \frac{F_a(Y_1, Y_2)}{\sigma},$$

where we used (5) in the last passage. Recall that  $a = x_0$ , and that  $x$  and  $x_1$  lie on the geodesic from  $x_0$  to  $x_2$ . Hence  $X$  and  $X_1$  belong to the line segment between  $X_0 = 0$  and  $X_2$  in the tangent space  $T_a\mathcal{M}$ . Since  $x_1$  is the midpoint of the geodesic between  $x_0$  and  $x_2$ , then  $X_2 = 2X_1$ . Consequently, by (5) and Lemma 2.3.3,

$$(35) \quad \overrightarrow{XX_2} = X_2 - X = |X_2 - X| \cdot \frac{X_1}{|X_1|} = |X_2 - X| \cdot \frac{X_2 - X_1}{|X_1|} = d(x, x_2) \cdot \frac{\overrightarrow{X_1X_2}}{\sigma}.$$

Recall that  $u_a = u \circ \exp_a$ . According to Lemma 2.3.2 and to (32), (34) and (35),

$$(36) \quad \nabla u_a(X) = \frac{\overrightarrow{XX_2}}{d(x, x_2)} = \frac{\overrightarrow{X_1X_2}}{\sigma} = \frac{X_2 - X_1}{\sigma} = \frac{X_1}{\sigma}, \quad \nabla u_a(Y_1) = \frac{\overrightarrow{Y_1Y_2}}{\sigma}.$$

It follows from (36) and from conclusion (15) of Lemma 2.4.3 that

$$(37) \quad |\nabla u_a(X) - \nabla u_a(Y_1)| = \frac{1}{\sigma} \cdot |\overrightarrow{X_1X_2} - \overrightarrow{Y_1Y_2}| \leq \frac{200}{\sigma} \cdot |X - Y_1|.$$

Conclusion (ii) of the proposition follows from (37) and the definition (3) of  $M$ . Conclusion (i) holds trivially, as  $|\nabla u_a(X)| = 1 \leq M$  while  $|u_a(X)| \leq M$  by definition. It remains to prove conclusion (iii). It follows from (33) that

$$(38) \quad u_a(tX_1/\sigma) = u(x_0) + t \quad \text{for } t \in [0, 2].$$

Recalling that  $X$  and  $X_1$  belong to the line segment between  $X_0 = 0$  and  $X_2 = 2X_1$ , we learn from (38) that

$$(39) \quad u_a(X) + \langle \nabla u_a(X), X_1 - X \rangle = u_a(X_1).$$

From (4) we know that  $u_a(X_1) = u(x_1) = u(y_1) = u_a(Y_1)$ . We will now use (36) and (39), together with conclusion (16) of Lemma 2.4.3. This yields

$$\begin{aligned} & |u_a(X) + \langle \nabla u_a(X), Y_1 - X \rangle - u_a(Y_1)| \\ &= |u_a(X) + \langle \nabla u_a(X), X_1 - X \rangle + \langle \nabla u_a(X), Y_1 - X_1 \rangle - u_a(Y_1)| \\ &= \left| u_a(X_1) + \left\langle \frac{X_1}{\sigma}, Y_1 - X_1 \right\rangle - u_a(Y_1) \right| = \left| \left\langle \frac{X_1}{\sigma}, Y_1 - X_1 \right\rangle \right| \leq \frac{10^4}{\sigma} |X - Y_1|^2, \end{aligned}$$

and (iii) is proven in view of the definition of  $M$ . This completes the proof of the proposition.  $\square$

By using a partition of unity and a standard argument, we may deduce from Theorem 2.1.13 the following corollary (which will not be needed here):

**COROLLARY 2.4.4.** *Let  $\mathcal{M}$  be a geodesically-convex Riemannian manifold. Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$  and let  $\varepsilon_0 > 0$ . Then there exists a  $C^{1,1}$ -function  $\tilde{u} : \mathcal{M} \rightarrow \mathbb{R}$  such that for any  $x \in \mathcal{M}$ ,*

$$x \in \text{Strain}_{\varepsilon_0}[u] \quad \implies \quad \tilde{u}(x) = u(x), \quad \nabla \tilde{u}(x) = \nabla u(x).$$



## Conditioning a measure with respect to a geodesic foliation

Let  $(\mathcal{M}, d, \mu)$  be a weighted Riemannian manifold of dimension  $n$  which is geodesically-convex. In this section we describe the conditioning of  $\mu$  with respect to the partition  $T^\circ[u]$  associated with a given 1-Lipschitz function  $u$ . The conditioning is based on “ray clusters” which are defined in Section 3.1. Analogous constructions appear in Caffarelli, Feldman and McCann [13], Evans and Gangbo [21], Feldman and McCann [22] and Trudinger and Wang [44]. Section 3.2 explains that the set  $Strain[u]$  may be partitioned into countably many ray clusters. The connection with curvature appears in Section 3.3.

### 3.1. Geodesics emanating from a $C^{1,1}$ -hypersurface

In what follows we prefer to work with a slightly different normalization of the exponential map. For  $t \in \mathbb{R}$  set

$$\text{Exp}_t(v) = \exp_p(tv) \quad (p \in \mathcal{M}, v \in T_p\mathcal{M}).$$

Then  $\text{Exp}_t : T\mathcal{M} \rightarrow \mathcal{M}$  is a partially-defined map which is well-defined and smooth on a maximal open set containing the zero section. That is, for any  $v \in T\mathcal{M}$  there is a maximal interval  $I \subseteq \mathbb{R}$  containing the origin such that  $\text{Exp}_t(v)$  is well-defined for  $t \in I$ . This maximal interval  $I$  is always open, and if  $t \in I$ , then  $\text{Exp}_s(w)$  is well-defined for any  $(w, s) \in T\mathcal{M} \times \mathbb{R}$  in a neighborhood of  $(v, t) \in T\mathcal{M} \times \mathbb{R}$ . Write  $d\text{Exp}_t : T(T\mathcal{M}) \rightarrow T\mathcal{M}$  for the differential of the map  $\text{Exp}_t : T\mathcal{M} \rightarrow \mathcal{M}$ . The maps  $\text{Exp}_t$  and  $d\text{Exp}_t$  are smooth in all of their variables, including the  $t$ -variable. Note that for any  $p \in \mathcal{M}$  and  $v \in T_p\mathcal{M}$ ,

$$(1) \quad \text{Exp}_0(v) = p \quad \text{and} \quad \left. \frac{d}{dt} \text{Exp}_t(v) \right|_{t=0} = v.$$

Let  $\gamma : (a, b) \rightarrow \mathcal{M}$  be a smooth curve with  $a, b \in \mathbb{R} \cup \{\pm\infty\}$ . We say that  $J : (a, b) \rightarrow T\mathcal{M}$  is a smooth vector field along  $\gamma$  if  $J$  is smooth and  $J(t) \in T_{\gamma(t)}\mathcal{M}$  for any  $t \in (a, b)$ . As in Cheeger and Ebin [16, Section 1.1], we may use the Riemannian connection and consider the *covariant derivative* of  $J$  along  $\gamma$ , denoted by

$$J' = \nabla_{\dot{\gamma}} J.$$

Then  $J' : (a, b) \rightarrow T\mathcal{M}$  is a well-defined, smooth vector field along  $\gamma$ . Assume that  $\gamma : (a, b) \rightarrow \mathcal{M}$  is a geodesic. We say that a smooth vector field  $J$  along  $\gamma$  is a *Jacobi field* if

$$(2) \quad J''(t) = R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) \quad \text{for } t \in (a, b),$$

where  $R$  is the *Riemann curvature tensor*. We refer the reader to Burns and Gidea [11, Chapter 5] and to Cheeger and Ebin [16, Chapter I] for background on the

Jacobi equation (2). The space of Jacobi fields along the fixed geodesic curve  $\gamma$  is a linear space of dimension  $2n$ . In fact, we may parameterize the space of Jacobi fields along  $\gamma$  by the  $(2n)$ -dimensional vector space  $T_{\dot{\gamma}(0)}(T\mathcal{M})$ , where we assume for simplicity that  $0 \in (a, b)$ . The parametrization is defined as follows: For  $\xi \in T_{\dot{\gamma}(0)}(T\mathcal{M})$  we define a Jacobi field  $J$  via

$$(3) \quad J(t) = \text{dExp}_t(\xi) \quad \text{for } t \in (a, b).$$

Let  $V : \mathcal{M} \rightarrow T\mathcal{M}$  be a vector field on  $\mathcal{M}$ , i.e.,  $V(p) \in T_p\mathcal{M}$  for any  $p \in \mathcal{M}$ . When we say that a vector field  $V$  on the manifold  $\mathcal{M}$  is differentiable at a point  $p \in \mathcal{M}$ , we mean that  $V : \mathcal{M} \rightarrow T\mathcal{M}$  is differentiable at  $p \in \mathcal{M}$  as a map between the differentiable manifolds  $\mathcal{M}$  and  $T\mathcal{M}$ . For a vector field  $V$  on  $\mathcal{M}$  that is differentiable at the point  $p \in \mathcal{M}$  and for  $w \in T_p\mathcal{M}$  we define

$$\partial_w V := \left. \frac{d}{ds} V(\beta(s)) \right|_{s=0} \in T_{V(p)}(T\mathcal{M})$$

where  $\beta : (-1, 1) \rightarrow \mathcal{M}$  is any smooth curve with  $\beta(0) = p$  and  $\dot{\beta}(0) = w$ . As is customary, we write  $\nabla_w V \in T_p\mathcal{M}$  for the covariant derivative of  $V$  with respect to the Riemannian connection. It is explained in [11, Section 5.8] that there exists a linear map  $\text{Vert}_{V(p)} : T_{V(p)}(T\mathcal{M}) \rightarrow T_p\mathcal{M}$  determined by the Riemannian connection, which is called the *vertical projection*, such that

$$(4) \quad \nabla_w V = \text{Vert}_{V(p)}(\partial_w V).$$

LEMMA 3.1.1. *Let  $a \in [-\infty, 0)$ ,  $b \in (0, +\infty]$ , let  $\gamma : (a, b) \rightarrow \mathcal{M}$  be a geodesic and let  $\xi \in T_{\dot{\gamma}(0)}(T\mathcal{M})$ . Let  $J(t)$  be the Jacobi field along  $\gamma$  that is given by (3). Assume that  $V$  is a vector field on  $\mathcal{M}$  that is differentiable at the point  $\gamma(0) \in \mathcal{M}$  and satisfies  $V(\gamma(0)) = \dot{\gamma}(0)$  and  $\partial_{J(0)} V = \xi$ . Then,*

$$J'(0) = \nabla_{J(0)} V.$$

PROOF. Denote  $p = \gamma(0)$ . Proposition 5.9.2 in [11] states that, assuming the relation (3),

$$(5) \quad J'(0) = \text{Vert}_{V(p)}(\xi).$$

Since  $\partial_{J(0)} V = \xi$ , the conclusion of the lemma follows from (4) and (5).  $\square$

Write  $\lambda_n$  for the Lebesgue measure in  $\mathbb{R}^n$ . Recall that given a measurable set  $A \subseteq \mathbb{R}^n$ , a point  $x \in A$  is called a Lebesgue density point of  $A$  if for any sequence of Euclidean balls  $(B_k)_{k=1, \dots, \infty}$  containing  $x$  with  $\text{Diam}(B_k) \rightarrow 0$ , we have

$$(6) \quad \frac{\lambda_n(A \cap B_k)}{\lambda_n(B_k)} \xrightarrow{k \rightarrow \infty} 1.$$

The Lebesgue theorem states that almost any point  $x \in A$  is a Lebesgue density point, see, e.g., [43, Section 3.1.2]. When  $x$  is a Lebesgue density point of  $A$ , conclusion (6) remains valid for any sequence of measurable sets  $(B_k)_{k=1, \dots, \infty}$  containing  $x$  with  $\text{Diam}(B_k) \rightarrow 0$ , provided that

$$(7) \quad \inf_k \frac{\lambda_n(B_k)}{\text{Diam}(B_k)^n} > 0.$$

Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at the point  $x \in \mathbb{R}^n$ . Assume that  $A \subseteq \mathbb{R}^n$  is a measurable set of which  $x$  is a Lebesgue density point, and that  $f$  is constant

on the set  $A$ . We claim that necessarily  $\nabla f(x) = 0$ : In fact, this is a standard exercise in real analysis, which may be solved by applying (6) with  $(B_k)_{k=1,2,\dots}$  being a sequence of truncated cones.

We say that a  $C^1$ -function  $f : \mathcal{M} \rightarrow \mathbb{R}$  is *twice differentiable with a symmetric Hessian* at the point  $p \in \mathcal{M}$  if the vector field  $\nabla f$  is differentiable at the point  $p$  and

$$\langle \nabla_v(\nabla f), w \rangle = \langle \nabla_w(\nabla f), v \rangle \quad \text{for } v, w \in T_p \mathcal{M}.$$

The notation of the next lemma will accompany us now for several pages. We will consider geodesics orthogonal to the level set  $\{\tilde{u} = r_0\}$ , where  $\tilde{u} : \mathcal{M} \rightarrow \mathbb{R}$  is usually twice differentiable with a symmetric Hessian. We will assume that this level set is locally parameterized by a  $C^1$ -function  $f : \Omega_0 \rightarrow \mathcal{M}$  where  $\Omega_0 \subseteq \mathbb{R}^{n-1}$  is an open set. The geodesics are denoted by  $\tilde{F}(y, t) = \text{Exp}_t(\nabla \tilde{u}(f(y)))$ . Later on, the restriction of  $\tilde{F}$  to a certain set will be denoted by  $F$ , while  $\tilde{u}$  will be the function provided by Theorem 2.1.13. By differentiating  $\tilde{F}(y, t)$  with respect to  $y_i$  we obtain a Jacobi field  $J_i$ , as is precisely stated in the following lemma:

LEMMA 3.1.2. *Let  $r_0 \in \mathbb{R}$  and let  $\tilde{u} : \mathcal{M} \rightarrow \mathbb{R}$  be a  $C^1$ -function. Let  $\Omega_0 \subseteq \mathbb{R}^{n-1}$  be an open set and let  $y_0 \in \Omega_0$ . Let  $f : \Omega_0 \rightarrow \mathcal{M}$  be a  $C^1$ -map, and assume that the function  $\tilde{u}$  is twice differentiable with a symmetric Hessian at the point  $f(y_0)$ . For  $y \in \Omega_0$  and  $t \in \mathbb{R}$  set*

$$\tilde{F}(y, t) = \text{Exp}_t(\nabla \tilde{u}(f(y))), \quad N(y, t) = \frac{\partial \tilde{F}}{\partial t}(y, t).$$

*Our Riemannian manifold is not necessarily complete, and we assume that  $t \mapsto \tilde{F}(y, t)$  is well-defined in a maximal interval  $(a_y, b_y) \subseteq \mathbb{R}$  containing the origin. Suppose that  $B_0 \subseteq \Omega_0$  is a measurable set containing  $y_0$ , such that  $y_0$  is a Lebesgue density point of  $B_0 \subseteq \mathbb{R}^{n-1}$ , and*

$$(8) \quad \tilde{u}(f(y)) = r_0, \quad |\nabla \tilde{u}(f(y))| = 1 \quad \text{for } y \in B_0.$$

*Then,*

- (i) *For any  $t \in (a_{y_0}, b_{y_0})$  the map  $\tilde{F}$  is differentiable at the point  $(y_0, t) \in \Omega_0 \times \mathbb{R}$ . (Note that  $\tilde{F}$  is well-defined in an open neighborhood of  $(y_0, t)$  in  $\mathbb{R}^{n-1} \times \mathbb{R}$ ).*
- (ii) *There exist Jacobi fields  $J_1(y_0, t), \dots, J_{n-1}(y_0, t)$  along the geodesic curve  $t \mapsto \tilde{F}(y_0, t)$ , which are well-defined in the entire interval  $t \in (a_{y_0}, b_{y_0})$ , such that*

$$J_i(y_0, t) = \frac{\partial \tilde{F}}{\partial y_i}(y_0, t) \quad \text{for all } i = 1, \dots, n-1, \quad t \in (a_{y_0}, b_{y_0}).$$

- (iii) *At the point  $(y_0, 0) \in \Omega_0 \times \mathbb{R}$  we have*

$$(9) \quad \langle J_i, N \rangle = \langle J'_i, N \rangle = 0 \quad (i = 1, \dots, n-1),$$

*and*

$$(10) \quad \langle J'_i, J_k \rangle = \langle J'_k, J_i \rangle \quad (i, k = 1, \dots, n-1).$$

*Here,  $J'_i(y_0, t)$  is the covariant derivative of the Jacobi field  $t \mapsto J_i(y_0, t)$  along the geodesic curve  $t \mapsto \tilde{F}(y_0, t)$  for  $t \in (a_{y_0}, b_{y_0})$ .*

PROOF. The curve  $t \mapsto \tilde{F}(y_0, t)$  is a geodesic curve of velocity one since  $|\nabla \tilde{u}(f(y_0))| = 1$  as follows from (8) and the fact that  $y_0 \in B_0$ . The vector field  $t \mapsto N(y_0, t)$  is the unit tangent along this geodesic, with  $N(y_0, 0) = \nabla \tilde{u}(f(y_0))$ , as follows from (1). The equation

$$(11) \quad \tilde{F}(y, t) = \text{Exp}_t(\nabla \tilde{u}(f(y)))$$

is valid in an open set in  $\Omega_0 \times \mathbb{R}$  containing  $\{y_0\} \times (a_{y_0}, b_{y_0})$ . The map  $f$  is a  $C^1$ -map and the vector field  $\nabla \tilde{u} : \mathcal{M} \rightarrow T\mathcal{M}$  is differentiable at the point  $f(y_0)$ . Hence the map  $y \mapsto \nabla \tilde{u}(f(y))$  is differentiable at  $y_0$ . Since  $(t, v) \mapsto \text{Exp}_t(v)$  is a smooth map, we conclude that for any  $t \in (a_{y_0}, b_{y_0})$ , the map  $\tilde{F}$  is differentiable at  $(y_0, t)$ , and (i) is proven. Consequently, differentiating (11) at the point  $y = y_0$  yields

$$(12) \quad J_i(y_0, t) := \frac{\partial \tilde{F}}{\partial y_i}(y_0, t) = d\text{Exp}_t(\xi_{y_0, i}) \quad \text{for } t \in (a_{y_0}, b_{y_0}), i = 1, \dots, n-1,$$

where

$$(13) \quad \xi_{y_0, i} = \frac{\partial[(\nabla \tilde{u}) \circ f]}{\partial y_i}(y_0) \in T_{N(y_0, 0)}(T\mathcal{M}).$$

Note that  $\tilde{F}(y, 0) = f(y)$  for  $y \in \Omega_0$ , and hence

$$(14) \quad \frac{\partial f}{\partial y_i}(y_0) = \frac{\partial \tilde{F}}{\partial y_i}(y_0, 0).$$

From (12), (13) and (14) we obtain

$$(15) \quad \xi_{y_0, i} = \partial_{J_i(y_0, 0)} \nabla \tilde{u} \quad \text{for } i = 1, \dots, n-1.$$

From (12) we learn that the vector fields  $J_1(y_0, t), \dots, J_{n-1}(y_0, t)$  have the form (3), and hence they are Jacobi fields along the geodesic  $t \mapsto \tilde{F}(y_0, t)$ . This proves (ii). Thanks to (12) and (15) we may apply Lemma 3.1.1 with  $V = \nabla \tilde{u}$ ,  $\xi = \xi_{y_0, i}$ ,  $\gamma(t) = \tilde{F}(y_0, t)$  and  $J(t) = J_i(y_0, t)$ , and conclude that

$$(16) \quad J'_i(y_0, 0) = \nabla_{J_i(y_0, 0)} \nabla \tilde{u} \quad \text{for } i = 1, \dots, n-1.$$

Since  $y_0$  is a Lebesgue density point of  $B_0$ , then (8) entails that for  $i = 1, \dots, n-1$ ,

$$(17) \quad \left. \frac{\partial \tilde{u}(f(y))}{\partial y_i} \right|_{y=y_0} = 0 \quad \text{and} \quad \left. \frac{\partial |\nabla \tilde{u}(f(y))|}{\partial y_i} \right|_{y=y_0} = 0.$$

Since  $J_i(y_0, 0) = \frac{\partial \tilde{F}}{\partial y_i}(y_0, 0) = \frac{\partial f}{\partial y_i}(y_0)$  and  $N(y_0, 0) = \nabla \tilde{u}(f(y_0))$ , we may rewrite (17) as

$$(18) \quad \langle N(y_0, 0), J_i(y_0, 0) \rangle = 0 \quad \text{and} \quad \langle \nabla_{J_i(y_0, 0)} \nabla \tilde{u}, N(y_0, 0) \rangle = 0,$$

for  $i = 1, \dots, n-1$ . Now (9) follows from (16) and (18). As for the proof of (10): in view of (16) we actually need to prove that

$$\langle \nabla_{J_i(y_0, 0)} \nabla \tilde{u}, J_k(y_0, 0) \rangle = \langle \nabla_{J_k(y_0, 0)} \nabla \tilde{u}, J_i(y_0, 0) \rangle \quad \text{for } i, k = 1, \dots, n-1.$$

The latter relations hold as  $\tilde{u}$  is twice differentiable with a symmetric Hessian at the point  $f(y_0) = \tilde{F}(y_0, 0)$ .  $\square$

Recall the definitions of  $\text{Strain}[u]$ ,  $\text{Strain}_{\varepsilon_0}[u]$  and  $\alpha_u, \beta_u$  from Section 2.1.

DEFINITION 3.1.3. Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$  and let  $R_0 \subseteq \mathcal{M}$  be a Borel set. We say that  $R_0$  is a “seed of a ray cluster” associated with  $u$  if there exist numbers  $r_0 \in \mathbb{R}, \varepsilon_0 > 0$ , open sets  $U \subseteq \mathcal{M}, \Omega_0 \subseteq \mathbb{R}^{n-1}$  and  $C^{1,1}$ -functions  $\tilde{u} : U \rightarrow \mathbb{R}, f : \Omega_0 \rightarrow \mathcal{M}$  for which the following hold:

- (i) For any  $x \in U \cap \text{Strain}_{\varepsilon_0}[u]$  we have that  $\tilde{u}(x) = u(x)$  and  $\nabla \tilde{u}(x) = \nabla u(x)$ .
- (ii) The  $C^{1,1}$ -map  $f : \Omega_0 \rightarrow \mathcal{M}$  is one-to-one with  $f(\Omega_0) = \{x \in U; \tilde{u}(x) = r_0\}$ . The inverse map  $f^{-1} : f(\Omega_0) \rightarrow \Omega_0$  is continuous.
- (iii) For almost any point  $y \in \Omega_0$ , the function  $\tilde{u}$  is twice differentiable with a symmetric Hessian at the point  $f(y)$ .
- (iv)  $R_0 \subseteq \{x \in U \cap \text{Strain}_{\varepsilon_0}[u]; \tilde{u}(x) = r_0\}$ .

If the functions  $\alpha_u, \beta_u : R_0 \rightarrow \mathbb{R} \cup \{\pm\infty\}$  are continuous, then we say that  $R_0$  is a “seed of a ray cluster of continuous length”.

Note that any Borel set which is contained in a seed of a ray cluster, is in itself a seed of a ray cluster. Recall from Lemma 2.1.11 that  $T^\circ[u]$  is the collection of all relative interiors of non-degenerate transport rays associated with  $u$ , and that  $T^\circ[u]$  is a partition of  $\text{Strain}[u]$ .

DEFINITION 3.1.4. Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . A subset  $R \subseteq \text{Strain}[u]$  is a “ray cluster” associated with  $u$  if there exists  $R_0 \subseteq \mathcal{M}$  which is a seed of a ray cluster such that

$$(19) \quad R = \{x \in \mathcal{M}; \exists \mathcal{I} \in T^\circ[u] \text{ such that } x \in \mathcal{I} \text{ and } \mathcal{I} \cap R_0 \neq \emptyset\}.$$

We say that  $R$  is a “ray cluster of continuous length” if  $R_0$  is a seed of a ray cluster of continuous length.

When  $A \subseteq \mathbb{R}^n$  is a measurable set and  $f : A \rightarrow \mathbb{R}^m$  is locally-Lipschitz, the function  $f$  maps measurable sets to measurable sets: Indeed, any measurable set equals the union of a Lebesgue-null set and countably many compacts, hence also its image under a locally-Lipschitz map is the union of a Lebesgue-null set and countably many compacts. Therefore, the concept of a measurable subset of a differentiable manifold  $\mathcal{M}$  is well-defined. Similarly, the concepts of a Lebesgue-null set and a Lebesgue density point of a measurable set in a differentiable manifold  $\mathcal{M}$  are well-defined. The Lebesgue theorem, stating that almost any point of a measurable set  $A$  is a Lebesgue density point of  $A$ , also applies in the context of an abstract differentiable manifold.

For an arbitrary subset  $A \subseteq \mathbb{R}^n$ , a function  $f : A \rightarrow \mathbb{R}^m$  and a point  $x_0 \in A$ , we say that  $f$  is differentiable at  $x_0$  if there is a *unique* linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{A \ni x \rightarrow x_0} |f(x_0) + T(x - x_0) - f(x)|/|x - x_0| = 0.$$

In this case we may speak of the differential of  $f$  at  $x_0$ . When  $A \subseteq \mathbb{R}^n$  is open, our definition coincides with the familiar notion of differentiability. If  $f : A \rightarrow \mathbb{R}^m$  is differentiable at the point  $x \in A \subseteq \mathbb{R}^n$ , and  $B \subseteq A$  is a measurable set containing  $x$  such that  $x$  is a Lebesgue density point of  $B$ , then  $f|_B$  is differentiable at  $x$ .

Similarly, given differentiable manifolds  $\mathcal{M}$  and  $\mathcal{N}$ , an arbitrary subset  $A \subseteq \mathcal{M}$  and a function  $f : A \rightarrow \mathcal{N}$ , we may speak about the differentiability of  $f$  at the point  $p \in A$ . When  $f$  is differentiable at  $p$ , we may consider the differential of  $f$  at the point  $p$ .

A function  $f : A \rightarrow \mathbb{R}^m$ , where  $A \subseteq \mathbb{R}^n$  is an arbitrary set, is locally-Lipschitz if any point in  $A$  has a neighborhood in which  $f$  is Lipschitz. Similarly, when  $\mathcal{M}$  and  $\mathcal{N}$  are differentiable manifolds and  $A \subseteq \mathcal{M}$  is an arbitrary set, we say that  $f : A \rightarrow \mathcal{N}$  is locally-Lipschitz when it is locally-Lipschitz in any chart. By the Rademacher theorem and the Kirszbraun theorem (see, e.g., Evans and Gariepy [21, Section 3.1]), any locally-Lipschitz function defined on a measurable subset  $A$  of a differentiable manifold, is differentiable almost-everywhere in  $A$ .

A *parallel line-cluster* is a subset  $B \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$  of the following form: There exist a measurable set  $B_0 \subseteq \mathbb{R}^{n-1}$  and continuous functions  $a : B_0 \rightarrow [-\infty, 0)$  and  $b : B_0 \rightarrow (0, +\infty]$  such that

$$(20) \quad B = \{(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}; y \in B_0, a_y < t < b_y\},$$

where  $a_y = a(y)$  and  $b_y = b(y)$  for  $y \in B_0$ . Note that when  $y \in B_0$  is a Lebesgue density point of  $B_0$ , the point  $(y, t) \in B$  is a Lebesgue density point of  $B$  for any  $t \in (a_y, b_y)$ .

An *almost line-cluster* is a subset  $B \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$  of the form (20) where  $B_0 \subseteq \mathbb{R}^{n-1}$  is measurable and the functions  $a : B_0 \rightarrow [-\infty, 0)$  and  $b : B_0 \rightarrow (0, +\infty]$  are only assumed to be measurable, and not continuous. Note that a parallel line-cluster is always measurable, as well as an almost line-cluster. We say that a map  $F$  is invertible if it is one-to-one and onto.

**PROPOSITION 3.1.5.** *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Suppose that  $R \subseteq \text{Strain}[u]$  is a non-empty ray cluster of continuous length. Then there exist a parallel line-cluster  $B \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$ , a measurable set  $B_0 \subseteq \mathbb{R}^{n-1}$ , functions  $a, b : B_0 \rightarrow \mathbb{R} \cup \{\pm\infty\}$  and a locally-Lipschitz, invertible map  $F : B \rightarrow R$  with the following properties:*

(i) *The relation (20) holds true. Denoting  $f(y) = F(y, 0)$  for  $y \in B_0$  we have that the set  $R_0 = f(B_0)$  is a seed of a ray cluster satisfying (19). The inverse map  $f^{-1} : R_0 \rightarrow B_0$  is a continuous map from  $R_0$  to  $B_0$ . Additionally,*

$$(21) \quad a_y = -\alpha_u(f(y)), \quad b_y = \beta_u(f(y)) \quad \text{for all } y \in B_0.$$

(ii) *For any  $y \in B_0$ , the curve*

$$t \mapsto F(y, t) \quad t \in (a_y, b_y)$$

*is a minimizing geodesic, and the set  $\{F(y, t); t \in (a_y, b_y)\}$  is the relative interior of a transport ray associated with  $u$ . Furthermore, there exists  $r_0 \in \mathbb{R}$  such that*

$$(22) \quad u(F(y, t)) = t + r_0 \quad \text{for all } (y, t) \in B.$$

(iii) *For almost any Lebesgue density point  $y_0 \in B_0$  the following hold: The map  $F$  is differentiable at  $(y_0, t)$  for all  $t \in (a_{y_0}, b_{y_0})$ , and there exist Jacobi fields  $J_1(y_0, t), \dots, J_{n-1}(y_0, t)$  along the geodesic  $t \mapsto F(y_0, t)$  in the entire interval  $t \in (a_{y_0}, b_{y_0})$  such that for  $i = 1, \dots, n-1$ ,*

$$(23) \quad J_i(y_0, t) = \frac{\partial F}{\partial y_i}(y_0, t) \quad \text{for all } t \in (a_{y_0}, b_{y_0}).$$

*Denoting  $N(y_0, t) = \frac{\partial F}{\partial t}(y_0, t)$  we have, at the point  $(y_0, 0) \in B$ ,*

$$(24) \quad \langle J_i, N \rangle = \langle J'_i, N \rangle = 0 \quad (i = 1, \dots, n-1),$$

and

$$(25) \quad \langle J'_i, J_k \rangle = \langle J'_k, J_i \rangle \quad (i, k = 1, \dots, n-1).$$

Here,  $J'_i(y_0, 0)$  is the covariant derivative at  $t = 0$  of the Jacobi field  $t \mapsto J_i(y_0, t)$  along the geodesic curve  $t \mapsto F(y_0, t)$ .

(iv) For  $(y, t) \in B$  denote  $T(y, t) = \{\langle J_i(y, t), J_k(y, t) \rangle\}_{i,k=1,\dots,n}$ , where  $J_n := N$ . Then the symmetric matrix  $T(y, t)$  is well-defined and positive semi-definite almost everywhere in  $B$ , and for any Borel set  $A \subseteq R$ ,

$$(26) \quad \lambda_{\mathcal{M}}(A) = \int_{F^{-1}(A)} \sqrt{\det T(y, t)} dy dt,$$

where  $\lambda_{\mathcal{M}}$  is the Riemannian volume measure in  $\mathcal{M}$ .

PROOF. Let  $R_0 \subseteq \mathcal{M}$  be the seed of a ray cluster of continuous length given by Definition 3.1.4. Then  $R_0$  is a Borel set with

$$(27) \quad R = \{x \in \mathcal{M}; \exists \mathcal{I} \in T^\circ[u] \text{ such that } x \in \mathcal{I} \text{ and } \mathcal{I} \cap R_0 \neq \emptyset\}.$$

Since  $R_0$  is a seed of a ray cluster, Definition 3.1.3 provides us with certain numbers  $r_0 \in \mathbb{R}, \varepsilon_0 > 0$ , open sets  $U \subseteq \mathcal{M}, \Omega_0 \subseteq \mathbb{R}^{n-1}$  and  $C^{1,1}$ -functions  $\tilde{u} : U \rightarrow \mathbb{R}, f : \Omega_0 \rightarrow \mathcal{M}$  such that

$$(28) \quad R_0 \subseteq \{x \in U \cap \text{Strain}_{\varepsilon_0}[u]; \tilde{u}(x) = r_0\}.$$

Moreover,  $f$  is a one-to-one map with  $f(\Omega_0) = \{x \in U; \tilde{u}(x) = r_0\}$ , and the inverse function  $f^{-1} : f(\Omega_0) \rightarrow \Omega_0$  is continuous. We see that  $R_0 \subseteq f(\Omega_0)$ . Denote

$$B_0 := f^{-1}(R_0) \subseteq \Omega_0.$$

Since  $R_0 \subseteq f(\Omega_0)$  then

$$(29) \quad f(B_0) = R_0.$$

Since  $B_0$  is the preimage of the Borel set  $R_0$  under the continuous map  $f$ , then  $B_0 \subseteq \mathbb{R}^{n-1}$  is measurable. According to (28) and (29), for each  $y \in B_0$ , the point  $f(y)$  belongs to  $\text{Strain}_{\varepsilon_0}[u] \subseteq \text{Strain}[u]$ . Since  $T^\circ[u]$  is a partition of  $\text{Strain}[u]$ , then for any  $y \in B_0$  there exists a unique  $\mathcal{I} = \mathcal{I}(y) \in T^\circ[u]$  for which  $f(y) \in \mathcal{I}$ . In view of (29), we may rewrite (27) as follows:

$$(30) \quad R = \bigcup_{y \in B_0} \mathcal{I}(y).$$

For any  $y \in B_0$ , the set  $\mathcal{I}(y)$  is the relative interior of a non-degenerate transport ray. According to Proposition 2.1.10 there exists an open set  $(a_y, b_y) \subseteq \mathbb{R}$  containing the origin, with  $a_y = -\alpha_u(f(y)), b_y = \beta_u(f(y))$ , such that

$$(31) \quad \mathcal{I}(y) = \{\text{Exp}_t(\nabla u(f(y))) ; t \in (a_y, b_y)\} \quad \text{for } y \in B_0,$$

and such that  $t \mapsto \text{Exp}_t(\nabla u(f(y)))$  is a minimizing geodesic in  $t \in (a_y, b_y)$  with

$$(32) \quad u(\text{Exp}_t(\nabla u(f(y)))) = u(f(y)) + t \quad \text{for } y \in B_0, t \in (a_y, b_y).$$

The curve  $t \mapsto \text{Exp}_t(\nabla u(f(y)))$  is a geodesic of velocity one, so

$$(33) \quad |\nabla u(f(y))| = 1 \quad \text{for } y \in B_0.$$

Since  $R_0$  is a seed of a ray cluster of continuous length, then the functions  $\alpha_u, \beta_u : R_0 \rightarrow (0, +\infty]$  are continuous. Therefore  $b_y = \beta_u(f(y))$  and  $a_y = -\alpha_u(f(y))$  are

continuous functions of  $y \in B_0$ , thanks to (29) and the continuity of  $f$ . Consequently,

$$(34) \quad B := \{(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}; y \in B_0, a_y < t < b_y\}$$

is a parallel line-cluster. According to (28), (29) and item (i) of Definition 3.1.3,

$$(35) \quad u(f(y)) = \tilde{u}(f(y)) = r_0, \quad \nabla \tilde{u}(f(y)) = \nabla u(f(y)) \quad \text{for } y \in B_0.$$

For  $y \in \Omega_0$  and  $t \in \mathbb{R}$  define

$$(36) \quad \tilde{F}(y, t) = \text{Exp}_t(\nabla \tilde{u}(f(y))), \quad N(y, t) = \frac{\partial \tilde{F}}{\partial t}(y, t).$$

Since  $\mathcal{M}$  is not necessarily complete, then  $(y, t) \mapsto \tilde{F}(y, t)$  and  $(y, t) \mapsto N(y, t)$  are well-defined on a maximal open subset of  $\Omega_0 \times \mathbb{R}$  that contains  $\Omega_0 \times \{0\}$ . The functions  $\tilde{u}$  and  $f$  are  $C^{1,1}$ -maps, and hence

$$\Omega_0 \ni y \mapsto \nabla \tilde{u}(f(y)) \in T\mathcal{M}$$

is locally-Lipschitz. The exponential map is smooth, and from (36) we learn that  $\tilde{F}$  is locally-Lipschitz. According to (31), (34) and (35), the map  $\tilde{F}$  is well-defined on the entire set  $B$ . Set

$$F = \tilde{F}|_B,$$

a well-defined, locally-Lipschitz map. From (34), (35) and (36), for all  $(y, t) \in B$ ,

$$(37) \quad F(y, t) = \tilde{F}(y, t) = \text{Exp}_t(\nabla \tilde{u}(f(y))) = \text{Exp}_t(\nabla u(f(y))).$$

We conclude from (30), (31), (34) and (37) that

$$R = F(B).$$

Thus  $F : B \rightarrow R$  is onto. We argue that for any  $y_1, y_2 \in B_0$ ,

$$(38) \quad y_1 \neq y_2 \quad \implies \quad f(y_1) \notin \mathcal{I}(y_2).$$

Indeed,  $u(f(y_1)) = u(f(y_2)) = r_0$  according to (35). Hence, if  $f(y_1) \in \mathcal{I}(y_2)$  then by (31) and (32) necessarily  $f(y_1) = \text{Exp}_t(\nabla u(f(y_2)))$  for  $t = 0$ . Therefore  $f(y_1) = f(y_2)$  and consequently  $y_1 = y_2$  as the function  $f$  is one-to-one. This establishes (38). Recalling that  $T^\circ[u]$  is a partition, we deduce from (38) that the union in (30) is a disjoint union. Glancing at (31) and (37), we now see that the locally-Lipschitz map  $F : B \rightarrow R$  is one-to-one and hence invertible, as required.

Let us verify conclusion (i) of the proposition: The relation (20) holds true in view of (34). It follows from (37) that  $F(y, 0) = f(y)$  for all  $y \in B_0$ . By (27) and (29), the set  $R_0 = f(B_0)$  is a seed of a ray cluster satisfying (19). The definition of  $a_y$  and  $b_y$  above implies (21). Since  $f^{-1} : f(\Omega_0) \rightarrow \Omega_0$  is continuous and  $R_0 \subseteq f(\Omega_0)$ , the verification of conclusion (i) is complete. We move on to the proof of conclusion (ii) of the proposition: The fact that  $t \mapsto F(y, t)$  is a minimizing geodesic whose image is the relative interior of a transport ray follows from (31) and (37). The relation (22) follows from (32), (35) and (37). Thus conclusion (ii) is proven as well.

In order to obtain conclusion (iii) we would like to apply Lemma 3.1.2. To this end, observe that our definition (36) of  $\tilde{F}(y, t)$  and  $N(y, t)$  coincides with that of Lemma 3.1.2. According to Definition 3.1.3(iii), for almost any  $y_0 \in B_0 \subseteq \Omega_0$ , the function  $\tilde{u}$  is twice differentiable with a symmetric Hessian at  $f(y_0)$ . Note that the



requirement (8) of Lemma 3.1.2 is satisfied in view of (33) and (35). Thus, from conclusion (ii) of Lemma 3.1.2, for almost any Lebesgue density point  $y_0 \in B_0$ ,

$$(39) \quad J_1(y_0, t) = \frac{\partial \tilde{F}}{\partial y_1}(y_0, t), \dots, J_{n-1}(y_0, t) = \frac{\partial \tilde{F}}{\partial y_{n-1}}(y_0, t),$$

are well-defined Jacobi fields along the entire geodesic  $t \mapsto \tilde{F}(y_0, t)$  for  $t \in (a_{y_0}, b_{y_0})$ . In fact,  $(y_0, t)$  is a Lebesgue density point of  $B$  for any  $t \in (a_{y_0}, b_{y_0})$ . Recalling that  $F = \tilde{F}|_B$  we conclude from Lemma 3.1.2(i) that the map  $F : B \rightarrow R$  is differentiable at  $(y_0, t)$  whenever  $t \in (a_{y_0}, b_{y_0})$ . The relation (23) thus follows from the validity of (39) for all  $t \in (a_{y_0}, b_{y_0})$ . The Jacobi fields  $t \mapsto J_1(y_0, t), \dots, t \mapsto J_{n-1}(y_0, t)$  also satisfy (24) and (25), thanks to Lemma 3.1.2(iii), and the proof of (iii) is complete.

We continue with the proof of (iv). First of all, the function  $F$  is locally-Lipschitz and hence differentiable almost everywhere in  $B$ . According to conclusion (iii) which was proven above, for almost any  $(y, t) \in B$ ,

$$(40) \quad T(y, t) = \{ \langle J_i(y, t), J_k(y, t) \rangle \}_{i,k=1,\dots,n} = \left\{ \left\langle \frac{\partial F}{\partial y_i}(y, t), \frac{\partial F}{\partial y_k}(y, t) \right\rangle \right\}_{i,k=1,\dots,n}$$

where  $\partial F / \partial y_n := \partial F / \partial t$ . We will use the area formula for Lipschitz maps from Evans and Gariepy [21]. Let us recall the relevant theory. Let  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a Lipschitz function. The Jacobian of  $H$ , denoted by  $J_H$ , is well-defined almost everywhere. According to [21, Section 3.3.3], for any measurable function  $g : \mathbb{R}^n \rightarrow [0, \infty)$  and a measurable set  $D \subseteq \mathbb{R}^n$ ,

$$(41) \quad \int_D g(x) J_H(x) dx = \int_{\mathbb{R}^n} \left[ \sum_{x \in D \cap H^{-1}(y)} g(x) \right] dy,$$

where an empty sum is defined to be zero. We claim that in order to define the left-hand side and the right-hand side of (41), it suffices to know the values of  $H$  in the set  $D$  alone. Indeed, the Jacobian  $J_H(x)$  is determined by  $H|_D$  at any Lebesgue density point  $x \in D$  in which  $H$  is differentiable. The Kirszbraun theorem [21, Section 3.3.1] states that any Lipschitz map from  $D$  to  $\mathbb{R}^n$  may be extended to a Lipschitz map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . It therefore suffices to assume that  $H : D \rightarrow \mathbb{R}^n$  is a Lipschitz function in order for (41) to hold true. In fact, it is enough to assume that  $H : D \rightarrow \mathbb{R}^n$  is only locally-Lipschitz. Indeed, there exist compacts  $K_1 \subseteq K_2 \subseteq \dots$  that are contained in  $D$  with

$$\lambda_n \left( D \setminus \bigcup_{i=1}^{\infty} K_i \right) = 0,$$

where  $\lambda_n$  is the Lebesgue measure on  $\mathbb{R}^n$ . The function  $H$  is Lipschitz in the compact  $K_i$  for any  $i$ . We now apply (41) with the compact set  $K_i$  playing the role of  $D$  and use the monotone convergence theorem. This yields (41) for the original set  $D$ , even though  $H$  is only locally-Lipschitz. To summarize, when  $D \subseteq \mathbb{R}^n$  is a measurable set and  $H : D \rightarrow \mathbb{R}^n$  is a locally-Lipschitz, one-to-one map, then for any measurable function  $g : \mathbb{R}^n \rightarrow [0, \infty)$ ,

$$(42) \quad \int_D g(x) J_H(x) dx = \int_{H(D)} g(H^{-1}(y)) dy.$$

Next, what happens if the range of  $H$  is not a Euclidean space, but a Riemannian manifold  $\mathcal{M}$ ? In this case, we claim that for any measurable set  $D \subseteq \mathbb{R}^n$  and a locally-Lipschitz map  $H : D \rightarrow \mathcal{M}$  which is one-to-one,

$$(43) \quad \int_D \varphi(x) \sqrt{\det T(x)} dx = \int_{H(D)} \varphi(H^{-1}(y)) d\lambda_{\mathcal{M}}(y),$$

for any measurable  $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ . Here,  $T(x) = (\langle \partial H / \partial x_i, \partial H / \partial x_j \rangle)_{i,j=1,\dots,n}$ . Note that (iv) follows from (40) and (43), with  $D = B$ ,  $H = F$  and  $\varphi = 1_{H^{-1}(A)}$ . In order to deduce (43) from (42) we need to work in a local chart, and observe that  $\sqrt{\det T(x)}$  is the Riemannian volume of the parallelepiped spanned by the tangent vectors

$$\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n}.$$

The usual Jacobian  $J_H(x)$  is the Euclidean volume of this parallelepiped in our local chart. We conclude that  $\sqrt{\det T(x)} / J_H(x)$  is precisely the density of the Riemannian volume measure  $\lambda_{\mathcal{M}}$  at the point  $H(x)$  in our local chart. By setting

$$g(x) = \varphi(x) \sqrt{\det T(x)} / J_H(x),$$

we deduce (43) from (42).  $\square$

REMARK 3.1.6. It suffices to assume that  $A \subseteq R$  is a measurable set in order for (26) to hold true. In fact, denote by  $\theta$  the complete measure on the set  $B$  whose density is  $(y, t) \mapsto \sqrt{\det T(y, t)}$ . Note also that the restriction of  $\lambda_{\mathcal{M}}$  to  $R$  is a complete measure on  $R$ . The validity of (26) for all Borel subsets of  $R$  and a standard measure-theoretic argument show that a subset  $A \subseteq R$  is  $\lambda_{\mathcal{M}}$ -measurable if and only if  $F^{-1}(A)$  is  $\theta$ -measurable. Therefore,  $F$  pushes forward the measure  $\theta$  to the restriction of  $\lambda_{\mathcal{M}}$  to the ray cluster  $R$ .

REMARK 3.1.7. What happens if the ray cluster  $R$  from Proposition 3.1.5 is not assumed to be of *continuous length*? The assumption that the ray cluster  $R$  is of continuous length was mainly used to prove that the set  $B$  defined in (34) is a parallel line-cluster. Without the assumption that  $R$  is of continuous length, the functions

$$b_y = \beta_u(f(y)), \quad a_y = -\alpha_u(f(y))$$

are still measurable functions of  $y \in B_0$ , thanks to Lemma 2.1.12 and the continuity of  $f$ . Therefore  $B$  is an *almost line-cluster*. We thus see that only minor changes will occur in the conclusion of the proposition, if the ray cluster  $R$  is not assumed to be of continuous length. One obvious change would be that  $B$  becomes an almost line-cluster, and not a parallel line-cluster. The only additional change is that

$$\text{“for all } t \in (a_{y_0}, b_{y_0})\text{”}$$

in the second line of (iii) and also in (23) will be replaced by

$$\text{“for almost all } t \in (a_{y_0}, b_{y_0})\text{”}.$$

Indeed, the function  $F = \tilde{F}|_B$  is differentiable at  $(y_0, t)$  and it satisfies the equality in (23) at any point  $(y_0, t) \in B$  which is a Lebesgue density point of  $B$ . By the Lebesgue density theorem, for almost any  $y_0 \in B$  and for almost any  $t \in (a_{y_0}, b_{y_0})$ , the point  $(y_0, t) \in B$  is a Lebesgue density point of  $B$ . To conclude, we are allowed to apply Proposition 3.1.5, with the aforementioned tiny changes, even if the ray cluster  $R$  is not assumed to be of continuous length.

For a subset  $A \subseteq \mathcal{M}$  define  $Ends(A) \subseteq \mathcal{M}$  to be the union of all relative boundaries of transport rays intersecting  $A$ . In other words, a point  $x \in \mathcal{M}$  belongs to  $Ends(A)$  if and only if there exists a transport ray  $\mathcal{I} \in T[u]$ , whose relative boundary contains  $x$ , such that  $A \cap \mathcal{I} \neq \emptyset$ .

LEMMA 3.1.8. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$  and let  $R \subseteq Strain[u]$  be a ray cluster. Then,*

$$\lambda_{\mathcal{M}}(Ends(R)) = 0.$$

PROOF. We can assume that  $R \neq \emptyset$ . We may apply Proposition 3.1.5(ii) thanks to Remark 3.1.7. Whence,

$$R = \{F(y, t); y \in B_0, a_y < t < b_y\}.$$

Furthermore,  $F = \tilde{F}|_B$  where  $\tilde{F}$  as defined in (36) is a locally-Lipschitz map which is well-defined in a maximal open subset of  $\Omega_0 \times \mathbb{R}$  containing  $\Omega_0 \times \{0\}$ . We claim that

$$(44) \quad Ends(R) = \left\{ \tilde{F}(y, t); y \in B_0, t \in \mathbb{R} \cap \{a_y, b_y\}, \tilde{F}(y, t) \text{ is well-defined} \right\}.$$

Indeed, fix an arbitrary point  $x \in R$ . Since  $R \subseteq Strain[u]$ , then according to Lemma 2.1.7, there is a unique transport ray  $\mathcal{I} \in T[u]$  containing  $x$ . The transport ray  $\mathcal{I} \subseteq \mathcal{M}$  is a closed set, and the relative interior of  $\mathcal{I}$  contains the point  $x$ . By Proposition 3.1.5(ii), the relative interior of  $\mathcal{I}$  must take the form

$$\{F(y, t); t \in (a_y, b_y)\}$$

for a certain  $y \in B_0$ . Recall that  $F = \tilde{F}|_B$ , and that the curve  $t \mapsto F(y, t)$  is a minimizing geodesic in  $t \in (a_y, b_y)$ . In the case where this minimizing geodesic may be extended to a geodesic defined on a larger interval  $A \supset (a_y, b_y)$ , this extended geodesic is given by  $t \mapsto \tilde{F}(y, t)$  for  $t \in A$ . We thus deduce from Lemma 2.1.3 that

$$(45) \quad \mathcal{I} = \left\{ \tilde{F}(y, t); t \in \mathbb{R} \cap [a_y, b_y], \tilde{F}(y, t) \text{ is well-defined} \right\}.$$

Since  $x \in R$  was an arbitrary point, the relation (44) follows from the representation (45) of the unique transport ray  $\mathcal{I}$  containing  $x$ . Consider the set

$$(46) \quad \left\{ (y, t) \in B_0 \times \mathbb{R}; t \in \mathbb{R} \cap \{a_y, b_y\}, \tilde{F}(y, t) \text{ is well-defined} \right\} \subseteq \mathbb{R}^{n-1} \times \mathbb{R}.$$

This set is contained in the union of two graphs of measurable functions, and hence it is a set of measure zero in  $\mathbb{R}^{n-1} \times \mathbb{R}$ . Since  $Ends(R)$  is the image of the set in (46) under the locally-Lipschitz map  $\tilde{F}$ , then  $Ends(R)$  is a null-set in the  $n$ -dimensional manifold  $\mathcal{M}$ .  $\square$

### 3.2. Decomposition into ray clusters

As before, we write  $\lambda_{\mathcal{M}}$  for the Riemannian volume measure on the geodesically-convex, Riemannian manifold  $\mathcal{M}$ . Our main result in this section is the following:

PROPOSITION 3.2.1. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Then there exists a countable family  $\{R_i\}_{i=1, \dots, \infty}$  of disjoint ray clusters of continuous length such that*

$$\lambda_{\mathcal{M}} \left( Strain[u] \setminus \left( \bigcup_{i=1}^{\infty} R_i \right) \right) = 0.$$

We begin the proof of Proposition 3.2.1 with the following lemma.

LEMMA 3.2.2. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Let  $R \subseteq \text{Strain}[u]$  be any ray cluster associated with  $u$ . Then  $R$  is a Borel subset of  $\mathcal{M}$ .*

PROOF. We may assume that  $R \neq \emptyset$ . According to Proposition 3.1.5 and Remark 3.1.7 we know that  $R = F(B)$  where  $B$  is an almost-line cluster. Let  $R_0 \subseteq \mathcal{M}$  and  $r_0 \in \mathbb{R}$  be as in Proposition 3.1.5. We claim that a given point  $x \in \text{Strain}[u]$  belongs to  $R$  if and only if the following two conditions are met:

$$(A) \quad r_0 - u(x) \in (-\alpha_u(x), \beta_u(x)).$$

$$(B) \quad \text{Exp}_{r_0 - u(x)}(\nabla u(x)) \in R_0.$$

In order to prove this claim, assume that  $x \in \text{Strain}[u]$  satisfies conditions (A) and (B). Since  $T^\circ[u]$  is a partition of  $\text{Strain}[u]$ , there exists  $\mathcal{I} \in T^\circ[u]$  such that  $x \in \mathcal{I}$ . From (A) and Proposition 2.1.10 the point  $\text{Exp}_{r_0 - u(x)}(\nabla u(x))$  belongs to  $\mathcal{I}$ , while condition (B) shows that this point belongs to  $R_0$ . Hence  $\mathcal{I} \cap R_0 \neq \emptyset$ . From Definition 3.1.4 we obtain that  $\mathcal{I} \subseteq R$  and consequently  $x \in R$ . Conversely, assume that  $x \in R$ . According to Proposition 3.1.5 there exists  $(y, t) \in B$  for which  $F(y, t) = x$  and  $u(x) = t + r_0$ . Additionally,

$$\alpha_u(x) = t - a_y, \quad \beta_u(x) = b_y - t,$$

in the notation of Proposition 3.1.5. Since  $B$  is an almost-line cluster, then  $0 \in (a_y, b_y)$  and consequently  $r_0 - u(x) = -t \in (a_y - t, b_y - t) = (-\alpha_u(x), \beta_u(x))$ . We have thus verified condition (A). By Proposition 3.1.5 and Proposition 2.1.10, we have

$$R_0 \ni F(y, 0) = \text{Exp}_{-t}(\nabla u(x)) = \text{Exp}_{r_0 - u(x)}(\nabla u(x)),$$

and (B) follows as well.

Recall that the set  $\text{Strain}[u]$  is Borel according to Lemma 2.1.12, as well as the functions  $\alpha_u, \beta_u : \mathcal{M} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ . Since  $u$  is continuous, then the collection of all  $x \in \text{Strain}[u]$  satisfying condition (A) is a Borel set. As for condition (B), the set  $R_0$  is a seed of a ray cluster and by definition it is a Borel set. Consider the partially-defined function

$$(1) \quad \text{Strain}[u] \ni x \mapsto \text{Exp}_{r_0 - u(x)}(\nabla u(x)) \in \mathcal{M}.$$

We claim that this function is well-defined on a Borel subset of  $\text{Strain}[u]$ , and that it is a Borel map. Indeed, Lemma 2.1.6 shows that the Lipschitz function  $u$  is differentiable in the Borel set  $\text{Strain}[u]$ . Consequently  $\nabla u : \text{Strain}[u] \rightarrow T\mathcal{M}$  is a well-defined Borel map, as it may be represented as a pointwise limit of Borel maps. The exponential map is continuous and the domain of definition of the partially-defined map

$$T\mathcal{M} \times \mathbb{R} \ni (v, t) \mapsto \text{Exp}_t(v) \in \mathcal{M}$$

is an open set. Hence the map in (1) is a Borel map which is defined on a Borel subset of  $\text{Strain}[u]$ . We conclude that the collection of all  $x \in \text{Strain}[u]$  satisfying condition (B) is Borel, being the preimage of the Borel set  $R_0$  under the Borel map (1). Therefore the set  $R \subseteq \text{Strain}[u]$ , which is defined by conditions (A) and (B), is a Borel set.  $\square$

LEMMA 3.2.3. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Let  $R, R_1, R_2, \dots, R_L \subseteq \text{Strain}[u]$  be ray clusters. Then also  $R \setminus (\bigcup_{i=1}^L R_i)$  is a ray cluster.*

PROOF. Denote by  $R_0$  the seed of the ray cluster  $R$  provided by Definition 3.1.4. Then  $R_0$  is a Borel set. Lemma 3.2.2 implies that  $\tilde{R}_0 = R_0 \setminus (\cup_{i=1}^L R_i)$  is a Borel set as well. By the remark following Definition 3.1.3, the set  $\tilde{R}_0$  is a seed of a ray cluster associated with  $u$ . In fact, the set  $\tilde{R}_0$  is the seed of the ray cluster  $R \setminus (\cup_{i=1}^L R_i)$ , as follows from Definition 3.1.4 and the fact that  $T^\circ[u]$  is a partition of  $\text{Strain}[u]$ .  $\square$

The equality of the mixed second derivatives of  $C^{1,1}$ -functions, stated in the following lemma, is of great importance to us.

LEMMA 3.2.4. *Let  $U \subseteq \mathbb{R}^n$  be an open set and let  $f : U \rightarrow \mathbb{R}$  be a  $C^{1,1}$ -function. Then for  $i, j = 1, \dots, n$ , the functions  $\partial_i f$  and  $\partial_j f$  are differentiable almost everywhere in  $U$ , with*

$$(2) \quad \partial_i (\partial_j f) = \partial_j (\partial_i f) \quad \text{almost everywhere in } U.$$

PROOF. Let  $x_0 \in U$ . It suffices to prove the lemma in an open neighborhood of  $x_0$ , in which  $f$  and  $\partial_1 f, \dots, \partial_n f$  are Lipschitz functions. By the Rademacher theorem, the functions  $\partial_1 f, \dots, \partial_n f$  are differentiable almost everywhere in  $U$ . By considering slices of  $U$ , we see that it suffices to prove (2) assuming that  $n = 2$  and that  $U$  is a rectangle parallel to the axes, of the form

$$U = \{(x, y) \in \mathbb{R}^2; a < x < b, c < y < d\}.$$

We may also assume that  $\partial f / \partial x$  and  $\partial f / \partial y$  are Lipschitz functions in  $U$ . Denote

$$h = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right).$$

Since  $\partial f / \partial y$  is Lipschitz, then  $h$  is an  $L^\infty$ -function. Furthermore, for any  $(x, y) \in U$ ,

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial y}(a, y) + \int_a^x h(t, y) dt.$$

Integrating with respect to the  $y$ -variable we see that for any  $(x, y) \in U$ ,

$$(3) \quad f(x, y) = f(x, c) + \int_c^y \frac{\partial f}{\partial y}(x, s) ds = f(x, c) + \int_c^y \frac{\partial f}{\partial y}(a, s) ds + \int_{[a, x] \times [c, y]} h,$$

where the use of Fubini's theorem is legitimate as  $h$  is an  $L^\infty$ -function on  $U$ . Differentiating (3) with respect to  $x$ , we deduce that the Lipschitz function  $\partial f / \partial x$  satisfies

$$(4) \quad \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial x}(x, c) + \int_c^y h(x, s) ds$$

almost everywhere in  $U$ . Both the left-hand side and the right-hand side of (4) are differentiable with respect to  $y$  almost everywhere in  $U$ . Therefore, by differentiating (4) with respect to  $y$  we obtain

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = h$$

almost everywhere in  $U$ . Thus (2) is proven.  $\square$

COROLLARY 3.2.5. *Let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a  $C^{1,1}$ -function. Then the vector field  $\nabla f$  is differentiable almost-everywhere in  $\mathcal{M}$ , and for almost any  $p \in \mathcal{M}$ ,*

$$(5) \quad \langle \nabla_v(\nabla f), w \rangle = \langle \nabla_w(\nabla f), v \rangle \quad \text{for } v, w \in T_p\mathcal{M}.$$

Here, by “almost-everywhere” we refer to the Riemannian volume measure  $\lambda_{\mathcal{M}}$ .

PROOF. Working in a local chart, we may replace  $\mathcal{M}$  by an open set  $U \subseteq \mathbb{R}^n$  equipped with a Riemannian metric tensor. Since  $f : U \rightarrow \mathbb{R}$  is a  $C^{1,1}$ -function, Lemma 3.2.4 implies that the functions  $\partial_1 f, \dots, \partial_n f$  are differentiable almost everywhere, and

$$(6) \quad \partial_i(\partial_j f) = \partial_j(\partial_i f)$$

almost everywhere in  $U$ . The Leibnitz rule applies at any point where the involved functions are differentiable and hence,

$$\begin{aligned} & \langle \nabla_{\partial_i}(\nabla f), \partial_j \rangle - \langle \nabla_{\partial_j}(\nabla f), \partial_i \rangle \\ &= \partial_i \langle \nabla f, \partial_j \rangle - \partial_j \langle \nabla f, \partial_i \rangle - \langle \nabla f, \nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i \rangle = \partial_i(\partial_j f) - \partial_j(\partial_i f) \end{aligned}$$

at any point in which  $\partial_1 f, \dots, \partial_n f$  are differentiable. Now (5) follows from the validity of (6) almost everywhere in  $U$ .  $\square$

LEMMA 3.2.6. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$  and let  $\varepsilon > 0$  and  $p \in \text{Strain}_{\varepsilon}[u]$ . Then there exist an open set  $V \subseteq \mathcal{M}$  containing  $p$  and a ray cluster  $R \subseteq \mathcal{M}$  such that*

$$(7) \quad \text{Strain}_{\varepsilon}[u] \cap V \subseteq R.$$

PROOF. Set  $\varepsilon_0 = \varepsilon/2$ . Applying Theorem 2.1.13, we find  $\delta > 0$  and a  $C^{1,1}$ -function  $\tilde{u} : B_{\mathcal{M}}(p, \delta) \rightarrow \mathbb{R}$  such that

$$(8) \quad x \in B_{\mathcal{M}}(p, \delta) \cap \text{Strain}_{\varepsilon_0}[u] \implies \tilde{u}(x) = u(x), \quad \nabla \tilde{u}(x) = \nabla u(x).$$

We would like to apply the implicit function theorem, in the form of Lemma 2.2.1(iii). Decreasing  $\delta$  if necessary, we may assume that  $B_{\mathcal{M}}(p, \delta)$  is contained in a single chart of the differentiable manifold  $\mathcal{M}$ . Since  $p \in \text{Strain}_{\varepsilon}[u] \subseteq \text{Strain}_{\varepsilon_0}[u]$  then  $p$  belongs to the relative interior of some transport ray. From (8) and Lemma 2.1.6,

$$(9) \quad \nabla \tilde{u}(p) = \nabla u(p) \neq 0 \quad \text{and} \quad \tilde{u}(p) = u(p).$$

We may apply Lemma 2.2.1(iii) in the local chart, thanks to (9). We conclude from Lemma 2.2.1(iii) that there exist an open set

$$(10) \quad U \subseteq B_{\mathcal{M}}(p, \delta)$$

containing  $p$ , an open set  $\Omega = \Omega_0 \times (a, b) \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$  and a  $C^{1,1}$ -diffeomorphism  $G : \Omega \rightarrow U$  with

$$(11) \quad \tilde{u}(G(y, t)) = t \quad \text{for } (y, t) \in \Omega_0 \times (a, b).$$

Since  $p \in U$  and  $G : \Omega \rightarrow U$  is onto, then (9) and (11) imply that

$$(12) \quad u(p) = \tilde{u}(p) \in (a, b).$$

The set  $U$  is an open neighborhood of  $p$ , hence there exists  $0 < \eta < \varepsilon$  with

$$(13) \quad B_{\mathcal{M}}(p, \eta) \subseteq U.$$

According to Corollary 3.2.5, for almost any  $x \in U$ , the  $C^{1,1}$ -function  $\tilde{u}$  is twice differentiable with a symmetric Hessian at  $x$ . Since  $G$  is a  $C^1$ -diffeomorphism, then for almost any  $(y, t) \in \Omega_0 \times (a, b)$ , the function  $\tilde{u}$  is twice differentiable with a symmetric Hessian at the point  $G(y, t)$ . From the latter fact and from (12) we conclude that there exists

$$(14) \quad t_0 \in (a, b) \cap \left( u(p) - \frac{\eta}{2}, u(p) + \frac{\eta}{2} \right)$$

with the following property: For almost any  $y \in \Omega_0 \subseteq \mathbb{R}^{n-1}$ , the function  $\tilde{u}$  is twice differentiable with a symmetric Hessian at the point  $G(y, t_0)$ . Denote

$$(15) \quad R_0 = \{x \in U \cap \text{Strain}_{\varepsilon_0}[u]; \tilde{u}(x) = t_0\}.$$

Lemma 2.1.12 implies that  $\text{Strain}_{\varepsilon_0}[u] = \{x \in \mathcal{M}; \ell_u(x) > \varepsilon_0\}$  is a Borel set. From (15), the set  $R_0 \subseteq \mathcal{M}$  is also Borel. We claim that  $R_0$  is a seed of a ray cluster in the sense of Definition 3.1.3. In order to prove our claim we define  $r_0 := t_0$  and set

$$f(y) := G(y, t_0) \quad (y \in \Omega_0).$$

Since  $G$  is a  $C^{1,1}$ -diffeomorphism onto  $U$ , then the  $C^{1,1}$ -function  $f$  is one-to-one with a continuous inverse. The relation (11) implies that

$$(16) \quad f(\Omega_0) = \{x \in U; \tilde{u}(x) = t_0\} = \{x \in U; \tilde{u}(x) = r_0\}.$$

Let us verify that the numbers  $r_0 \in \mathbb{R}$ ,  $\varepsilon_0 > 0$ , the open sets  $U \subseteq \mathcal{M}$ ,  $\Omega_0 \subseteq \mathbb{R}^{n-1}$  and the  $C^{1,1}$ -functions  $\tilde{u} : U \rightarrow \mathbb{R}$ ,  $f : \Omega_0 \rightarrow \mathcal{M}$  satisfy the requirements of Definition 3.1.3. Indeed, by the choice of  $t_0$  we verify requirement (iii) of Definition 3.1.3. By using (16) and the preceding sentence we obtain Definition 3.1.3(ii). The relation (15) and the fact that  $r_0 = t_0$  show that Definition 3.1.3(iv) holds as well. From (8) and (10) we deduce Definition 3.1.3(i). Thus  $R_0$  is a seed of a ray cluster associated with  $u$ . Set

$$(17) \quad R = \{x \in \mathcal{M}; \exists \mathcal{I} \in T^\circ[u] \text{ such that } x \in \mathcal{I} \text{ and } \mathcal{I} \cap R_0 \neq \emptyset\}.$$

Then  $R \subseteq \text{Strain}[u]$  is a ray cluster, according to Definition 3.1.4. We still need to find an open set  $V \subseteq \mathcal{M}$  containing  $p$  for which (7) holds true. Let us define

$$(18) \quad V = \left\{ x \in B_{\mathcal{M}} \left( p, \frac{\eta}{2} \right); |u(x) - t_0| < \eta/2 \right\},$$

which is an open set containing  $p$  in view of (14). In order to prove (7), we recall that  $\varepsilon = 2\varepsilon_0$  and let  $x \in \text{Strain}_\varepsilon[u] \cap V$  be an arbitrary point. Since  $\ell_u(x) > \varepsilon$ , then Proposition 2.1.10 implies that there exist  $\mathcal{I} \in T^\circ[u]$  and a minimizing geodesic  $\gamma : [-\varepsilon, \varepsilon] \rightarrow \mathcal{M}$  with

$$(19) \quad \gamma(0) = x$$

such that

$$(20) \quad \gamma([-\varepsilon, \varepsilon]) \subseteq \mathcal{I},$$

and such that

$$(21) \quad u(\gamma(t)) = u(x) + t \quad \text{for } t \in [-\varepsilon, \varepsilon].$$

It follows from (21) and from Lemma 2.1.9 that

$$(22) \quad \ell_u(\gamma(t)) \geq \varepsilon - |t| \quad \text{for } t \in (-\varepsilon, \varepsilon).$$

Since  $x \in V$ , then  $|u(x) - t_0| < \eta/2$  according to (18). Denoting  $t_1 = t_0 - u(x)$ , we have

$$(23) \quad |t_1| = |u(x) - t_0| < \eta/2 < \varepsilon/2,$$

where  $\eta < \varepsilon$  according to the line before (13). From (21) and (23) we see that  $u(\gamma(t_1)) = u(x) + t_1 = t_0$ . From (22) and (23) it follows that  $\ell_u(\gamma(t_1)) > \varepsilon/2 = \varepsilon_0$ . Therefore,

$$(24) \quad \gamma(t_1) \in \text{Strain}_{\varepsilon_0}[u] \cap \{x \in \mathcal{M}; u(x) = t_0\}.$$

Furthermore,  $x \in V$  and hence  $d(x, p) < \eta/2$  by (18). Since  $\gamma$  is a unit speed geodesic, then from (19) and (23),

$$(25) \quad d(\gamma(t_1), p) \leq d(\gamma(0), p) + |t_1| = d(x, p) + |t_1| < \eta/2 + \eta/2 = \eta.$$

We learn from (13) and (25) that  $\gamma(t_1) \in U$ . From (8), (10) and (24), we thus obtain that  $\tilde{u}(\gamma(t_1)) = u(\gamma(t_1)) = t_0$ . By using (15) and (24), we finally obtain that

$$\gamma(t_1) \in R_0.$$

Note also that  $\gamma(t_1) \in \mathcal{I}$ , thanks to (20) and (23). We have thus found a point  $\gamma(t_1) \in \mathcal{I} \cap R_0$ , and hence  $\mathcal{I} \cap R_0 \neq \emptyset$ . Recalling that  $\mathcal{I} \in T^\circ[u]$  we learn from (17) that  $\mathcal{I} \subseteq R$ . Since  $x = \gamma(0) \in \mathcal{I}$  by (19) and (20), then  $x \in R$ . However,  $x$  was an arbitrary point in  $\text{Strain}_\varepsilon[u] \cap V$ , and hence the proof of (7) is complete.  $\square$

LEMMA 3.2.7. *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Then there exists a countable family  $\{R_i\}_{i=1, \dots, \infty}$  of disjoint ray clusters associated with  $u$  such that*

$$(26) \quad \text{Strain}[u] = \bigcup_{i=1}^{\infty} R_i.$$

PROOF. In order to prove the lemma, it suffices to find ray clusters  $\tilde{R}_i \subseteq \mathcal{M}$  for  $i = 1, 2, \dots$  which are not necessarily disjoint, such that

$$(27) \quad \text{Strain}[u] \subseteq \bigcup_{i=1}^{\infty} \tilde{R}_i.$$

Indeed, any ray cluster  $R$  is automatically contained in  $\text{Strain}[u]$ . By setting  $R_i = \tilde{R}_i \setminus \bigcup_{j < i} \tilde{R}_j$  and using Lemma 3.2.3, we deduce (26) from (27). We thus focus on the proof of (27). Recall from Section 2.1 that  $\text{Strain}[u] = \bigcup_{k=1}^{\infty} \text{Strain}_{1/k}[u]$ . Hence, in order to prove (27), it suffices to fix  $\varepsilon > 0$  and to find ray clusters  $R_1, R_2, \dots$  with

$$(28) \quad \text{Strain}_\varepsilon[u] \subseteq \bigcup_{i=1}^{\infty} R_i.$$

Let us fix  $\varepsilon > 0$ . We need to find ray clusters  $R_1, R_2, \dots$  satisfying (28). For  $p \in \text{Strain}_\varepsilon[u]$  let us write  $V_{p, \varepsilon} = V \subseteq \mathcal{M}$  for the open set containing  $p$  that is provided by Lemma 3.2.6. Then for any  $p \in \text{Strain}_\varepsilon[u]$  there is a ray cluster  $R = R_{p, \varepsilon} \subseteq \mathcal{M}$  such that

$$(29) \quad \text{Strain}_\varepsilon[u] \cap V_{p, \varepsilon} \subseteq R_{p, \varepsilon}.$$



Consider all open sets of the form  $V_{p,\varepsilon}$  where  $p \in \text{Strain}_\varepsilon[u]$ . This collection is an open cover of  $\text{Strain}_\varepsilon[u]$ . Recall that  $\mathcal{M}$  is second-countable, being a geodesically-convex Riemannian manifold. Hence we may find an open sub-cover of  $\text{Strain}_\varepsilon[u]$  which is countable. That is, there exist points  $p_1, p_2, \dots \in \text{Strain}_\varepsilon[u]$  such that

$$(30) \quad \text{Strain}_\varepsilon[u] \subseteq \bigcup_{i=1}^{\infty} V_{p_i, \varepsilon}.$$

From (29) and (30) we conclude that the ray clusters  $R_i = R_{p_i, \varepsilon}$  satisfy (28), and the lemma is proven.  $\square$

**PROOF OF PROPOSITION 3.2.1.** In view of Lemma 3.2.2 and Lemma 3.2.7, all that remains is to prove the following: For any ray cluster  $R \subseteq \mathcal{M}$  with  $\lambda_{\mathcal{M}}(R) > 0$ , there exist disjoint ray clusters of *continuous length*  $\{R_i\}_{i=1, \dots, \infty}$ , all contained in  $R$ , such that

$$(31) \quad \lambda_{\mathcal{M}} \left( R \setminus \left( \bigcup_{i=1}^{\infty} R_i \right) \right) = 0.$$

According to Remark 3.1.7, we may apply Proposition 3.1.5 for the ray cluster  $R$ . Let  $B$  be the almost-line cluster that is provided by Remark 3.1.7 and Proposition 3.1.5, and let  $F, B_0, a, b$  be as in Proposition 3.1.5. The set  $B_0 \subseteq \mathbb{R}^{n-1}$  is a measurable set, and  $a : B_0 \rightarrow [-\infty, 0)$  and  $b : B_0 \rightarrow (0, +\infty]$  are measurable functions. By Luzin's theorem from real analysis (e.g., [43, Section 1.4.3]), there exist disjoint  $\sigma$ -compact subsets  $\tilde{B}_0^{(k)} \subseteq B_0$  for  $k = 1, 2, \dots$  such that

$$(32) \quad \lambda_{n-1} \left( B_0 \setminus \left( \bigcup_{k=1}^{\infty} \tilde{B}_0^{(k)} \right) \right) = 0,$$

while for any  $k \geq 1$ , the functions  $a|_{\tilde{B}_0^{(k)}}$  and  $b|_{\tilde{B}_0^{(k)}}$  are continuous. Here,  $\lambda_{n-1}$  is the Lebesgue measure on  $\mathbb{R}^{n-1}$ . From Proposition 3.1.5(i), the set  $R_0 = f(B_0)$  is a seed of a ray cluster, where  $f(y) = F(y, 0)$  for  $y \in B_0$ . Note that  $\tilde{R}^{(k)} := f(\tilde{B}_0^{(k)})$  is a  $\sigma$ -compact set for any  $k \geq 1$ , being the image of a  $\sigma$ -compact set under a continuous map. By the remark following Definition 3.1.3, the set  $\tilde{R}^{(k)} \subseteq R_0$  is a seed of a ray cluster.

From our construction the functions  $a_y = -\alpha_u(f(y))$  and  $b_y = \beta_u(f(y))$  are continuous functions of  $y \in \tilde{B}_0^{(k)}$ , for any  $k \geq 1$ . From Proposition 3.1.5(i), the function  $f^{-1}$  is continuous on  $R_0$ , and therefore the functions  $\alpha_u, \beta_u$  are continuous on  $\tilde{R}^{(k)} = f(\tilde{B}_0^{(k)})$  for any  $k \geq 1$ . This shows that  $\tilde{R}^{(k)}$  is actually a seed of a ray cluster of continuous length. The function  $f$  is one-to-one, and therefore  $\tilde{R}^{(1)}, \tilde{R}^{(2)}, \dots$  are pairwise-disjoint.

For  $k \geq 1$ , define  $R_k$  to be the union of all relative interiors of transport rays intersecting  $\tilde{R}^{(k)}$ . The sets  $R_1, R_2, \dots$  are pairwise-disjoint and are contained in  $R$ , according to Proposition 3.1.5(ii). From Definition 3.1.4, the sets  $R_1, R_2, \dots$  are ray clusters of continuous length, while Lemma 3.2.2 implies the measurability of these sets. The desired relation (31) holds true in view of (32) and Proposition 3.1.5(iv). This completes the proof.  $\square$

### 3.3. Needles and Ricci curvature

We begin this section with an addendum to Proposition 3.1.5.

LEMMA 3.3.1. *We work under the notation and assumptions of Proposition 3.1.5. Let  $y = y_0 \in B_0$  be a Lebesgue density point of  $B_0$  for which the conclusions of Proposition 3.1.5(iii) hold true. Then either for all  $t \in (a_y, b_y)$  the vectors*

$$J_1(y, t), \dots, J_{n-1}(y, t) \in T_{F(y,t)}\mathcal{M}$$

*are linearly independent, or else for all  $t \in (a_y, b_y)$ , these vectors are linearly dependent.*

PROOF. Fix  $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{R}$  and denote

$$J(y, t) = \sum_{i=1}^{n-1} \lambda_i J_i(y, t) \quad \text{for } t \in (a_y, b_y).$$

We would like to show that the set  $\{t \in (a_y, b_y); J(y, t) = 0\}$  is an open set. Assume that  $t_1 \in (a_y, b_y)$  satisfies

$$(1) \quad J(y, t_1) = 0.$$

We need to prove that  $J(y, t) = 0$  for  $t$  in a small neighborhood of  $t_1$ . To this end, denote  $v = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$ . Since  $y \in B_0$  is a Lebesgue density point of  $B_0 \subseteq \mathbb{R}^{n-1}$ , then there exists a  $C^1$ -curve  $\gamma : (-1, 1) \rightarrow \mathbb{R}^{n-1}$  with  $\gamma(0) = y$  and  $\dot{\gamma}(0) = v$ , such that the set

$$I = \{s \in (-1, 1); \gamma(s) \in B_0\}$$

has an accumulation point at zero. We are going to view  $\gamma$  as a map from  $I$  to  $B_0$ , and we will never use the values of  $\gamma$  outside  $I$ . Thus, from now on when we write  $\dot{\gamma}(0) = v$ , we actually mean that

$$\lim_{I \ni s \rightarrow 0} \frac{\gamma(s) - \gamma(0)}{s} = v.$$

We plan to apply the geometric lemma of Feldman and McCann, which is Lemma 2.4.1 above. Set

$$(2) \quad p = F(y, t_1) \in \mathcal{M}.$$

Let  $\delta_1 = \delta_1(p) > 0$  be the parameter provided by Lemma 2.4.1. Fix  $\varepsilon > 0$  with

$$(3) \quad \varepsilon < \min\{\delta_1, b_y - t_1, t_1 - a_y\}.$$

Then  $a_y < t_1 - \varepsilon$  while  $b_y > t_1 + \varepsilon$ . Since  $B$  is a parallel line cluster, then the functions  $a$  and  $b$  are continuous on  $B_0$ . Since  $\gamma$  is continuous with  $\gamma(0) = y$ , then for some  $\eta > 0$ ,

$$(4) \quad a_{\gamma(s)} < t_1 - \varepsilon, \quad b_{\gamma(s)} > t_1 + \varepsilon \quad \text{for all } s \in I \cap (-\eta, \eta).$$

According to Proposition 3.1.5(iii) and the chain rule, for any  $t \in (t_1 - \varepsilon, t_1 + \varepsilon)$ ,

$$(5) \quad J(y, t) = \sum_{i=1}^{n-1} \lambda_i \frac{\partial F}{\partial y_i}(y, t) = \left. \frac{d}{ds} F(\gamma(s), t) \right|_{s=0}$$

where we only consider values  $s \in I$  when computing the limit defining the derivative with respect to  $s$ . Note that the use of the chain rule is legitimate, as  $F$  is

differentiable at  $(y, t)$  while  $\gamma(0) = y$  and  $\dot{\gamma}(0) = v = (\lambda_1, \dots, \lambda_{n-1})$ . From (5), for any  $t \in (t_1 - \varepsilon, t_1 + \varepsilon)$ ,

$$(6) \quad |J(y, t)| = \lim_{I \ni s \rightarrow 0} \frac{d(F(\gamma(0), t), F(\gamma(s), t))}{|s|} = \lim_{I \ni s \rightarrow 0} \frac{d(F(y, t), F(\gamma(s), t))}{|s|}.$$

Fix  $0 < \delta < \varepsilon$ . For  $s \in (-\eta, \eta) \cap I$  and  $i = 0, 1, 2$  define

$$(7) \quad x_i = F(y, t_1 + \delta(i-1)), \quad z_i(s) = F(\gamma(s), t_1 + \delta(i-1)).$$

The points  $x_0, x_1, x_2, z_0(s), z_1(s), z_2(s) \in \mathcal{M}$  are well-defined due to (3) and (4). According to Proposition 3.1.5(ii),

$$(8) \quad u(x_i) = t_1 + \delta(i-1) + r_0 = u(z_i(s)) \quad \text{for } i = 0, 1, 2, s \in I \cap (-\eta, \eta).$$

Recall that  $\|u\|_{Lip} \leq 1$  and that  $t \mapsto F(y, t)$  is a minimizing geodesic, as well as  $t \mapsto F(\gamma(s), t)$ . We thus conclude from (7) and (8) that for any  $s \in I \cap (-\eta, \eta)$  and  $i, j = 0, 1, 2$ ,

$$(9) \quad d(x_i, x_j) = d(z_i(s), z_j(s)) = \delta|i-j| = |u(x_i) - u(z_j(s))| \leq d(x_i, z_j(s)).$$

Furthermore, since  $d(x_i, x_1) \leq \delta < \varepsilon$  for  $i = 0, 1, 2$ , then thanks to (2) and (3),

$$(10) \quad x_0, x_1, x_2 \in \mathcal{B}_{\mathcal{M}}(x_1, \varepsilon) = \mathcal{B}_{\mathcal{M}}(p, \varepsilon) \subseteq \mathcal{B}_{\mathcal{M}}(p, \delta_1).$$

The map  $F$  is continuous, while  $\gamma(s) \rightarrow y$  as  $I \ni s \rightarrow 0$ . Therefore, for  $i = 0, 1, 2$  we have that  $z_i(s) \rightarrow x_i$  as  $I \ni s \rightarrow 0$ . From (10) we thus conclude that  $z_0(s), z_1(s), z_2(s) \in \mathcal{B}_{\mathcal{M}}(p, \delta_1)$  for any  $s \in I \cap (-\tilde{\eta}, \tilde{\eta})$  for some  $0 < \tilde{\eta} < \eta$ . Thanks to (9) we may apply Lemma 2.4.1 for the six points

$$x_0, x_1, x_2, z_0(s), z_1(s), z_2(s) \in \mathcal{B}_{\mathcal{M}}(p, \delta_1),$$

when  $s \in I \cap (-\tilde{\eta}, \tilde{\eta})$ . From the conclusion of Lemma 2.4.1,

$$(11) \quad \limsup_{I \ni s \rightarrow 0} \frac{d(x_0, z_0(s)) + d(x_2, z_2(s))}{|s|} \leq 20 \cdot \limsup_{I \ni s \rightarrow 0} \frac{d(x_1, z_1(s))}{|s|}.$$

By using (6), (7) and (11) we obtain

$$(12) \quad |J(y, t_1 - \delta)| + |J(y, t_1 + \delta)| \leq 20 \cdot |J(y, t_1)|.$$

However,  $\delta > 0$  was an arbitrary number in  $(0, \varepsilon)$ . From (1) and (12) we therefore conclude that

$$|J(y, t)| = 0 \quad \text{for all } t \in (t_1 - \varepsilon, t_1 + \varepsilon).$$

This completes the proof that the set  $\{t \in (a_y, b_y); J(y, t) = 0\}$  is an open set. Since  $J$  is a smooth Jacobi field, then this set is also closed. Therefore, either  $t \mapsto J(y, t)$  never vanishes on  $(a_y, b_y)$ , or else it is the zero function. In other words, for any  $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{R}$ ,

$$\exists t \in (a_y, b_y), \sum_{i=1}^{n-1} \lambda_i J_i(y, t) = 0 \quad \implies \quad \forall t \in (a_y, b_y), \sum_{i=1}^{n-1} \lambda_i J_i(y, t) = 0.$$

By linear algebra, either  $J_1(y, t), \dots, J_{n-1}(y, t)$  are linearly independent for all  $t \in (a_y, b_y)$ , or else they are linearly dependent for all  $t \in (a_y, b_y)$ .  $\square$

Recall that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold which is geodesically-convex. Recall also that  $\lambda_{\mathcal{M}}$  is the Riemannian volume measure on the Riemannian manifold  $\mathcal{M}$ . Let  $\rho : \mathcal{M} \rightarrow \mathbb{R}$  be the smooth function for which

$$(13) \quad \frac{d\mu}{d\lambda_{\mathcal{M}}} = e^{-\rho}.$$

**DEFINITION 3.3.2.** *A measure  $\nu$  on  $\mathcal{M}$  is called a “Jacobi needle” of the weighted Riemannian manifold  $(\mathcal{M}, d, \mu)$  and the 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  if there exist a non-empty interval  $(a, b) \subseteq \mathbb{R}$  with  $a, b \in \mathbb{R} \cup \{\pm\infty\}$ , a measure  $\theta$  on  $(a, b)$ , a minimizing geodesic  $\gamma : (a, b) \rightarrow \mathcal{M}$  and Jacobi fields  $J_1(t), \dots, J_{n-1}(t)$  along  $\gamma$  with the following properties:*

- (i) *The measure  $\nu$  is the push-forward of  $\theta$  under the map  $\gamma$ .*
- (ii) *Denote  $J_n = \dot{\gamma}$ . Then the measure  $\theta$  is absolutely-continuous in the interval  $(a, b) \subseteq \mathbb{R}$ , and its density is proportional to*

$$t \mapsto e^{-\rho(\gamma(t))} \cdot \sqrt{\det (\langle J_i(t), J_k(t) \rangle)_{i,k=1,\dots,n}}.$$

- (iii) *There exists  $t \in (a, b)$  with*

$$(14) \quad \langle J_i(t), \dot{\gamma}(t) \rangle = \langle J'_i(t), \dot{\gamma}(t) \rangle = 0 \quad (i = 1, \dots, n-1),$$

and

$$(15) \quad \langle J'_i(t), J_k(t) \rangle = \langle J'_k(t), J_i(t) \rangle \quad (i, k = 1, \dots, n-1).$$

- (iv) *Either for all  $t \in (a, b)$  the vectors*

$$J_1(t), \dots, J_{n-1}(t) \in T_{\gamma(t)}\mathcal{M}$$

*are linearly independent, or else for all  $t \in (a, b)$  these vectors are linearly dependent.*

- (v) *Denote  $A = (a, b) \subseteq \mathbb{R}$ . Then the set  $\gamma(A)$  is the relative interior of a transport ray associated with  $u$  and*

$$u(\gamma(t)) = t \quad \text{for all } t \in A.$$

Assume that  $\Omega_1, \Omega_2, \dots$  are certain disjoint sets. Let  $\nu_i$  be a measure defined on  $\Omega_i$  for  $i \geq 1$ . We may clearly consider the measure  $\nu = \sum_{i \geq 1} \nu_i$  defined on  $\Omega = \cup_{i \geq 1} \Omega_i$ . A subset  $A \subseteq \Omega$  is  $\nu$ -measurable if and only if  $A \cap \Omega_i$  is  $\nu_i$ -measurable for any  $i \geq 1$ .

Recall that  $T^\circ[u]$  is a partition of  $Strain[u]$  and that  $\pi : Strain[u] \rightarrow T^\circ[u]$  is the partition map, i.e.,  $x \in \pi(x) \in T^\circ[u]$  for any  $x \in Strain[u]$ . According to Lemma 2.1.7, for any  $x \in Strain[u]$ , the set  $\pi(x)$  is the relative interior of the unique transport ray containing  $x$ .

**PROPOSITION 3.3.3.** *Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Then there exist a measure  $\nu$  on  $T^\circ[u]$  and a family  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$  of measures on  $\mathcal{M}$ , such that the following hold true:*

- (i) *If  $G \subseteq T^\circ[u]$  is  $\nu$ -measurable then  $\pi^{-1}(G) \subseteq Strain[u]$  is a measurable subset of  $\mathcal{M}$ . For any measurable set  $A \subseteq \mathcal{M}$ , the map  $\mathcal{I} \mapsto \mu_{\mathcal{I}}(A)$  is well-defined  $\nu$ -almost everywhere and is a  $\nu$ -measurable map.*

(ii) For any measurable set  $A \subseteq \mathcal{M}$ ,

$$(16) \quad \mu(A \cap \text{Strain}[u]) = \int_{T^\circ[u]} \mu_{\mathcal{I}}(A) d\nu(\mathcal{I}).$$

(iii) For  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ , the measure  $\mu_{\mathcal{I}}$  is a Jacobi needle of  $(\mathcal{M}, d, \mu)$  and  $u$  that is supported on  $\mathcal{I}$  and it satisfies  $\mu_{\mathcal{I}}(\mathcal{M}) > 0$ . Furthermore, the interval  $A = (a, b)$  and the geodesic curve  $\gamma$  from Definition 3.3.2 satisfy  $\mathcal{I} = \gamma(A)$ .

PROOF. The measure  $\mu$  is assumed to be absolutely-continuous with respect to  $\lambda_{\mathcal{M}}$ . According to Proposition 3.2.1, there exist disjoint ray clusters of continuous length  $\{R_i\}_{i=1,2,\dots}$  with

$$(17) \quad \mu \left( \text{Strain}[u] \setminus \left( \bigcup_{i=1}^{\infty} R_i \right) \right) = 0.$$

Recall from Definition 3.1.4 and Lemma 3.2.2 that each ray cluster  $R_i$  is a measurable set contained in  $\text{Strain}[u]$  of the form  $R_i = \cup_{\mathcal{I} \in S_i} \mathcal{I}$  for some subset  $S_i \subseteq T^\circ[u]$ . Fix  $i \geq 1$ . Let us apply Proposition 3.1.5 for  $R_i$ , which is a ray cluster of continuous length. Proposition 3.1.5 provides us with a certain parallel line cluster  $B \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$ , a locally-Lipschitz, invertible map  $F : B \rightarrow R_i$ , and also with vector fields

$$J_1(y, t), \dots, J_{n-1}(y, t).$$

Let  $J_n, r_0, B_0, a_y$  and  $b_y$  be as in Proposition 3.1.5. Then for almost any Lebesgue density point  $y \in B_0$ , the vector fields  $J_1(y, t), \dots, J_{n-1}(y, t)$  are well-defined Jacobi fields along the entire geodesic  $t \mapsto F(y, t)$  for  $t \in (a_y, b_y)$ . Consider the measure on  $B$  whose density with respect to the Lebesgue measure on  $B$  is

$$(18) \quad (y, t) \mapsto \sqrt{\det (\langle J_\ell(y, t), J_k(y, t) \rangle)_{\ell, k=1, \dots, n}}.$$

According to Proposition 3.1.5(iv) and Remark 3.1.6, the map  $F$  pushes forward the measure whose density is given by (18) to the restriction of  $\lambda_{\mathcal{M}}$  to the ray cluster  $R_i$ . Next, consider the measure on  $B$  with density

$$(19) \quad (y, t) \mapsto e^{-\rho(F(y, t))} \cdot \sqrt{\det (\langle J_\ell(y, t), J_k(y, t) \rangle)_{\ell, k=1, \dots, n}}.$$

Glancing at (13), we see that the map  $F$  pushes forward the measure whose density is given by (19) to the restriction of  $\mu$  to  $R_i$ . From Proposition 3.1.5(ii), for any  $y \in B_0$  there exists  $\mathcal{I}(y) \in T^\circ[u]$  such that

$$\mathcal{I}(y) = \{F(y, t); a_y < t < b_y\}.$$

Furthermore,  $\mathcal{I}(y) \subseteq R_i$ , and since  $F$  is invertible then  $\mathcal{I}(y_1) \cap \mathcal{I}(y_2) = \emptyset$  for  $y_1 \neq y_2$ . By Proposition 3.1.5(ii), for all  $y \in B_0$  the map  $t \mapsto F(y, t)$  is a minimizing geodesic. Define the measure

$$\tilde{\mu}_{\mathcal{I}(y)}$$

to be the push-forward under the map  $t \mapsto F(y, t)$  of the measure on  $(a_y, b_y)$  whose density is given by

$$t \mapsto e^{-\rho(F(y, t))} \cdot \sqrt{\det (\langle J_\ell(y, t), J_k(y, t) \rangle)_{\ell, k=1, \dots, n}}.$$

Then  $\tilde{\mu}_{\mathcal{I}(y)}$  is a well-defined measure supported on  $\mathcal{I}(y)$  for almost any  $y \in B_0$ . Recall that the map  $F$  pushes forward the measure whose density is given by (19) to the restriction of  $\mu$  to  $R_i$ . By Fubini's theorem, for any measurable set  $A \subseteq R_i$ ,

$$(20) \quad \mu(A) = \int_{B_0} \tilde{\mu}_{\mathcal{I}(y)}(A) dy = \int_{B_0} \mu_{\mathcal{I}(y)}(A) e^{-|y|} dy,$$

where  $\mu_{\mathcal{I}(y)} := e^{|y|} \tilde{\mu}_{\mathcal{I}(y)}$ . (The  $e^{-|y|}$  factor in (20) might seem artificial, the integrability of  $e^{-|y|}$  will shortly imply that a certain measure  $\nu_i$  is a finite measure). Denote

$$(21) \quad \tilde{B}_0 = \{y \in B_0; \mu_{\mathcal{I}(y)}(\mathcal{M}) > 0\},$$

which is a measurable subset of  $B_0 \subseteq \mathbb{R}^{n-1}$ . Define the measure  $\nu_i$  to be the push-forward under the map  $y \mapsto \mathcal{I}(y)$  of the measure on  $\tilde{B}_0$  whose density is  $y \mapsto e^{-|y|}$ . Then  $\nu_i$  is a finite measure supported on  $T^\circ[u]$ . In fact,  $\nu_i$  is supported on  $S_i \subseteq T^\circ[u]$  since  $\mathcal{I}(y) \in S_i$  for all  $y \in B_0$ . From (20) and (21), for any measurable set  $A \subseteq \mathcal{M}$ ,

$$(22) \quad \mu(A \cap R_i) = \int_{S_i} \mu_{\mathcal{I}}(A \cap R_i) d\nu_i(\mathcal{I}) = \int_{S_i} \mu_{\mathcal{I}}(A) d\nu_i(\mathcal{I}).$$

Furthermore,  $\mu_{\mathcal{I}}(\mathcal{M}) > 0$  for  $\nu_i$ -almost any  $\mathcal{I} \in S_i$ , by the definition of  $\tilde{B}_0$ . Recall that when we push-forward a measure, we also push-forward its  $\sigma$ -algebra. Therefore if a subset  $G \subseteq S_i$  is  $\nu_i$ -measurable, then  $\{y \in \tilde{B}_0; \mathcal{I}(y) \in G\}$  is a measurable subset of  $B_0$ . Since  $B$  is a parallel line cluster, also  $\{(y, t) \in B; y \in \tilde{B}_0, \mathcal{I}(y) \in G\}$  is measurable in  $\mathbb{R}^{n-1} \times \mathbb{R}$ . The image of the latter measurable set under  $F$  equals  $\pi^{-1}(G)$ , up to a set of  $\lambda_{\mathcal{M}}$ -measure zero. Since  $F$  is locally-Lipschitz, then  $\pi^{-1}(G)$  is a measurable subset of  $\text{Strain}[u]$ , whenever  $G \subseteq S_i$  is  $\nu_i$ -measurable.

Let us show that  $\mu_{\mathcal{I}}$  is a Jacobi needle for  $\nu_i$ -almost any  $\mathcal{I} \in S_i$ . Since  $\mu_{\mathcal{I}}$  is proportional to  $\tilde{\mu}_{\mathcal{I}}$ , it suffices to prove that  $\tilde{\mu}_{\mathcal{I}(y)}$  is a Jacobi needle for almost any  $y \in \tilde{B}_0$ . Properties (i) and (ii) from Definition 3.3.2 hold by the definition of  $\tilde{\mu}_{\mathcal{I}(y)}$ , where we set

$$J_i(t) = J_i(y, t - r_0), \quad \gamma(t) = F(y, t - r_0), \quad a = a_y + r_0, \quad b = b_y + r_0.$$

Property (v) follows from Proposition 3.1.5(ii). We deduce property (iii) of Definition 3.3.2 (with  $t = r_0$ ) from Proposition 3.1.5(iii). Property (iv) follows from Lemma 3.3.1. Note also that setting  $A = (a, b)$  we have

$$(23) \quad \mathcal{I}(y) = \gamma(A).$$

Hence  $\tilde{\mu}_{\mathcal{I}(y)}$  is a Jacobi needle supported on  $\mathcal{I}(y)$  for almost any  $y \in \tilde{B}_0$ , and consequently  $\mu_{\mathcal{I}}$  is a Jacobi needle supported on  $\mathcal{I}$  for  $\nu_i$ -almost any  $\mathcal{I} \in S_i$ . Write  $\tilde{S}_i \subseteq S_i$  for the collection of all  $\mathcal{I} \in S_i$  for which  $\mu_{\mathcal{I}}$  is a Jacobi needle supported on  $\mathcal{I}$  with  $\mu_{\mathcal{I}}(\mathcal{M}) > 0$ . Then  $\nu_i(S_i \setminus \tilde{S}_i) = 0$ . For completeness, let us redefine  $\mu_{\mathcal{I}} \equiv 0$  for  $\mathcal{I} \in S_i \setminus \tilde{S}_i$ . Note that (22) still holds true for any measurable set  $A \subseteq \mathcal{M}$ , since we altered the definition of  $\mu_{\mathcal{I}}$  only on a  $\nu_i$ -null set.

To summarize, we found a family of measures  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in S_i}$  such that (22) holds true for any measurable set  $A \subseteq \mathcal{M}$ . We now let  $i$  vary. Since the ray clusters  $\{R_i\}_{i=1,2,\dots}$  are disjoint, then  $S_1, S_2, \dots \subseteq T^\circ[u]$  are also disjoint. Denoting  $\nu = \sum_i \nu_i$ , we deduce (16) from (17) and (22). This completes the proof of (ii), and also of the second assertion in (i). Furthermore, for  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ , we have that  $\mathcal{I} \in S_i$  for some  $i$ , and the measure  $\mu_{\mathcal{I}}$  is a Jacobi needle supported

on  $\mathcal{I}$  with  $\mu_{\mathcal{I}}(\mathcal{M}) > 0$ . It thus follows from (23) that conclusion (iii) holds true. Note that if a subset  $G \subseteq T^\circ[u]$  is  $\nu$ -measurable, then  $G \cap S_i$  is  $\nu_i$ -measurable for any  $i$ , and hence  $\pi^{-1}(G \cap S_i) \subseteq R_i$  is measurable in  $\mathcal{M}$ . Consequently  $\pi^{-1}(G)$  is  $\lambda_{\mathcal{M}}$ -measurable whenever  $G \subseteq T^\circ[u]$  is  $\nu$ -measurable. This completes the proof of (i). The proposition is therefore proven.  $\square$

Recall from Section 1 the definition of the generalized Ricci tensor  $Ric_{\mu,N}$  of the weighted Riemannian manifold  $(\mathcal{M}, d, \mu)$ .

**DEFINITION 3.3.4.** *Let  $n \geq 2$  and let  $(\mathcal{M}, d, \mu)$  be an  $n$ -dimensional weighted Riemannian manifold. We say that a measure  $\nu$  on the Riemannian manifold  $\mathcal{M}$  is a ‘‘Ricci needle’’ if there exist a non-empty, open interval  $A \subseteq \mathbb{R}$ , a smooth function  $\Psi : A \rightarrow \mathbb{R}$  and a minimizing geodesic  $\gamma : A \rightarrow \mathcal{M}$  such that:*

- (i) *Denote by  $\theta$  the measure on  $A \subseteq \mathbb{R}$  whose density with respect to the Lebesgue measure is  $e^{-\Psi}$ . Then  $\nu$  is the push-forward of  $\theta$  under the map  $\gamma$ .*
- (ii) *For any  $N \in (-\infty, 1) \cup [n, +\infty]$ , the following inequality holds in the entire interval  $A$ :*

$$\Psi'' \geq Ric_{\mu,N}(\dot{\gamma}, \dot{\gamma}) + \frac{(\Psi')^2}{N-1},$$

where in the case  $N = \infty$ , we interpret the term  $(\Psi')^2/(N-1)$  as zero.

The following proposition asserts that any Jacobi needle in the sense of Definition 3.3.2 which is not the zero measure is in fact a *Ricci needle*.

**PROPOSITION 3.3.5.** *Let  $n \geq 2, N \in (-\infty, 1) \cup [n, +\infty]$  and let  $(\mathcal{M}, d, \mu)$  be an  $n$ -dimensional weighted Riemannian manifold which is geodesically-convex. Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Let  $\nu$  be a Jacobi needle of  $(\mathcal{M}, d, \mu)$  and  $u$ . Then either  $\nu$  is the zero measure, or else  $\nu$  is a Ricci needle.*

The proof of Proposition 3.3.5 boils down to a classical estimate in Riemannian geometry from Heintze and Karcher [30] that was generalized to the case of *weighted* Riemannian manifolds by Bayle [5, Appendix E.1] and by Morgan [39]. According to Gromov [26], the estimate stems from the work of Paul Levy on the isoperimetric inequality in 1919. We begin the proof of Proposition 3.3.5 with the following trivial lemma:

**LEMMA 3.3.6.** *Let  $a, b \in \mathbb{R}$  with  $b > 0$  and  $a \notin [-b, 0]$ . Then,*

$$\frac{x^2}{a} + \frac{y^2}{b} \geq \frac{(x-y)^2}{a+b} \quad (x, y \in \mathbb{R}).$$

**PROOF.** We use the inequality  $|b/a| \cdot x^2 \pm 2xy + |a/b| \cdot y^2 \geq 0$  to deduce that

$$\frac{x^2}{a} + \frac{y^2}{b} - \frac{(x-y)^2}{a+b} = \frac{1}{a+b} \left( \frac{b}{a}x^2 + 2xy + \frac{a}{b}y^2 \right) \geq 0,$$

whenever  $b > 0$  and  $a \notin [-b, 0]$ .  $\square$

Let us recall the familiar formulas for differentiating a determinant. If  $A_t$  is an invertible  $n \times n$  matrix that depends smoothly on  $t \in \mathbb{R}$ , then

$$(24) \quad \frac{d}{dt} \log |\det(A_t)| = \text{Trace}[A_t^{-1} \cdot \dot{A}_t],$$

and

$$(25) \quad \frac{d^2}{dt^2} \log |\det(A_t)| = \text{Trace}[A_t^{-1} \cdot \ddot{A}_t] - \text{Trace} \left[ \left( A_t^{-1} \cdot \dot{A}_t \right)^2 \right].$$

PROOF OF PROPOSITION 3.3.5. Let  $\nu$  be a Jacobi needle of  $(\mathcal{M}, d, \mu)$  and  $u$ . We may assume that  $\nu$  is not the zero measure. Let  $a, b, \theta, \gamma$  and  $J_1, \dots, J_{n-1}$  be as in Definition 3.3.2. For  $t \in (a, b)$  denote

$$(26) \quad f(t) = e^{-\rho(\gamma(t))} \cdot \sqrt{\det(\langle J_i(t), J_k(t) \rangle)_{i,k=1,\dots,n}}$$

where  $J_n = \dot{\gamma}$ . According to Definition 3.3.2(ii), the density of the measure  $\theta$  on  $(a, b) \subseteq \mathbb{R}$  is proportional to the function  $f$ . We will prove that the smooth function  $f$  is positive in  $(a, b)$ , and that for any  $N \in (-\infty, 1) \cup [n, +\infty]$ , the function  $\Psi := -\log f$  satisfies

$$(27) \quad \Psi'' \geq \text{Ric}_{\mu, N}(\dot{\gamma}, \dot{\gamma}) + \frac{(\Psi')^2}{N-1},$$

where in the case  $N = +\infty$  we interpret the term  $(\Psi')^2/(N-1)$  as zero. Comparing Definition 3.3.4 of Ricci needles and Definition 3.3.2 of Jacobi needles, we see that the proposition would follow from (27). The rest of the proof is therefore devoted to establishing (27). The Jacobi fields  $J_1, \dots, J_{n-1}$  satisfy the Jacobi equation:

$$(28) \quad J_i''(t) = R(\dot{\gamma}(t), J_i(t))\dot{\gamma}(t) \quad \text{for } t \in (a, b), i = 1, \dots, n-1.$$

Since  $\gamma$  is a geodesic then  $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ , and for any  $i = 1, \dots, n-1$  and  $t \in (a, b)$ ,

$$(29) \quad \frac{d}{dt} \langle J_i, \dot{\gamma} \rangle = \langle J_i', \dot{\gamma} \rangle, \quad \frac{d^2}{dt^2} \langle J_i, \dot{\gamma} \rangle = \langle J_i'', \dot{\gamma} \rangle.$$

From (28) and the symmetries of the Riemann curvature tensor we deduce that  $\langle J_i'', \dot{\gamma} \rangle \equiv 0$ . Therefore  $\langle J_i(t), \dot{\gamma}(t) \rangle$  is an affine function of  $t \in (a, b)$ . It thus follows from (14) and (29) that for any  $t \in (a, b)$ ,

$$(30) \quad J_1(t), \dots, J_{n-1}(t) \perp \dot{\gamma}(t).$$

From (26) and (30) we obtain

$$(31) \quad f(t) = e^{-\rho(\gamma(t))} \cdot \sqrt{\det(\langle J_i(t), J_k(t) \rangle)_{i,k=1,\dots,n-1}}.$$

(The indices run only up to  $n-1$ , as  $\dot{\gamma} = J_n$  is a unit vector orthogonal to  $J_1, \dots, J_{n-1}$ ). Since  $\theta$  is not the zero measure, there exists  $t_1 \in (a, b)$  for which  $f(t_1) \neq 0$ . From (31) we learn that the vectors

$$J_1(t_1), \dots, J_{n-1}(t_1) \in T_{\gamma(t_1)}\mathcal{M}$$

are linearly independent. By Definition 3.3.2(iv), the vectors  $J_1(t), \dots, J_{n-1}(t)$  are linearly independent for all  $t \in (a, b)$ . Hence, (31) yields

$$(32) \quad \forall t \in (a, b), \quad f(t) > 0.$$

From the Jacobi equation (28), for any  $t \in (a, b)$  and  $i, k = 1, \dots, n-1$ ,

$$(33) \quad \frac{d}{dt} (\langle J_i', J_k \rangle - \langle J_i, J_k' \rangle) = \langle J_i'', J_k \rangle - \langle J_i, J_k'' \rangle = \langle R(\dot{\gamma}, J_i)\dot{\gamma}, J_k \rangle - \langle J_i, R(\dot{\gamma}, J_k)\dot{\gamma} \rangle = 0,$$

by the symmetries of the Riemann curvature tensor. By using (15) and (33) we deduce that in the entire interval  $(a, b) \subseteq \mathbb{R}$ ,

$$(34) \quad \langle J_i', J_k \rangle = \langle J_i, J_k' \rangle \quad \text{for } i, k = 1, \dots, n.$$



Let  $G_t = (G_t(i, k))_{i, k=1, \dots, n-1}$  be the symmetric, positive-definite  $(n-1) \times (n-1)$  matrix whose entries are  $G_t(i, k) = \langle J_i(t), J_k(t) \rangle$ . According to (31) and (32), the function  $\Psi = -\log f$  satisfies,

$$(35) \quad \Psi(t) = \rho(\gamma(t)) - \frac{1}{2} \log \det G_t \quad \text{for } t \in (a, b).$$

Denote  $H(t) = \dot{\gamma}(t)^\perp \subset T_{\dot{\gamma}(t)}\mathcal{M}$ , the orthogonal complement to the vector  $\dot{\gamma}(t)$ . From (29) and (30),

$$(36) \quad J_i(t), J'_i(t) \in H(t) \quad \text{for all } t \in (a, b), i = 1, \dots, n-1.$$

For any  $t \in (a, b)$  the linearly-independent vectors  $J_1(t), \dots, J_{n-1}(t) \in H(t)$  constitute a basis of the  $(n-1)$ -dimensional space  $H(t)$ . In view of (36), we may define an  $(n-1) \times (n-1)$  matrix  $A_t = (A_t(i, k))_{i, k=1, \dots, n-1}$  by requiring that

$$(37) \quad J'_i(t) = \sum_{k=1}^{n-1} A_t(i, k) J_k(t) \quad \text{for } t \in (a, b), i = 1, \dots, n-1.$$

Recall that  $G_t(i, k) = \langle J_i(t), J_k(t) \rangle$ . From (34) and (37), for any  $t \in (a, b)$ ,

$$\begin{aligned} \dot{G}_t(i, k) &= \langle J'_i, J_k \rangle + \langle J_i, J'_k \rangle = 2\langle J'_i, J_k \rangle \\ &= 2 \left\langle \sum_{\ell=1}^{n-1} A_t(i, \ell) J_\ell, J_k \right\rangle = 2 \sum_{\ell=1}^{n-1} A_t(i, \ell) G_t(\ell, k). \end{aligned}$$

Equivalently,  $\dot{G}_t = 2A_t G_t$ . Since  $G_t$  is a symmetric matrix then also  $A_t G_t = \dot{G}_t/2$  is a symmetric matrix. Since  $G_t$  is a positive-definite matrix, then from (24),

$$(38) \quad \frac{d}{dt} \log \det(G_t) = \text{Trace} [G_t^{-1} \dot{G}_t] = 2 \text{Trace} [G_t^{-1} A_t G_t] = 2 \text{Trace}[A_t].$$

As for the second derivative, we use (37) and the Jacobi equation (28) and obtain,

$$(39) \quad \begin{aligned} \ddot{G}_t(i, k) &= \langle J''_i, J_k \rangle + 2\langle J'_i, J'_k \rangle + \langle J''_k, J_i \rangle \\ &= 2\langle R(\dot{\gamma}, J_i)\dot{\gamma}, J_k \rangle + 2 \sum_{\ell, m=1}^{n-1} A_t(i, \ell) A_t(k, m) G_t(\ell, m) \end{aligned}$$

where we used the symmetries of the Riemann curvature tensor in the last passage. Recall that  $\text{Ric}_{\mathcal{M}}(\dot{\gamma}, \dot{\gamma})$  is the trace of the linear transformation  $V \mapsto -R(\dot{\gamma}, V)\dot{\gamma}$  in the linear space  $H(t)$ . By linear algebra, (39) entails that

$$(40) \quad \text{Trace} [G_t^{-1} \ddot{G}_t] = -2\text{Ric}_{\mathcal{M}}(\dot{\gamma}(t), \dot{\gamma}(t)) + \text{Trace} [2G_t^{-1} A_t^2 G_t],$$

where we used the fact that  $A_t G_t A_t^* = A_t (A_t G_t)^* = A_t^2 G_t$  in the last passage, as  $A_t G_t$  is symmetric. Since  $\dot{G}_t = 2A_t G_t$  then from (25) and (40),

$$(41) \quad \frac{d^2}{dt^2} \log \det(G_t) = -2\text{Ric}_{\mathcal{M}}(\dot{\gamma}(t), \dot{\gamma}(t)) + 2 \text{Trace} [A_t^2] - 4 \text{Trace} [G_t^{-1} A_t^2 G_t].$$

Applying (35) and (38) yields

$$(42) \quad \Psi'(t) = \partial_{\dot{\gamma}(t)} \rho - \text{Trace}[A_t].$$

Since  $\gamma$  is a geodesic, the equations (35) and (41) lead to

$$(43) \quad \Psi''(t) = \text{Hess}_\rho(\dot{\gamma}(t), \dot{\gamma}(t)) + \text{Ric}_{\mathcal{M}}(\dot{\gamma}(t), \dot{\gamma}(t)) + \text{Trace} [A_t^2].$$

We will now utilize the definition of the generalized Ricci tensor with parameter  $N$ . From (43), for any  $N \in (-\infty, 1) \cup [n, +\infty]$ ,

$$\Psi''(t) \geq Ric_{\mu, N}(\dot{\gamma}(t), \dot{\gamma}(t)) + \frac{(\partial_{\dot{\gamma}(t)}\rho)^2}{N-n} + Trace[A_t^2],$$

where in the case where  $N = \infty$  we interpret the term  $(\partial_{\dot{\gamma}(t)}\rho)^2/(N-n)$  as zero. In the case where  $N = n$ , we require  $\rho$  to be a constant function and the latter term is again interpreted as zero. The matrix  $\dot{G}_t = 2A_tG_t$  is symmetric, and hence  $G_t^{-1/2}A_tG_t^{1/2}$  is also symmetric. Thus the  $(n-1) \times (n-1)$  matrix  $A_t$  is conjugate to a symmetric matrix and consequently it has  $n-1$  real eigenvalues, repeated according to their multiplicity. The Cauchy-Schwartz inequality yields  $[Trace(A_t)]^2 \leq (n-1)Trace[A_t^2]$ . Therefore, for any  $t \in (a, b)$  and  $N \in (-\infty, 1) \cup [n, +\infty]$ ,

$$(44) \quad \Psi''(t) \geq Ric_{\mu, N}(\dot{\gamma}(t), \dot{\gamma}(t)) + \frac{(\partial_{\dot{\gamma}(t)}\rho)^2}{N-n} + \frac{(Trace[A_t])^2}{n-1}.$$

In the case where  $N = \infty$  or  $N = n$ , we deduce (27) from (42) and (44). Otherwise, we have  $N \in \mathbb{R} \setminus [1, n]$  and from (44) and Lemma 3.3.6,

$$(45) \quad \Psi''(t) \geq Ric_{\mu, N}(\dot{\gamma}(t), \dot{\gamma}(t)) + \frac{(\partial_{\dot{\gamma}(t)}\rho - Trace[A_t])^2}{N-1}.$$

From (42) and (45) we conclude that (27) holds true for any  $t \in (a, b)$  and  $N \in (-\infty, 1) \cup [n, +\infty]$ . The proof of the proposition is complete.  $\square$

**EXAMPLE 3.3.7.** Consider the example where  $\rho \equiv Const$  and where  $\mathcal{M} \subseteq \mathbb{R}^n$  is an open, convex set. Equations (42) and (43) along with simple manipulations show that in this case,

$$(46) \quad \Psi'(t) = -Trace[A_t], \quad \Psi''(t) = Trace[A_t^2] \quad \text{and} \quad \dot{A}_t = -A_t^2.$$

The eigenvalues of  $A_t$  may be viewed as ‘‘principal curvatures’’ or as ‘‘eigenvalues of the second fundamental form’’ of a level set of  $u$ . Solving (46), we see that the density  $f(t) = e^{-\Psi(t)}$  is proportional to the function

$$(47) \quad t \mapsto \prod_{i=1}^k |t - \lambda_i| \quad \text{for } t \in (a, b),$$

where  $k \leq n-1$  and  $\lambda_1, \dots, \lambda_k \in \mathbb{R} \setminus (a, b)$  are some numbers. An empty product is defined to be one. We learn from (47) that the positive function  $f : (a, b) \rightarrow \mathbb{R}$  is a polynomial of degree at most  $n-1$ , all of whose roots lie in  $\mathbb{R} \setminus (a, b)$ .

**THEOREM 3.3.8.** *Let  $n \geq 2$  and assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold which is geodesically-convex. Let  $u : \mathcal{M} \rightarrow \mathbb{R}$  satisfy  $\|u\|_{Lip} \leq 1$ . Then there exist a measure  $\nu$  on the set  $T^\circ[u]$  and a family  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$  of measures on  $\mathcal{M}$  such that:*

- (i) *For any Lebesgue-measurable set  $A \subseteq \mathcal{M}$ , the map  $\mathcal{I} \mapsto \mu_{\mathcal{I}}(A)$  is well-defined  $\nu$ -almost everywhere and is a  $\nu$ -measurable map. When a subset  $S \subseteq T^\circ[u]$  is  $\nu$ -measurable then  $\pi^{-1}(S) \subseteq Strain[u]$  is a measurable subset of  $\mathcal{M}$ , where  $\pi : Strain[u] \rightarrow T^\circ[u]$  is the partition map.*

(ii) For any Lebesgue-measurable set  $A \subseteq \mathcal{M}$ ,

$$\mu(A \cap \text{Strain}[u]) = \int_{T^\circ[u]} \mu_{\mathcal{I}}(A) d\nu(\mathcal{I}).$$

(iii) For  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ , the measure  $\mu_{\mathcal{I}}$  is a Ricci needle supported on  $\mathcal{I} \subseteq \mathcal{M}$ . Furthermore, the set  $A \subseteq \mathbb{R}$  and the minimizing geodesic  $\gamma : A \rightarrow \mathcal{M}$  from Definition 3.3.4 may be selected so that  $\mathcal{I} = \gamma(A)$  and so that

$$u(\gamma(t)) = t \quad \text{for all } t \in A.$$

PROOF. Apply Proposition 3.3.3 to obtain certain measures  $\nu$  and  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$ . Applying Proposition 3.3.3(iii) and Proposition 3.3.5, we learn that  $\mu_{\mathcal{I}}$  is a Ricci needle supported on  $\mathcal{I}$  for  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ . Together with Definition 3.3.2(v), this proves conclusion (iii). Conclusions (i) and (ii) follow from Proposition 3.3.3(i) and Proposition 3.3.3(ii), respectively.  $\square$

PROOF OF THEOREM 1.4. Recall from Section 1 that the weighted Riemannian manifold  $(\mathcal{M}, d, \mu)$  satisfies the curvature-dimension condition  $CD(\kappa, N)$  when

$$\text{Ric}_{\mu, N}(v, v) \geq \kappa \quad \text{for any } p \in \mathcal{M}, v \in T_p\mathcal{M}, |v| = 1.$$

Glancing at Definition 1.1 and Definition 3.3.4, we see that under the curvature-dimension condition  $CD(\kappa, N)$ , any Ricci needle is in fact a  $CD(\kappa, N)$ -needle. The theorem thus follows from Theorem 3.3.8.  $\square$

## The Monge-Kantorovich problem

In this section we prove Theorem 1.5, following the approach of Evans and Gangbo [21]. We assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional, geodesically-convex, weighted Riemannian manifold of class  $CD(\kappa, N)$ , where  $n \geq 2, \kappa \in \mathbb{R}$  and  $N \in (-\infty, 1) \cup [n, +\infty)$ . Suppose that  $f : \mathcal{M} \rightarrow \mathbb{R}$  is a  $\mu$ -integrable function with

$$\int_{\mathcal{M}} f d\mu = 0.$$

Assume also that there exists a point  $x_0 \in \mathcal{M}$  with

$$(1) \quad \int_{\mathcal{M}} |f(x)| \cdot d(x_0, x) d\mu(x) < \infty.$$

It follows from (1) that for any 1-Lipschitz function  $v : \mathcal{M} \rightarrow \mathbb{R}$ ,

$$\int_{\mathcal{M}} |fv| d\mu \leq |v(x_0)| \int_{\mathcal{M}} |f| d\mu + \int_{\mathcal{M}} |f(x)| d(x_0, x) d\mu(x) < \infty,$$

as  $|v(x)| \leq |v(x_0)| + d(x_0, x)$  for all  $x \in \mathcal{M}$ . Conclusion (A) of Theorem 1.5 follows from the following standard lemma:

LEMMA 4.1. *There exists a 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  with*

$$\int_{\mathcal{M}} u f d\mu = \sup \left\{ \int_{\mathcal{M}} v f d\mu ; v : \mathcal{M} \rightarrow \mathbb{R}, \|v\|_{Lip} \leq 1 \right\}.$$

PROOF. Recall that  $(\mathcal{M}, d)$  is a locally-compact, separable, metric space (see, e.g., Section 2.1). For  $k = 1, 2, \dots$  let  $v_k : \mathcal{M} \rightarrow \mathbb{R}$  be a 1-Lipschitz function such that

$$\int_{\mathcal{M}} v_k f d\mu \xrightarrow{k \rightarrow \infty} \sup_{\|v\|_{Lip} \leq 1} \int_{\mathcal{M}} v f d\mu.$$

Since  $\int_{\mathcal{M}} f d\mu = 0$ , then we may add a constant to  $v_k$  and assume that  $v_k(x_0) = 0$  for all  $k$ . By the Arzela-Ascoli theorem, there exists a subsequence  $v_{k_i}$  that converges locally-uniformly to a 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  with  $u(x_0) = 0$ . Since  $|v_k(x)| \leq d(x_0, x)$  for all  $x \in \mathcal{M}$  and  $k \geq 1$ , then we may apply the dominated convergence theorem thanks to (1). We conclude that

$$\int_{\mathcal{M}} u f d\mu = \lim_{i \rightarrow \infty} \int_{\mathcal{M}} v_{k_i} f d\mu = \sup_{\|v\|_{Lip} \leq 1} \int_{\mathcal{M}} v f d\mu,$$

and the lemma is proven. □

The maximization problem in Lemma 4.1 is dual to the  $L^1$ -Monge-Kantorovich problem in the theory of optimal transportation. For information about the Monge-Kantorovich  $L^1$ -transportation problem, we refer the reader to the book by Kantorovich and Akilov [32, Section VIII.4] and to the papers by Ambrosio [1], Evans and Gangbo [21] and Gangbo [25].

Most of the remainder of this section is devoted to the proof of conclusions (B) and (C) of Theorem 1.5. To that end, let us fix a 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  such that

$$(2) \quad \int_{\mathcal{M}} u f d\mu = \sup_{\|v\|_{L^1} \leq 1} \int_{\mathcal{M}} v f d\mu.$$

Recall the definition of a transport ray from Section 2.1. The set  $T[u]$  is the collection of all transport rays associated with  $u$ . From the definition of a transport ray, for any  $x, y \in \mathcal{M}$ ,

$$|u(x) - u(y)| = d(x, y) \quad \iff \quad \exists \mathcal{I} \in T[u], x, y \in \mathcal{I}.$$

A transport ray is called degenerate when it is a singleton. By the maximality property of transport rays (see Definition 2.1.2), for any  $x \in \mathcal{M}$ ,

$$\{x\} \in T[u] \quad \iff \quad \forall x \neq y \in \mathcal{M}, |u(y) - u(x)| < d(x, y).$$

Define  $Loose[u] \subseteq \mathcal{M}$  to be the union of all *degenerate* transport rays associated with  $u$ . Thus,

$$Loose[u] = \{x \in \mathcal{M}; \{x\} \in T[u]\}.$$

By the maximality property of transport rays, for any  $\mathcal{I} \in T[u]$ ,

$$(3) \quad \mathcal{I} \cap Loose[u] \neq \emptyset \quad \iff \quad \exists x \in Loose[u], \mathcal{I} = \{x\}.$$

From Lemma 2.1.3, any transport ray  $\mathcal{I} \in T[u]$  is a minimizing geodesic. The relative interior of  $\mathcal{I} \in T[u]$  is empty if and only if  $\mathcal{I}$  is a singleton. Recall from Lemma 2.1.11 that  $T^\circ[u]$  is the collection of all relative interiors of non-degenerate transport rays associated with  $u$ , while

$$(4) \quad Strain[u] = \bigcup_{\mathcal{I} \in T^\circ[u]} \mathcal{I}.$$

It follows from (3) and (4) that

$$Strain[u] \cap Loose[u] = \emptyset.$$

Finally, let us set  $Ends[u] = \mathcal{M} \setminus (Loose[u] \cup Strain[u])$ . Thus,  $Strain[u]$ ,  $Ends[u]$  and  $Loose[u]$  are three disjoint sets whose union equals  $\mathcal{M}$ .

LEMMA 4.2.  $\mu(Ends[u]) = \lambda_{\mathcal{M}}(Ends[u]) = 0$ .

PROOF. Recall from Section 3.1 that for a subset  $A \subseteq \mathcal{M}$ , we define  $Ends(A) \subseteq \mathcal{M}$  to be the union of all relative *boundaries* of transport rays intersecting  $A$ . We claim that

$$(5) \quad Ends[u] \subseteq Ends(Strain[u]).$$

Indeed, if  $x \in Ends[u]$ , then  $\{x\}$  is not a transport ray as  $x \notin Loose[u]$ . From Definition 2.1.2, there exists a non-degenerate transport ray  $\mathcal{I} \in T[u]$  that contains  $x$ . Since  $x \notin Strain[u]$ , then the point  $x \in \mathcal{I}$  does not belong to the relative interior of  $\mathcal{I}$ . Consequently,  $x$  belongs to the relative boundary of  $\mathcal{I}$ . Since the relative interior of  $\mathcal{I}$  is non-empty, then  $\mathcal{I} \cap Strain[u] \neq \emptyset$  and consequently  $x \in$

$Ends(Strain[u])$ . Thus (5) is proven. Next, according to Lemma 3.2.7, there exist ray clusters  $R_1, R_2, \dots$  such that  $Strain[u] = \cup_i R_i$ . Hence,

$$(6) \quad Ends(Strain[u]) = \bigcup_{i=1}^{\infty} Ends(R_i).$$

However, Lemma 3.1.8 asserts that  $\lambda_{\mathcal{M}}(Ends(R_i)) = 0$  for any  $i \geq 1$ . Consequently, from (5) and (6),

$$\lambda_{\mathcal{M}}(Ends[u]) = 0.$$

Since  $\mu$  is absolutely-continuous with respect to  $\lambda_{\mathcal{M}}$ , the lemma is proven.  $\square$

The following lemma, just like our entire proof of conclusion (B), is similar to the mass balance lemma of Evans and Gangbo [21, Lemma 5.1]. For a set  $K$  we write  $1_K$  for the function that equals one on  $K$  and vanishes elsewhere.

LEMMA 4.3. *Let  $K \subseteq \mathcal{M}$  be a compact set. For  $\delta > 0$  denote*

$$(7) \quad u_{\delta}(x) = \inf_{y \in \mathcal{M}} [u(y) + d(x, y) - \delta \cdot 1_K(y)] \quad \text{for } x \in \mathcal{M}.$$

*Let  $A \subseteq \mathcal{M}$  be the union of all transport rays  $\mathcal{I} \in T[u]$  that intersect  $K$ . Then there exists a function  $v : \mathcal{M} \rightarrow [0, 1]$  such that*

$$(8) \quad \lim_{\delta \rightarrow 0^+} \frac{u(x) - u_{\delta}(x)}{\delta} = \begin{cases} 0 & x \in \mathcal{M} \setminus A \\ v(x) & x \in A \setminus K \\ 1 & x \in K \end{cases}$$

*Moreover, for any  $x \in \mathcal{M}$  and  $\delta > 0$  we have that  $0 \leq u(x) - u_{\delta}(x) \leq \delta$ .*

PROOF. Since  $\|u\|_{Lip} \leq 1$  then for all  $x \in \mathcal{M}$ ,

$$(9) \quad u_{\delta}(x) = \inf_{y \in \mathcal{M}} [u(y) + d(x, y) - \delta \cdot 1_K(y)] \geq \inf_{y \in \mathcal{M}} [u(y) + d(x, y)] - \delta \geq u(x) - \delta.$$

The ‘‘Moreover’’ part of the lemma follows from (9) and from the simple inequality  $u_{\delta}(x) \leq u(x)$ . For any  $x, y \in \mathcal{M}$  we have that  $u(x) - u(y) - d(x, y) \leq 0$  as  $u$  is 1-Lipschitz. Therefore, for any  $x \in \mathcal{M}$ , the function

$$\delta \mapsto \frac{u(x) - u_{\delta}(x)}{\delta} = \sup_{y \in \mathcal{M}} \left[ \frac{u(x) - u(y) - d(x, y)}{\delta} + 1_K(y) \right]$$

is non-decreasing in  $\delta > 0$ . Hence the limit in (8) exists and belongs to  $[0, 1]$  for all  $x \in \mathcal{M}$ . Next, fix a point  $x \in \mathcal{M} \setminus A$ . Then for any  $y \in K$ , the points  $x$  and  $y$  do not belong to the same transport ray. Therefore  $|u(x) - u(y)| < d(x, y)$  and hence  $u(y) + d(x, y) > u(x)$  for any  $y \in K$ . By the compactness of  $K$ , there exists  $\delta_x > 0$  such that

$$(10) \quad \inf_{y \in K} [u(y) + d(x, y)] = \min_{y \in K} [u(y) + d(x, y)] > u(x) + \delta_x.$$

Since  $u$  is 1-Lipschitz, then  $u(y) + d(x, y) \geq u(x)$  for all  $y \in \mathcal{M}$ . Consequently, from (7) and (10),

$$u_{\delta}(x) = u(x) \quad \text{when } 0 < \delta < \delta_x.$$

This proves (8) in the case where  $x \in \mathcal{M} \setminus A$ . Consider now the case where  $x \in K$ . Then,

$$(11) \quad u_{\delta}(x) = \inf_{y \in \mathcal{M}} [u(y) + d(x, y) - \delta \cdot 1_K(y)] \leq u(x) + d(x, x) - \delta = u(x) - \delta.$$

From (9) and (11) we learn that  $u_{\delta}(x) = u(x) - \delta$  for any  $x \in K$  and  $\delta > 0$ . This proves (8) for the case where  $x \in K$ .  $\square$

Following Evans and Gangbo [21, Lemma 5.1], we say that a measurable subset  $A \subseteq \mathcal{M}$  is a *transport set* associated with  $u$  if for any  $x \in A \setminus \text{Ends}[u]$  and  $\mathcal{I} \in T[u]$ ,

$$(12) \quad x \in \mathcal{I} \quad \implies \quad \mathcal{I} \subseteq A.$$

In other words, a transport set  $A$  is a measurable set that contains all transport rays intersecting  $A \setminus \text{Ends}[u]$ .

LEMMA 4.4. *Let  $A \subseteq \mathcal{M}$  be a transport set associated with  $u$ . Then,*

$$\int_A f d\mu \geq 0.$$

PROOF. It suffices to prove that  $\int_A f d\mu > -\varepsilon$  for any  $\varepsilon > 0$ . To this end, let us fix  $\varepsilon > 0$ . According to Lemma 4.2, the set  $\text{Ends}[u]$  is of  $\mu$ -measure zero. Therefore,

$$(13) \quad \int_{A \setminus \text{Ends}[u]} |f| d\mu = \int_A |f| d\mu < \infty.$$

Since  $\mu$  is a Radon measure, it follows from (13) that there exists a compact  $K \subseteq A \setminus \text{Ends}[u]$  such that

$$(14) \quad \int_{A \setminus K} |f| d\mu < \varepsilon.$$

For  $\delta > 0$  we define  $u_\delta : \mathcal{M} \rightarrow \mathbb{R}$  as in (7). Then  $u_\delta$  is a 1-Lipschitz function, since it is the infimum of a family of 1-Lipschitz functions. From (2),

$$(15) \quad \int_{\mathcal{M}} \frac{u - u_\delta}{\delta} \cdot f \cdot d\mu \geq 0 \quad \text{for all } \delta > 0.$$

For  $k = 1, 2, \dots$  denote

$$v_k(x) = \frac{u(x) - u_{1/k}(x)}{1/k} \quad (x \in \mathcal{M}).$$

From the ‘‘Moreover’’ part of Lemma 4.3 we know that  $0 \leq v_k(x) \leq 1$  for all  $x \in \mathcal{M}$  and  $k \geq 1$ . According to Lemma 4.3, there exists a function  $v : \mathcal{M} \rightarrow [0, 1]$  such that  $v_k(x) \rightarrow v(x)$  for all  $x \in \mathcal{M}$ . Furthermore, by (8),

$$(16) \quad v(x) = \begin{cases} 0 & x \in \mathcal{M} \setminus A \\ 1 & x \in K \end{cases}$$

where we used the fact that  $A$  is a transport set and hence  $A$  contains all transport rays intersecting  $K \subseteq A \setminus \text{Ends}[u]$ . Since  $f$  is  $\mu$ -integrable and  $|v_k(x)| \leq 1$  for all  $k$  and  $x$ , then we may use the dominated convergence theorem and conclude from (15) and (16) that

$$(17) \quad 0 \leq \int_{\mathcal{M}} v_k f d\mu \xrightarrow{k \rightarrow \infty} \int_{\mathcal{M}} v f d\mu = \int_A v f d\mu = \int_{A \setminus K} v f d\mu + \int_K f d\mu.$$

Since  $v(x) \in [0, 1]$  for all  $x \in \mathcal{M}$ , then according to (14) and (17),

$$\int_A f d\mu > \int_K f d\mu - \varepsilon \geq - \int_{A \setminus K} v f d\mu - \varepsilon \geq - \int_{A \setminus K} |f| d\mu - \varepsilon > -2\varepsilon,$$

and the lemma is proven.  $\square$

COROLLARY 4.5. *Let  $A \subseteq \mathcal{M}$  be a transport set associated with  $u$ . Then,*

$$\int_A f d\mu = 0.$$

PROOF. In view of Lemma 4.4 we only need to prove that  $\int_A f d\mu \leq 0$ . Note that the supremum of  $\int v(-f)d\mu$  over all 1-Lipschitz functions  $v$  is attained for  $v = -u$ . Furthermore,  $T[u] = T[-u]$  and  $Ends[u] = Ends[-u]$ . Therefore  $A$  is also a transport set associated with  $-u$ . We may therefore apply Lemma 4.4 with  $f$  replaced by  $-f$  and with  $u$  replaced by  $-u$ . By the conclusion of Lemma 4.4,  $\int_A (-f)d\mu \geq 0$ , and the corollary is proven.  $\square$

Recall that  $T^\circ[u]$  is a partition of  $Strain[u]$ , and that  $\pi : Strain[u] \rightarrow T^\circ[u]$  is the partition map, i.e.,  $x \in \pi(x) \in T^\circ[u]$  for all  $x \in Strain[u]$ .

LEMMA 4.6. *Let  $S \subseteq T^\circ[u]$ . Assume that  $\pi^{-1}(S) \subseteq Strain[u]$  is a measurable subset of  $\mathcal{M}$ . Then,*

$$\int_{\pi^{-1}(S)} f d\mu = 0.$$

PROOF. Recall that  $Strain[u]$ ,  $Loose[u]$  and  $Ends[u]$  are three disjoint sets whose union equals  $\mathcal{M}$ . In view of Lemma 4.2 and Corollary 4.5, it suffices to show that there exists a transport set  $A \subseteq \mathcal{M}$  with

$$(18) \quad \pi^{-1}(S) \subseteq A \quad \text{and} \quad A \setminus \pi^{-1}(S) \subseteq Ends[u].$$

Any  $\mathcal{J} \in T^\circ[u]$  is the relative interior of a non-degenerate transport ray. Since transport rays are closed sets, it follows from Lemma 2.1.3 that the closure  $\overline{\mathcal{J}}$  of any  $\mathcal{J} \in T^\circ[u]$  is a transport ray. We claim that for any  $\mathcal{J} \in T^\circ[u]$ ,

$$(19) \quad \overline{\mathcal{J}} \setminus \mathcal{J} \subseteq \mathcal{M} \setminus (Loose[u] \cup Strain[u]) = Ends[u].$$

Indeed, it follows from (3) that  $\overline{\mathcal{J}}$  is contained in  $\mathcal{M} \setminus Loose[u]$  since it is a transport ray whose relative interior is non-empty. Any point  $x \in \overline{\mathcal{J}}$  belonging to  $Strain[u]$  must lie in  $\mathcal{J}$ , according to Lemma 2.1.7. Hence  $\overline{\mathcal{J}} \setminus \mathcal{J}$  is disjoint from  $Strain[u]$ , and (19) is proven. Denote

$$(20) \quad A = \bigcup_{\mathcal{J} \in S} \overline{\mathcal{J}}.$$

Clearly  $A \supseteq \bigcup_{\mathcal{J} \in S} \mathcal{J} = \pi^{-1}(S)$ . It follows from (19) that

$$(21) \quad A \setminus \pi^{-1}(S) = \left\{ \bigcup_{\mathcal{J} \in S} \overline{\mathcal{J}} \right\} \setminus \left\{ \bigcup_{\mathcal{J} \in S} \mathcal{J} \right\} \subseteq \bigcup_{\mathcal{J} \in S} (\overline{\mathcal{J}} \setminus \mathcal{J}) \subseteq Ends[u].$$

Now (18) follows from (21) and from the fact that  $A \supseteq \bigcup_{\mathcal{J} \in S} \mathcal{J} = \pi^{-1}(S)$ . All that remains is to show that  $A \subseteq \mathcal{M}$  is a transport set. Since  $\pi^{-1}(S)$  is assumed to be measurable and  $Ends[u]$  is a null set, then the measurability of  $A$  follows from (18). In order to prove condition (12) and conclude that  $A$  is a transport set, we choose  $x \in A \setminus Ends[u]$  and  $\mathcal{I} \in T[u]$  with

$$(22) \quad x \in \mathcal{I}.$$

Since  $x \in A \setminus Ends[u]$ , then necessarily  $x \in \pi^{-1}(S) \subseteq Strain[u]$  according to (21). Denote by  $\mathcal{J}$  the relative interior of the transport ray  $\mathcal{I}$ . From (22) and Lemma



2.1.7 we deduce that  $\mathcal{I}$  is the unique transport ray containing  $x$ , and that  $x \in \mathcal{J}$ . Since  $x \in \pi^{-1}(S)$ , we learn that  $\mathcal{J} \in S$ . From (20) we conclude that  $\mathcal{I} = \overline{\mathcal{J}} \subseteq A$ . We have thus verified condition (12) and proved that  $A$  is a transport set associated with  $u$ . The lemma is proven.  $\square$

PROOF OF THEOREM 1.5(B). The set  $Strain[u]$  is measurable, as follows from Lemma 2.1.12. We would like to show that

$$(23) \quad f(x) = 0 \quad \text{for } \mu\text{-almost any point } x \in \mathcal{M} \setminus Strain[u].$$

We learn from (3) and from the definition (12) that any measurable set  $S \subseteq Loose[u]$  is a transport set associated with  $u$ . From Corollary 4.5, for any measurable set  $S \subseteq Loose[u]$ ,

$$\int_S f d\mu = 0.$$

This implies that  $f$  vanishes  $\mu$ -almost everywhere in  $Loose[u]$ . Recall that  $\mathcal{M} \setminus Strain[u] = Loose[u] \cup Ends[u]$ . In view of Lemma 4.2, we conclude (23).

Next, let  $\nu$  and  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$  be measures on  $T^\circ[u]$  and  $\mathcal{M}$ , respectively, satisfying conclusions (i), (ii) and (iii) of Theorem 1.4. Thus, for  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ , the measure  $\mu_{\mathcal{I}}$  is a  $CD(\kappa, N)$ -needle supported on  $\mathcal{I}$ . Additionally, for any measurable set  $A \subseteq \mathcal{M}$ ,

$$(24) \quad \mu(A \cap Strain[u]) = \int_{T^\circ[u]} \mu_{\mathcal{I}}(A) d\nu(\mathcal{I}),$$

and in particular, the map  $\mathcal{I} \mapsto \mu_{\mathcal{I}}(A)$  is  $\nu$ -measurable. It follows from (24) that for any  $\mu$ -integrable function  $g : \mathcal{M} \rightarrow \mathbb{R}$ ,

$$(25) \quad \int_{Strain[u]} g d\mu = \int_{T^\circ[u]} \left( \int_{\mathcal{I}} g(x) d\mu_{\mathcal{I}}(x) \right) d\nu(\mathcal{I}).$$

In order to complete the proof, we need to show that

$$(26) \quad \int_{\mathcal{I}} f d\mu_{\mathcal{I}} = 0 \quad \text{for } \nu\text{-almost any } \mathcal{I} \in T^\circ[u].$$

Since  $f$  is  $\mu$ -integrable, from (25) the map  $\mathcal{I} \mapsto \int_{\mathcal{I}} f d\mu_{\mathcal{I}}$  is  $\nu$ -integrable, and in particular, it is well-defined for  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ . The desired conclusion (26) would follow once we show that for any  $\nu$ -measurable subset  $S \subseteq T^\circ[u]$ ,

$$(27) \quad \int_S \left( \int_{\mathcal{I}} f d\mu_{\mathcal{I}} \right) d\nu(\mathcal{I}) = 0.$$

Thus, let us fix a  $\nu$ -measurable subset  $S \subseteq T^\circ[u]$ . From Theorem 1.4(i), the set  $\pi^{-1}(S)$  is a measurable subset of  $\mathcal{M}$ . According to Lemma 4.6,

$$(28) \quad 0 = \int_{\pi^{-1}(S)} f d\mu = \int_{Strain[u]} f(x) \cdot \mathbf{1}_{\pi^{-1}(S)}(x) d\mu(x).$$

By using (25) and (28),

$$0 = \int_{Strain[u]} f \cdot \mathbf{1}_{\pi^{-1}(S)} d\mu = \int_{T^\circ[u]} \mathbf{1}_S(\mathcal{I}) \cdot \left( \int_{\mathcal{I}} f d\mu_{\mathcal{I}} \right) d\nu(\mathcal{I}) = \int_S \left( \int_{\mathcal{I}} f d\mu_{\mathcal{I}} \right) d\nu(\mathcal{I}).$$

Recalling that  $S \subseteq T^\circ[u]$  was an arbitrary  $\nu$ -measurable set, we see that (27) is proven. The proof is complete.  $\square$

PROOF OF THEOREM 1.2. From Theorem 1.4, Theorem 1.5(A) and Theorem 1.5(B) we obtain a 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$ , a certain measure  $\nu$  on  $T^\circ[u]$  and a family of measures  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$  on the manifold  $\mathcal{M}$ . We make the following formal manipulations: Let  $\Omega$  be the partition of  $\mathcal{M}$  obtained by adding the singletons  $\{\{x\}; x \in \mathcal{M} \setminus \text{Strain}[u]\}$  to the partition  $T^\circ[u]$  of  $\text{Strain}[u]$ . Let  $\tilde{\nu}$  be the push-forward of  $\mu|_{\mathcal{M} \setminus \text{Strain}[u]}$  under the map  $x \mapsto \{x\}$  to the set  $\Omega$ . Define

$$\nu_1 = \nu + \tilde{\nu},$$

a measure on  $\Omega$ . Finally, for  $x \in \mathcal{M} \setminus \text{Strain}[u]$  write  $\mu_{\{x\}}$  for Dirac's delta measure at  $x$ . From Theorem 1.4, for any measurable subset  $A \subseteq \mathcal{M}$ ,

$$\begin{aligned} \mu(A) &= \mu(A \cap \text{Strain}[u]) + \mu(A \setminus \text{Strain}[u]) \\ &= \int_{T^\circ[u]} \mu_{\mathcal{I}}(A) d\nu(\mathcal{I}) + \int_{\mathcal{M} \setminus \text{Strain}[u]} \mu_{\{x\}}(A) d\mu(x) = \int_{\Omega} \mu_{\mathcal{I}}(A) d\nu_1(\mathcal{I}). \end{aligned}$$

Thus conclusion (i) holds true with  $\nu$  replaced by  $\nu_1$ . For  $\nu_1$ -almost any  $\mathcal{I} \in \Omega$ , we have that either  $\mathcal{I}$  is a singleton, or else  $\mathcal{I}$  is the relative interior of a transport ray on which the  $CD(\kappa, N)$ -needle  $\mu_{\mathcal{I}}$  is supported. We have thus verified conclusion (ii). Theorem 1.5(B) shows that  $f$  vanishes almost everywhere in  $\mathcal{M} \setminus \text{Strain}[u]$ . Conclusion (iii) thus follows from Theorem 1.5(B).  $\square$

PROOF OF THEOREM 1.5(C). This follows from Theorem 1.4(iii) and the previous proof.  $\square$

COROLLARY 4.7 (“Uniqueness of maximizer”). *Suppose that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional, geodesically-convex, weighted Riemannian manifold. Let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a  $\mu$ -integrable function with  $\int_{\mathcal{M}} f d\mu = 0$ , and assume that there exists  $x_0 \in \mathcal{M}$  with  $\int_{\mathcal{M}} d(x_0, x) |f(x)| d\mu(x) < +\infty$ . Assume furthermore that*

$$(29) \quad \mu(\{x \in \mathcal{M}; f(x) = 0\}) = 0.$$

Let  $u_1, u_2 : \mathcal{M} \rightarrow \mathbb{R}$  be 1-Lipschitz functions with

$$(30) \quad \int_{\mathcal{M}} u_1 f d\mu = \int_{\mathcal{M}} u_2 f d\mu = \sup \left\{ \int_{\mathcal{M}} u f d\mu; u : \mathcal{M} \rightarrow \mathbb{R}, \|u\|_{Lip} \leq 1 \right\}.$$

Then  $u_1 - u_2$  is a constant function.

PROOF. A 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  for which the supremum in (30) is attained is called here a *maximizer*. According to (29) and Theorem 1.5(B), the set  $\mathcal{M} \setminus \text{Strain}[u]$  is a Lebesgue-null set for any maximizer  $u$ . From Lemma 2.1.6 we deduce that for any maximizer  $u : \mathcal{M} \rightarrow \mathbb{R}$ ,

$$|\nabla u(x)| = 1 \quad \text{for almost any } x \in \mathcal{M}.$$

Suppose now that  $u_1$  and  $u_2$  are two maximizers. Then also  $(u_1 + u_2)/2$  is a maximizer. Therefore for almost any  $x \in \mathcal{M}$ ,

$$|\nabla u_1(x)| = |\nabla u_2(x)| = \left| \frac{\nabla u_1(x) + \nabla u_2(x)}{2} \right| = 1.$$

Consequently  $\nabla u_1 = \nabla u_2$  almost everywhere, and hence  $u_1 - u_2 \equiv \text{Const}$ .  $\square$

The  $CD(\kappa, N)$  curvature-dimension condition was used in our argument only in order to deduce that Ricci needles are  $CD(\kappa, N)$ -needles. The “Ricci needle” variant of Theorem 1.4 is rendered as Theorem 3.3.8 above. Next we formulate the corresponding variant of Theorem 1.5:

THEOREM 4.8. *Let  $n \geq 2$  and assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold which is geodesically-convex. Let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a  $\mu$ -integrable function with  $\int_{\mathcal{M}} f d\mu = 0$ . Assume that there exists a point  $x_0 \in \mathcal{M}$  with  $\int_{\mathcal{M}} |f(x)| \cdot d(x_0, x) d\mu(x) < \infty$ . Then,*

(A) *There exists a 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  such that*

$$\int_{\mathcal{M}} u f d\mu = \sup_{\|v\|_{Lip} \leq 1} \int_{\mathcal{M}} v f d\mu.$$

(B) *For any such function  $u$ , the function  $f$  vanishes  $\mu$ -almost everywhere in  $\mathcal{M} \setminus \text{Strain}[u]$ . Furthermore, let  $\nu$  and  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$  be measures on  $T^\circ[u]$  and  $\mathcal{M}$ , respectively, satisfying conclusions (i), (ii) and (iii) of Theorem 3.3.8. Then for  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ ,*

$$\int_{\mathcal{I}} f d\mu_{\mathcal{I}} = 0.$$

The proof of Theorem 4.8 is almost identical to the proof of Theorem 1.5. The only difference is that one needs to appeal to Theorem 3.3.8 rather than to Theorem 1.4 rather than, and to replace the words “ $CD(\kappa, N)$ -needle” by “Ricci needle” throughout the proof.

REMARK 4.9. Similarly, Theorem 1.2 and Theorem 1.3 remain valid without the  $CD(\kappa, N)$ -assumption, yet one has to replace the words “ $CD(\kappa, N)$ -needle” by “Ricci needle”.

## Some applications

One-dimensional log-concave needles are quite well-understood. Theorem 1.2 allows us to reduce certain questions pertaining to Riemannian manifolds whose Ricci curvature is non-negative, to analogous questions for one-dimensional log-concave needles.

### 5.1. The inequalities of Buser, Ledoux and E. Milman

Let  $\mathcal{M}$  be a Riemannian manifold with distance function  $d$ . For a subset  $S \subseteq \mathcal{M}$  and  $\varepsilon > 0$  denote

$$S_\varepsilon = \left\{ x \in \mathcal{M}; \inf_{y \in S} d(x, y) < \varepsilon \right\},$$

the  $\varepsilon$ -neighborhood of the set  $S$ . The next proposition was proven by E. Milman [38], improving upon earlier results by Buser [12] and by Ledoux [33]:

**PROPOSITION 5.1.1.** *Let  $n \geq 2, R > 0$ . Assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold of class  $CD(0, \infty)$  which is geodesically-convex with  $\mu(\mathcal{M}) = 1$ . Assume that for any 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$ ,*

$$(1) \quad \inf_{\alpha \in \mathbb{R}} \int_{\mathcal{M}} |u(x) - \alpha| d\mu(x) < R.$$

*Then for any measurable set  $S \subseteq \mathcal{M}$  and  $0 < \varepsilon < R$ ,*

$$\mu(S_\varepsilon \setminus S) \geq c \cdot \frac{\varepsilon}{R} \cdot \mu(S) \cdot (1 - \mu(S)),$$

*where  $c > 0$  is a universal constant.*

It is well-known that the optimal choice of  $\alpha$  in (1) is the median of the function  $u$ . The expectation  $E = \int_{\mathcal{M}} u d\mu$  is also a reasonable choice for the parameter  $\alpha$ , since  $\int_{\mathcal{M}} |u - E| d\mu$  is at most twice as large as the actual infimum in (1). We begin the proof of Proposition 5.1.1 with the following standard estimate from the theory of one-dimensional log-concave measures:

**LEMMA 5.1.2.** *Let  $R > 0$ , let  $A \subseteq \mathbb{R}$  be a non-empty, open interval, let  $\Psi : A \rightarrow \mathbb{R}$  be a convex function with  $\int_A e^{-\Psi} < \infty$ , and let  $\eta$  be the measure supported on  $A$  whose density is  $e^{-\Psi}$ . Suppose that  $R = \int_A |t| d\eta(t) / \eta(\mathbb{R})$ . Then for any  $0 < t < 1, 0 < \varepsilon < 2R$  and a measurable subset  $S \subseteq \mathbb{R}$ ,*

$$(2) \quad \eta(S) = t \cdot \eta(\mathbb{R}) \quad \implies \quad \eta(S_\varepsilon \setminus S) \geq c \cdot \frac{\varepsilon}{R} \cdot t(1-t) \cdot \eta(\mathbb{R}),$$

*where  $c > 0$  is a universal constant.*

**PROOF.** We may add a constant to  $\Psi$  and stipulate that  $\eta(\mathbb{R}) = 1$ . We may rescale and assume furthermore that  $R = \int_A |t| d\eta(t) = 1$ . According to Bobkov [6, Proposition 2.1], it suffices to prove (2) under the additional assumption that  $S$  is

a half-line in  $\mathbb{R}$  with  $\eta(S) = t$ . Reflecting  $\Psi$  if necessary, we may suppose that  $S$  takes the form  $S = (-\infty, a)$  for some  $a \in A$ . Furthermore, we may assume that

$$(3) \quad \eta((a, a + \varepsilon)) \leq \min\{t, 1 - t\}/2.$$

Indeed, if (3) fails then  $\eta(S_\varepsilon \setminus S) = \eta((a, a + \varepsilon)) \geq (\varepsilon/R) \cdot t(1 - t)/4$  and (2) holds true. For  $x \in \mathbb{R}$  and  $0 < s < 1$  denote

$$\Phi(x) = \int_{-\infty}^x e^{-\Psi}, \quad I(s) = \exp(-\Psi(\Phi^{-1}(s))).$$

Since  $\Psi$  is convex, then  $I : (0, 1) \rightarrow (0, \infty)$  is a well-defined concave function according to Bobkov [7, Lemma 3.2]. Furthermore, since  $\int_A |t| d\eta(t) = 1$  then  $I(1/2) \geq c$  where  $c > 0$  is a universal constant, as is shown in [7, Section 3]. Therefore, by the concavity of the positive function  $I : (0, 1) \rightarrow \mathbb{R}$ ,

$$(4) \quad I(t) \geq 2c \cdot \min\{t, 1 - t\} \quad \text{for all } 0 < t < 1.$$

Note that  $a \in A$  as  $0 < \eta((-\infty, a)) < \eta(\mathbb{R})$ . Similarly, from (3) we learn that  $a + \varepsilon \in A$ . Consequently  $(a, a + \varepsilon) \subseteq A$ . It now follows from (3) and (4) that

$$\eta((a, a + \varepsilon)) \geq \varepsilon \cdot \inf_{x \in (a, a + \varepsilon)} e^{-\Psi(x)} \geq \varepsilon \cdot \inf_{s \in [t, t + \min\{t, 1 - t\}/2]} I(s) \geq \varepsilon \cdot c \cdot \min\{t, 1 - t\},$$

and (2) is proven.  $\square$

PROOF OF PROPOSITION 5.1.1. Denote  $t = \mu(S) \in [0, 1]$ . We may assume that  $t \in (0, 1)$ , as otherwise there is nothing to prove. Set  $f(x) = 1_S(x) - t$  for  $x \in \mathcal{M}$ . Then  $\int_{\mathcal{M}} f d\mu = 0$ , and certainly for any  $x_0 \in \mathcal{M}$ ,

$$\int_{\mathcal{M}} |f(x)| \cdot d(x_0, x) d\mu(x) \leq |t + 1| \cdot \int_{\mathcal{M}} d(x_0, x) d\mu(x) < \infty,$$

where the integrability of the 1-Lipschitz function  $x \mapsto d(x_0, x)$  follows from (1). Applying Theorem 1.5, we obtain a certain 1-Lipschitz function  $u : \mathcal{M} \rightarrow \mathbb{R}$  and measures  $\nu$  and  $\{\mu_{\mathcal{I}}\}_{\mathcal{I} \in T^\circ[u]}$  on  $T^\circ[u]$  and  $\mathcal{M}$  respectively. It follows from (1) that after adding an appropriate constant to the 1-Lipschitz function  $u$ , we have

$$(5) \quad \int_{\mathcal{M}} |u| d\mu \leq R.$$

For  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$  we know that  $\int_{\mathcal{I}} f d\mu_{\mathcal{I}} = 0$ . Consequently, for  $\nu$ -almost any  $\mathcal{I} \in T^\circ[u]$ ,

$$(6) \quad \mu_{\mathcal{I}}(S) = t \cdot \mu_{\mathcal{I}}(\mathcal{M}) < \infty.$$

From Theorem 1.5(B), the function  $f$  vanishes  $\mu$ -almost everywhere outside  $Strain[u]$ , but our function  $f(x) = 1_S(x) - t$  never vanishes in  $\mathcal{M}$ . Hence  $Strain[u]$  is a set of a full  $\mu$ -measure. From Theorem 1.4(ii) and from (5) we thus obtain that

$$(7) \quad \int_{T^\circ[u]} \left( \int_{\mathcal{I}} |u| d\mu_{\mathcal{I}} \right) d\nu(\mathcal{I}) = \int_{Strain[u]} |u| d\mu = \int_{\mathcal{M}} |u| d\mu \leq R.$$

Denote

$$(8) \quad B = \left\{ \mathcal{I} \in T^\circ[u]; \int_{\mathcal{I}} |u| d\mu_{\mathcal{I}} \leq 2R \cdot \mu_{\mathcal{I}}(\mathcal{M}) \right\}.$$

Since  $\mu(\mathcal{M}) = \mu(\text{Strain}[u]) = 1$  then  $\int_{T^\circ[u]} \mu_{\mathcal{I}}(\mathcal{M}) d\nu(\mathcal{I}) = 1$ . From (7) and the Markov-Chebyshev inequality,

$$(9) \quad \int_B \mu_{\mathcal{I}}(\mathcal{M}) d\nu(\mathcal{I}) \geq \frac{1}{2}.$$

Furthermore,  $\mu_{\mathcal{I}}$  is a log-concave needle (i.e., a  $CD(0, \infty)$ -needle) for  $\nu$ -almost any  $\mathcal{I} \in B$ . We would like to show that for  $\nu$ -almost any  $\mathcal{I} \in B$  and any  $0 < \varepsilon < R$ ,

$$(10) \quad \mu_{\mathcal{I}}(S_\varepsilon \setminus S) \geq c \cdot \frac{\varepsilon}{R} \cdot t(1-t) \cdot \mu_{\mathcal{I}}(\mathcal{M}),$$

for a universal constant  $c > 0$ . Let us fix  $\mathcal{I} \in B$  such that  $\mu_{\mathcal{I}}$  is a log-concave needle for which (6) holds true. Let  $A \subseteq \mathbb{R}$ ,  $\Psi : A \rightarrow \mathbb{R}$  and  $\gamma : A \rightarrow \mathcal{M}$  be as in Definition 1.1. Then  $A \subseteq \mathbb{R}$  is a non-empty, open, interval and  $\Psi : A \rightarrow \mathbb{R}$  is smooth and convex. From Theorem 1.4(iii) we know that  $\mathcal{I} = \gamma(A)$  and

$$(11) \quad u(\gamma(t)) = t \quad \text{for all } t \in A.$$

Since  $\mathcal{I} \in B$ , we may apply Lemma 5.1.2 thanks to (6), (8) and (11). The conclusion of Lemma 5.1.2 implies (10). Consequently, for any  $0 < \varepsilon < R$ ,

$$\begin{aligned} \mu(S_\varepsilon \setminus S) &= \int_{T^\circ[u]} \mu_{\mathcal{I}}(S_\varepsilon \setminus S) d\nu(\mathcal{I}) \geq \int_B \mu_{\mathcal{I}}(S_\varepsilon \setminus S) d\nu(\mathcal{I}) \\ &\geq c \cdot \frac{\varepsilon}{R} \cdot t(1-t) \cdot \int_B \mu_{\mathcal{I}}(\mathcal{M}) d\nu(\mathcal{I}). \end{aligned}$$

The proposition now follows from (9).  $\square$

Proposition 5.1.1 is stated and proved in the particular case where  $\kappa = 0$  and  $N = \infty$ . In general, for values of  $\kappa$  and  $N$  for which there is an appropriate  $CD(\kappa, N)$ -variant of the one-dimensional Lemma 5.1.2, we also have a  $CD(\kappa, N)$ -variant of the  $n$ -dimensional Proposition 5.1.1.

## 5.2. A Poincaré inequality for geodesically-convex domains

For  $\kappa \in \mathbb{R}$ ,  $1 \neq N \in \mathbb{R} \cup \{+\infty\}$  and  $D \in (0, +\infty)$  write  $\mathcal{F}_{\kappa, N, D}$  for the collection of all measures  $\nu$  supported on the interval  $(0, D) \subseteq \mathbb{R}$  which are  $CD(\kappa, N)$ -needles. According to Definition 1.1, a measure  $\nu$  belongs to  $\mathcal{F}_{\kappa, N, D}$  if and only if  $\nu$  is supported on a non-empty, open interval  $A \subseteq (0, D)$  with density  $e^{-\Psi}$ , where  $\Psi : A \rightarrow \mathbb{R}$  is a smooth function that satisfies

$$(1) \quad \Psi'' \geq \kappa + \frac{(\Psi')^2}{N-1}.$$

The term  $(\Psi')^2/(N-1)$  in (1) is interpreted as zero when  $N = +\infty$ . In order to include the case  $D = +\infty$ , we write  $\mathcal{F}_{\kappa, N, +\infty}$  for the collection of all measures  $\nu$  on  $\mathbb{R}$  which are  $CD(\kappa, N)$ -needles. Define

$$\lambda_{\kappa, N, D} = \inf \left\{ \frac{\int_{\mathbb{R}} |u'|^2 d\nu}{\int_{\mathbb{R}} u^2 d\nu}; \nu \in \mathcal{F}_{\kappa, N, D}, u \in C^1 \cap L^{1 \cap 2}(\nu), \int_{\mathbb{R}} u d\nu = 0, \int_{\mathbb{R}} u^2 d\nu > 0 \right\}$$

where  $L^{1 \cap 2}(\nu)$  is an abbreviation for  $L^1(\nu) \cap L^2(\nu)$ . There are some cases where  $\lambda_{\kappa, N, D}$  may be computed explicitly. For example, for  $N \in (-\infty, -1] \cup (1, +\infty]$ , the simple one-dimensional lemma of Payne and Weinberger [41] shows that

$$(2) \quad \lambda_{0, N, D} = \frac{\pi^2}{D^2}.$$

We refer the reader to Bakry and Qian [3] and references therein for variants of the following proposition:

**PROPOSITION 5.2.1.** *Let  $n \geq 2, \kappa \in \mathbb{R}$  and  $N \in (-\infty, 1) \cup [n, +\infty]$ . Assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold of class  $CD(\kappa, N)$  which is geodesically-convex. Denote*

$$D = \text{Diam}(\mathcal{M}) = \sup_{x, y \in \mathcal{M}} d(x, y) \in (0, +\infty].$$

Then for any  $C^1$ -function  $f : \mathcal{M} \rightarrow \mathbb{R}$  with  $f \in L^1(\mu) \cap L^2(\mu)$ ,

$$(3) \quad \int_{\mathcal{M}} f d\mu = 0 \quad \implies \quad \lambda_{\kappa, N, D} \cdot \int_{\mathcal{M}} f^2 d\mu \leq \int_{\mathcal{M}} |\nabla f|^2 d\mu.$$

**PROOF.** Let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a  $C^1$ -function with  $f \in L^{1 \cap 2}(\mu)$  and  $\int_{\mathcal{M}} f d\mu = 0$ . Applying Theorem 1.2, we see that (3) would follow from the following inequality: for any measure  $\nu$  on  $\mathcal{M}$  which is a  $CD(\kappa, N)$ -needle,

$$(4) \quad \left[ f \in L^{1 \cap 2}(\nu) \text{ and } \int_{\mathcal{M}} f d\nu = 0 \right] \implies \lambda_{\kappa, N, D} \cdot \int_{\mathcal{M}} f^2 d\nu \leq \int_{\mathcal{M}} |\nabla f|^2 d\nu.$$

Thus, let us fix a  $CD(\kappa, N)$ -needle  $\nu$  for which  $f \in L^{1 \cap 2}(\nu)$  and  $\int_{\mathcal{M}} f d\nu = 0$ . Let  $A \subseteq \mathbb{R}, \Psi : A \rightarrow \mathbb{R}$  and  $\gamma : A \rightarrow \mathcal{M}$  be as in Definition 1.1. Denoting  $g = f \circ \gamma$ , we see that

$$|g'(t)| \leq |\nabla f(\gamma(t))| \quad \text{for } t \in A,$$

as  $\gamma$  is a unit speed geodesic. Hence (4) would follow from the inequality

$$(5) \quad \int_A g e^{-\Psi} = 0 \quad \implies \quad \lambda_{\kappa, N, D} \cdot \int_A g^2 e^{-\Psi} \leq \int_A (g')^2 e^{-\Psi},$$

where  $g : A \rightarrow \mathbb{R}$  is a  $C^1$ -function with  $\int_A (|g| + g^2) e^{-\Psi} < \infty$ . The set  $A$  is open and connected, and since  $\gamma : A \rightarrow \mathcal{M}$  is a minimizing geodesic then  $A$  is an open interval whose length is at most  $D$ . The smooth function  $\Psi : A \rightarrow \mathbb{R}$  satisfies (1), and the desired inequality (5) holds in view of the definition of  $\lambda_{\kappa, N, D}$ . This completes the proof.  $\square$

The case  $\kappa = 0, N = \infty$  of Proposition 5.2.1, with the constant  $\lambda_{0, \infty, D}$  given by (2), goes back to Payne-Weinberger [41] in the Euclidean case, and to Li-Yau [34], Yang-Zhong [47] and Yang [46] in the Riemannian case.

### 5.3. The isoperimetric inequality and its relatives

Recall the definition of  $\mathcal{F}_{\kappa, N, D}$  from the previous section. Recall that  $A_\varepsilon$  stands for the  $\varepsilon$ -neighborhood of the set  $A$ . For  $0 < t < 1$  and  $\varepsilon > 0$  define

$$(1) \quad I_{\kappa, N, D}(t, \varepsilon) = \inf \{ \nu(A_\varepsilon) ; \nu \in \mathcal{F}_{\kappa, N, D}, A \subseteq \mathbb{R}, \nu(\mathbb{R}) = 1, \nu(A) = t \}.$$

That is,  $I_{\kappa, N, D}(t, \varepsilon)$  is the infimal measure of an  $\varepsilon$ -neighborhood of a subset of measure  $t$ . There are cases where the function  $I_{\kappa, N, D}$  may be computed explicitly. For example, when  $\kappa > 0, N = D = \infty$ , the infimum in (1) is attained when  $A$  is a half-line and  $\nu$  is a Gaussian measure on the real line of variance  $1/\kappa$ . See E. Milman [37] and references therein for more information about the function  $I_{\kappa, N, D}$ .

PROPOSITION 5.3.1. *Let  $n \geq 2$ ,  $\kappa \in \mathbb{R}$  and  $N \in (-\infty, 1) \cup [n, +\infty]$ . Assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold of class  $CD(\kappa, N)$  which is geodesically-convex. Assume that  $\mu(\mathcal{M}) = 1$ . Denote  $D = \text{Diam}(\mathcal{M})$ , the diameter of  $\mathcal{M}$ . Then for any measurable set  $A \subseteq \mathcal{M}$  and  $\varepsilon > 0$ , denoting  $t = \mu(A)$ ,*

$$\mu(A_\varepsilon) \geq I_{\kappa, N, D}(t, \varepsilon).$$

PROOF. Denote  $f(x) = 1_A(x) - t$ . Then  $\int_{\mathcal{M}} f d\mu = 0$ . The proposition follows by applying Theorem 1.2 and arguing similarly to the proof of Proposition 5.2.1.  $\square$

Similarly, one may reduce the proof of log-Sobolev or transportation-cost inequalities to the one-dimensional case by using Theorem 1.2, as well as the proof of the inequalities of Cordero-Erausquin, McCann and Schmuckenschläger [17, 18]. These inequalities are sometimes stated under the additional assumption that  $\mathcal{M}$  is a complete Riemannian manifold, or that  $\mu$  is the Riemannian volume measure. The localization technique shows that these additional assumptions are superfluous. Let us also observe that by using Theorem 4.8, it is straightforward to reduce the proof of the Brascamp-Lieb inequality [10] to the one-dimensional case.

It was suggested to us by S. Bobkov to consider *dilations* of subsets of a Riemannian manifold. That is, let  $n \geq 2$  and  $N \in (-\infty, 0) \cup [n, +\infty]$ , and assume that  $(\mathcal{M}, d, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold of class  $CD(0, N)$  which is geodesically-convex. Assume furthermore that  $\mu(\mathcal{M}) = 1$ . Following Nazarov, Sodin and Volberg [40], Bobkov and Nazarov [8] and Fradelizi [23], we make the following definition: For a Borel subset  $A \subseteq \mathcal{M}$  with  $\mu(A) > 0$  and  $0 < \varepsilon < 1$  define the set  $\mathcal{N}_\varepsilon(A)$  to be the collection of all  $x \in \mathcal{M}$  for which there exists a minimizing geodesic  $\gamma : [a, b] \rightarrow \mathcal{M}$  with  $\gamma(a) = x$  such that

$$\lambda_1(\{t \in [a, b]; \gamma(t) \in A\}) \geq (1 - \varepsilon) \cdot (b - a),$$

where  $\lambda_1$  is the Lebesgue measure in the interval  $[a, b] \subseteq \mathbb{R}$ . We think of  $\mathcal{N}_\varepsilon(A)$  as a certain type of an  $\varepsilon$ -neighborhood of the set  $A$ . A direct generalization to the Riemannian setting of the proofs in [8] and [23] shows that

$$(2) \quad 1 - \mu(A) \geq \left[ (1 - \varepsilon) \cdot \mu(\mathcal{M} \setminus \mathcal{N}_\varepsilon(A))^{1/N} + \varepsilon \right]^N,$$

where in the case  $N = +\infty$ , the right-hand side of (2) is interpreted by continuity as  $\mu(\mathcal{M} \setminus \mathcal{N}_\varepsilon(A))^{1-\varepsilon}$ . Additionally, let us mention here the cross-ratio isoperimetric inequality of Lovász [35], which has a similar nature to the dilation inequality above, and also admits a generalization to geodesically-convex Riemannian manifolds of non-negative Ricci curvature. Perhaps one may devise an algorithm for estimating the volume of such manifolds, in a certain computational model of interest? We will end this section with a proof of the *four functions theorem*, rendered as Theorem 1.3 above.

PROOF OF THEOREM 1.3. By approximation, we may assume that the function  $f_3 : \mathcal{M} \rightarrow [0, +\infty)$  does not vanish in  $\mathcal{M}$  (for example, replace  $f_3$  by  $f_3 + \varepsilon g$  where  $g$  is a positive function with suitable integrability properties, and then let  $\varepsilon$  tend to zero). We claim that for any  $CD(\kappa, N)$ -needle  $\eta$  on the Riemannian



manifold  $\mathcal{M}$  for which  $f_1, f_2, f_3, f_4 \in L^1(\eta)$ ,

$$(3) \quad \left( \int_{\mathcal{M}} f_1 d\eta \right)^\alpha \left( \int_{\mathcal{M}} f_2 d\eta \right)^\beta \leq \left( \int_{\mathcal{M}} f_3 d\eta \right)^\alpha \left( \int_{\mathcal{M}} f_4 d\eta \right)^\beta.$$

Indeed, inequality (3) appears in the assumptions of the theorem, but under the additional assumption that the measure  $\eta$  is a probability measure. By homogeneity, (3) holds true under the additional assumption that  $\eta$  is a finite measure. In the general case, we may select a sequence of finite  $CD(\kappa, N)$ -needles  $\eta_\ell$  such that  $\eta_\ell \nearrow \eta$ , and use the monotone convergence theorem. Thus (3) is proven.

Next, denote  $\lambda = \int_{\mathcal{M}} f_1 d\mu / \int_{\mathcal{M}} f_3 d\mu$ , define  $f = f_1 - \lambda f_3$ , and apply Theorem 1.2. Let  $\Omega, \{\mu_{\mathcal{I}}\}_{\mathcal{I} \in \Omega}, \nu$  be as in Theorem 1.2. Then for  $\nu$ -almost any  $\mathcal{I} \in \Omega$  we have that  $f_1, f_2, f_3, f_4 \in L^1(\mu_{\mathcal{I}})$  and

$$(4) \quad \left( \int_{\mathcal{I}} f_1 d\mu_{\mathcal{I}} \right)^\alpha \left( \int_{\mathcal{I}} f_2 d\mu_{\mathcal{I}} \right)^\beta \leq \left( \int_{\mathcal{I}} f_3 d\mu_{\mathcal{I}} \right)^\alpha \left( \int_{\mathcal{I}} f_4 d\mu_{\mathcal{I}} \right)^\beta$$

as follows from (3) and from the pointwise inequality  $f_1^\alpha f_2^\beta \leq f_3^\alpha f_4^\beta$  that holds almost-everywhere in  $\mathcal{M}$ . However,  $\int_{\mathcal{I}} f_1 d\mu_{\mathcal{I}} = \lambda \int_{\mathcal{I}} f_3 d\mu_{\mathcal{I}}$  for  $\nu$ -almost any  $\mathcal{I} \in \Omega$ . Thus (4) implies that for  $\nu$ -almost any  $\mathcal{I} \in \Omega$ ,

$$(5) \quad \lambda^{\alpha/\beta} \int_{\mathcal{I}} f_2 d\mu_{\mathcal{I}} \leq \int_{\mathcal{I}} f_4 d\mu_{\mathcal{I}}.$$

Integrating (5) with respect to the measure  $\nu$  yields

$$\lambda^{\alpha/\beta} \int_{\mathcal{M}} f_2 d\mu = \lambda^{\alpha/\beta} \int_{\Omega} \left( \int_{\mathcal{I}} f_2 d\mu_{\mathcal{I}} \right) d\nu(\mathcal{I}) \leq \int_{\Omega} \left( \int_{\mathcal{I}} f_4 d\mu_{\mathcal{I}} \right) d\nu(\mathcal{I}) = \int_{\mathcal{M}} f_4 d\mu.$$

From the definition of  $\lambda$  we thus obtain

$$\left( \int_{\mathcal{M}} f_1 d\mu \right)^\alpha \left( \int_{\mathcal{M}} f_2 d\mu \right)^\beta \leq \left( \int_{\mathcal{M}} f_3 d\mu \right)^\alpha \left( \int_{\mathcal{M}} f_4 d\mu \right)^\beta,$$

and the theorem is proven.  $\square$

## CHAPTER 6

### Further research

This section contains ideas and conjectures for possible extensions of the results in this article. First, we conjecture that the results and the arguments presented above may be generalized to the case of a smooth Finsler manifold. Another interesting generalization involves *several constraints*. That is, suppose that we are given a weighted Riemannian manifold  $(\mathcal{M}, d, \mu)$  and a  $\mu$ -integrable function  $f : \mathcal{M} \rightarrow \mathbb{R}^k$  with

$$\int_{\mathcal{M}} f d\mu = 0.$$

We would like to understand whether the measure  $\mu$  may be decomposed into  $k$ -dimensional pieces in a way analogous to Theorem 1.2.

**DEFINITION 6.1.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be geodesically-convex Riemannian manifolds. We declare that “ $\mathcal{M} \rightarrow \mathcal{N}$  has the isometric extension property” if for any subset  $A \subseteq \mathcal{M}$  and a distance-preserving map  $f : A \rightarrow \mathcal{N}$ , there exists a geodesically-convex subset  $B \subseteq \mathcal{M}$  containing  $A$  and an extension of  $f$  to a distance-preserving map  $f : B \rightarrow \mathcal{N}$ .*

Lemma 2.1.1 shows that  $\mathbb{R} \rightarrow \mathcal{M}$  has the isometric extension property whenever  $\mathcal{M}$  is a geodesically-convex Riemannian manifold. If  $\mathcal{M} \subseteq \mathbb{R}^n$  is a convex set then for any  $k \leq n$ ,

$$\mathbb{R}^k \rightarrow \mathcal{M}$$

has the isometric extension property. Also  $S^k \rightarrow S^n$  has the isometric extension property, as well as  $S^k \rightarrow \mathcal{M}$  when  $\mathcal{M}$  is a geodesically-convex subset of the sphere  $S^n$ . These facts have direct proofs which do not rely on the Kirszbraun theorem. By considering small triangles, one may show the following: Only when  $\mathcal{M}$  and  $\mathcal{N}$  are Riemannian manifolds of constant sectional curvature, which is the same for  $\mathcal{M}$  and  $\mathcal{N}$ , it might be possible that  $\mathcal{M} \rightarrow \mathcal{N}$  has the isometric extension property.

Let us discuss in greater detail the case where  $\mathcal{M} \subseteq \mathbb{R}^n$  is an open, convex set. Suppose that  $u : \mathcal{M} \rightarrow \mathbb{R}^k$  is a 1-Lipschitz map. We may generalize Definition 2.1.2 as follows: A subset  $\mathcal{S} \subseteq \mathcal{M}$  is a *leaf* associated with  $u$  if

$$|u(x) - u(y)| = |x - y| \quad \text{for all } x, y \in \mathcal{S},$$

and if for any  $\mathcal{S}_1 \supsetneq \mathcal{S}$  there exist  $x, y \in \mathcal{S}_1$  with  $|u(x) - u(y)| < |x - y|$ . For any leaf  $\mathcal{S} \subseteq \mathcal{M}$ , the set

$$u(\mathcal{S}) = \{u(x); x \in \mathcal{S}\}$$

is a closed, convex subset of  $\mathbb{R}^k$ . This follows from the isometric extension property of  $\mathbb{R}^k \rightarrow \mathcal{M}$ . Let us define  $Strain[u]$  to be the union of all relative interiors of leaves. Write  $T^\circ[u]$  for the collection of all non-empty relative interiors of leaves. Suppose

that  $\mu$  is a measure on the convex set  $\mathcal{M} \subseteq \mathbb{R}^n$  such that  $(\mathcal{M}, |\cdot|, \mu)$  is an  $n$ -dimensional weighted Riemannian manifold of class  $CD(\kappa, N)$ . We conjecture that there exists a measure  $\nu$  on  $T^\circ[u]$  and a family of measures  $\{\mu_{\mathcal{S}}\}_{\mathcal{S} \in T^\circ[u]}$  such that

$$\mu(A \cap \text{Strain}[u]) = \int_{T^\circ[u]} \mu_{\mathcal{S}}(A) d\nu(\mathcal{S}) \quad \text{for any measurable } A \subseteq \mathcal{M}.$$

Additionally, for  $\nu$ -almost any  $\mathcal{S} \in T^\circ[u]$ , the measure  $\mu_{\mathcal{S}}$  is supported on  $\mathcal{S}$  and

$$(\mathcal{S}, |\cdot|, \mu_{\mathcal{S}})$$

is a weighted Riemannian manifold of class  $CD(\kappa, N)$ . In other words, at least in the Euclidean setting, we conjecture that Theorem 1.4 admits a direct generalization to functions  $u : \mathcal{M} \rightarrow \mathbb{R}^k$ . Maybe the generalization works whenever  $u : \mathcal{M} \rightarrow \mathcal{N}$  is 1-Lipschitz, where  $\mathcal{N} \rightarrow \mathcal{M}$  has the isometric extension property. Moreover, in the Euclidean setting, we believe that Theorem 1.5 may be generalized as follows: Assume that  $f : \mathcal{M} \rightarrow \mathbb{R}^k$  satisfies  $\int_{\mathcal{M}} f d\mu = 0$  and also  $\int_{\mathcal{M}} |f(x)| \cdot d(x_0, x) d\mu(x) < +\infty$  for a certain  $x_0 \in \mathcal{M}$ . Let us maximize

$$(1) \quad \int_{\mathcal{M}} \langle f, u \rangle d\mu$$

among all 1-Lipschitz functions  $u : \mathcal{M} \rightarrow \mathbb{R}^k$ . One may use Kirszbraun's theorem and prove that for any maximizer  $u : \mathcal{M} \rightarrow \mathbb{R}^k$  and for  $\nu$ -almost any leaf  $\mathcal{S} \in T^\circ[u]$ ,

$$\int_{\mathcal{S}} f d\mu_{\mathcal{S}} = 0 \quad \text{and} \quad \int_{\mathcal{M}} \langle f, u \rangle d\mu_{\mathcal{S}} = \sup \left\{ \int_{\mathcal{S}} \langle f, v \rangle d\mu_{\mathcal{S}} ; v : \mathcal{S} \rightarrow \mathbb{R}^k, \|v\|_{Lip} \leq 1 \right\}.$$

REMARK 6.2. The bisection method outlined in Section 1 has one significant advantage compared to our results. The methods discussed in this article are very much *linear*, or at least convex, as we obtain a geodesic foliation from the convex maximization problem (1). In comparison, the bisection method works only in symmetric spaces such as  $\mathbb{R}^n$  or  $S^n$ , but in these spaces it offers more flexibility, since one may devise various linear and non-linear rules for the bisection procedure. This flexibility is exploited artfully by Gromov [28]. It is currently unclear to us whether one may arrive at an *integrable* foliation in the situations considered by Gromov [28].

Another possible research direction is concerned with characterizations of measures  $\mu$  supported on a geodesically-convex Riemannian manifold  $\mathcal{M}$  which are of class  $CD(\kappa, N)$ . Ideally, one would like a characterization à la Borell [9]. That is, perhaps if  $\mu$  satisfies a certain inequality of Cordero-Erausquin, McCann and Schmuckenschläger, then there is a totally-geodesic submanifold  $\mathcal{S} \subseteq \mathcal{M}$  such that  $\mu$  is supported on  $\mathcal{S}$ , and  $(\mathcal{S}, d, \mu)$  is a Riemannian manifold of class  $CD(\kappa, N)$ , possibly non-smooth. This is also related to a remark by Gromov [28, Section 4.2].

Yet another possible research direction is concerned with  $CD(\kappa, N+1)$ -needles in one dimension. It seems that many concepts and results from convexity theory admit generalizations to the class of  $CD(\kappa, N+1)$ -needles. For example, when  $0 \neq N \in \mathbb{R}$  and  $\kappa/N > 0$ , we may define a Legendre-type transform of a function  $f : \mathbb{R} \rightarrow [0, +\infty]$  by setting

$$(2) \quad f^*(s) = \inf_{t: f(t) < +\infty} \frac{g(s+t)}{f(t)} \quad \text{for } s \in \mathbb{R},$$

where

$$g(t) = \left\{ \sin \left( \sqrt{\frac{\kappa}{N}} \cdot t \right) \cdot 1_{[0, \pi]} \left( \sqrt{\frac{\kappa}{N}} \cdot t \right) \right\}^N$$

and we agree that  $g(s+t)/0 \equiv +\infty$  and that  $0^N = 0$  when  $N \in (0, +\infty)$  and  $0^N = +\infty$  when  $N \in (-\infty, 0)$ . It seems that the function  $f^*$  is either a density of a  $CD(\kappa, N+1)$ -needle in  $\mathbb{R}$ , or else it is a limit of such densities. We say that a function  $f : \mathbb{R} \rightarrow [0, +\infty]$  is  $(\kappa, N+1)$ -concave if the set

$$\{t \in \mathbb{R}; f(t) > R \cdot g(s+t)\}$$

is an interval for all  $R > 0, s \in \mathbb{R}$ . Perhaps the transform (2) is an order-reversing involution on the class of upper semi-continuous  $(\kappa, N+1)$ -concave functions on  $\mathbb{R}$ .

In many cases, the  $CD(\kappa, N)$ -needles may be further decomposed into needles of a simpler form that satisfy a certain linear constraint. This was already discovered by Lovász and Simonovits [36] in the most interesting case  $\kappa = 0, N = n$ .

**DEFINITION 6.3.** *Let  $\kappa \in \mathbb{R}, 1 \neq N \in \mathbb{R} \cup \{\infty\}$  and let  $\nu$  be a measure on a certain Riemannian manifold  $\mathcal{M}$  which is a  $CD(\kappa, N)$ -needle. Let  $A, \Psi$  and  $\gamma$  be as in Definition 1.1. We say that  $\nu$  is a “ $CD(\kappa, N)$ -affine needle” if the following equality holds true in the entire set  $A$ :*

$$\Psi'' = \kappa + \frac{(\Psi')^2}{N-1},$$

where in the case  $N = \infty$ , we interpret the term  $(\Psi')^2/(N-1)$  as zero.

For  $x \in \mathbb{R}$  write  $x_+ = \max\{x, 0\}$ . The class of  $CD(\kappa, N)$ -affine needles may be described explicitly, as follows:

- (1) The *exponential needles* are  $CD(0, \infty)$ -affine needles, for which the function  $e^{-\Psi}$  is an exponential function restricted to the open interval  $A$ . That is, the function  $e^{-\Psi}$  takes the form

$$A \ni t \mapsto \alpha \cdot e^{\beta \cdot t}$$

for certain  $\beta \in \mathbb{R}, \alpha > 0$ . The  *$\kappa$ -log-affine needles* are  $CD(\kappa, \infty)$ -affine needles, for which  $\Psi(t) - \kappa t^2/2$  is an affine function in the open interval  $A$ .

- (2) The  *$N$ -affine needles* are  $CD(0, N+1)$ -affine needles with  $0 \neq N \in \mathbb{R}$ , for which  $f^{1/N}$  is an affine function in the open interval  $A$ .
- (3) For  $0 \neq \kappa \in \mathbb{R}$  and  $0 \neq N \in \mathbb{R}$ , the  $CD(\kappa, N+1)$ -affine needles satisfy, for all  $t \in A$ ,

$$e^{-\Psi(t)} = \begin{cases} \left\{ \alpha \cdot \sin \left( \sqrt{\frac{\kappa}{N}} t - \beta \right) \cdot 1_{[0, \pi]} \left( \sqrt{\frac{\kappa}{N}} t - \beta \right) \right\}_+^N & \kappa/N > 0 \\ (\alpha + t\beta)_+^N & \kappa = 0 \\ \left( \alpha \cdot \sinh \left( \sqrt{\left| \frac{\kappa}{N} \right|} \cdot t \right) + \beta \cdot \cosh \left( \sqrt{\left| \frac{\kappa}{N} \right|} \cdot t \right) \right)_+^N & \kappa/N < 0 \end{cases}$$

for some  $\alpha, \beta \in \mathbb{R}$ .

In the case where  $N \in (0, +\infty]$  and  $\kappa \geq 0$  it seems pretty safe to make the following:

**CONJECTURE 6.4.** *Let  $\mu$  be a probability measure on  $\mathbb{R}$  which is a  $CD(\kappa, N+1)$ -needle. Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous,  $\mu$ -integrable function with  $\int_{\mathbb{R}} \varphi d\mu = 0$ . Then there exist probability measures  $\{\mu_\alpha\}_{\alpha \in \Omega}$  on  $\mathbb{R}$  and a probability measure  $\nu$  on the set  $\Omega$  such that:*

- (i) For any Lebesgue-measurable set  $A \subseteq \mathbb{R}$  we have  $\mu(A) = \int_{\Omega} \mu_{\alpha}(A) d\nu(\alpha)$ .
- (ii) For  $\nu$ -almost any  $\alpha \in \Omega$ , the measure  $\mu_{\alpha}$  is either supported on a singleton, or else it is a  $CD(\kappa, N + 1)$ -affine needle with  $\int_{\mathbb{R}} \varphi d\mu_{\alpha} = 0$ .

Conjecture 6.4 reduces certain questions on  $CD(\kappa, N + 1)$ -needles to an inequality involving only two or three real parameters. A proof of Conjecture 6.4 in the case where  $N = +\infty$  or  $\kappa = 0$  follows from Choquet's integral representation theorem and the results of Fradelizi and Guédon [24]. We are not sure what should be the correct formulation of Conjecture 6.4 in the case where  $N < 0$  and  $\kappa < 0$ .

APPENDIX

## The Feldman-McCann proof of Lemma 2.4.1

In this appendix we describe the Feldman-McCann proof of Lemma 2.4.1. Let  $\mathcal{M}$  be a Riemannian manifold with distance function  $d$ . Fix  $p \in \mathcal{M}$  and let  $\delta_0 = \delta_0(p) > 0$  be the constant provided by Lemma 2.3.1. Thus,  $U = B_{\mathcal{M}}(p, \delta_0/2)$  is a strongly-convex set. As in Section 2.3, for  $a \in U$  we write

$$U_a = \{X \in T_a\mathcal{M}; \exp_a(tX) \in U \text{ for all } t \in [0, 1]\},$$

a convex subset of  $T_a\mathcal{M}$ . Recall from Section 2.3 that for  $a \in U$  and  $X, Y \in U_a$ , denoting  $x = \exp_a(X), y = \exp_a(Y)$ , we define  $F_a(X, Y) \in U_x$  via the requirement that

$$\exp_x(F_a(X, Y)) = y.$$

Next, for  $a \in U$  and  $X, Y \in U_a$  let us define

$$\Phi_a(X, Y) \in T_a\mathcal{M}$$

to be the parallel translate of  $F_a(X, Y)$  along the unique geodesic from  $x$  to  $a$ . The map  $\Phi_a : U_a \times U_a \rightarrow T_a\mathcal{M}$  satisfies

$$(1) \quad |\Phi_a(X, Y)| = |F_a(X, Y)| = d(\exp_a X, \exp_a Y).$$

The behavior of  $\Phi_a$  on lines through the origin is quite simple: Since  $\exp_a(sX)$  and  $\exp_a(tX)$  lie on the same geodesic emanating from  $a$ , then for any  $X \in T_a\mathcal{M}$  and  $s, t \in \mathbb{R}$ ,

$$(2) \quad \Phi_a(sX, tX) = (t - s)X \quad \text{when } sX, tX \in U_a.$$

See [22, Section 3.2] for more details about  $\Phi_a$ . Our next lemma is precisely Lemma 14 in [22]. The proof given in [22, Lemma 14] is very simple and uses essentially the same notation as ours, and it is not reproduced here. In fact, the argument is similar to the proof of Lemma 2.3.4 above, and it relies only on the smoothness of  $\Phi_a$  and on the relation  $\Phi_a(0, Y) = Y$  that follows from (2).

**Lemma A.1.** *Let  $a \in U$  and  $X, Y_1, Y_2 \in U_a$ . Then,*

$$|\Phi_a(X, Y_2) - \Phi_a(X, Y_1) - (Y_2 - Y_1)| \leq \bar{C}_p \cdot |X| \cdot |Y_1 - Y_2|,$$

where  $\bar{C}_p > 0$  is a constant depending only on  $p$ .

PROOF OF LEMMA 2.4.1 (DUE TO FELDMAN AND MCCANN [22]). Define

$$(3) \quad \delta_1 = \delta_1(p) = \min \left\{ \frac{1}{2000 \cdot \bar{C}_p}, \frac{\delta_0}{2} \right\},$$

where  $\bar{C}_p > 0$  is the constant from Lemma A.1. Both the assumptions and the conclusion of the lemma are not altered if we replace  $x_i, y_i$  by  $x_{2-i}, y_{2-i}$  for  $i = 0, 1, 2$ . Applying this replacement if necessary, we assume from now on that

$$(4) \quad d(x_0, y_0) \leq d(x_2, y_2).$$

The points  $x_0, x_1, x_2, y_0, y_1, y_2$  belong to  $B_{\mathcal{M}}(p, \delta_1) \subseteq U$ . Recall that the main assumption of the Lemma is that

$$(5) \quad d(x_i, x_j) = d(y_i, y_j) = \sigma|i - j| \leq d(x_i, y_j) \quad \text{for } i, j = 0, 1, 2.$$

Define

$$(6) \quad \varepsilon := d(x_1, y_1).$$

Denote  $a = x_0$  and let  $X_0, X_1, X_2, Y_0, Y_1, Y_2 \in U_a$  be such that  $x_i = \exp_a(X_i)$  and  $y_i = \exp_a(Y_i)$  for  $i = 0, 1, 2$ . Since  $a = x_0$  then

$$X_0 = 0.$$

For  $i = 0, 1, 2$  we know that  $x_i, y_i \in B_{\mathcal{M}}(p, \delta_1)$  and  $X_i, Y_i \in U_a$ . It follows from (1), (2) and (5) that

$$(7) \quad |X_i| = |\Phi_a(X_0, X_i)| = d(x_0, x_i) \leq 2\delta_1, \quad |Y_i| = |\Phi_a(X_0, Y_i)| = d(x_0, y_i) \leq 2\delta_1.$$

By using (7) and Lemma A.1, for any  $R, Z, W \in \{0 = X_0, X_1, X_2, Y_0, Y_1, Y_2\}$ ,

$$(8) \quad |\Phi_a(R, Z) - \Phi_a(R, W) - (Z - W)| \leq \bar{C}_p \cdot |R| \cdot |Z - W| \leq 2\bar{C}_p \delta_1 |Z - W| \leq \frac{|Z - W|}{10},$$

where we used (3) in the last passage. By using (1), (6) and also (8) with  $R = Z = X_1$  and  $W = Y_1$ ,

$$(9) \quad |Y_1 - X_1| \leq \frac{10}{9} \cdot |\Phi_a(X_1, Y_1) - \Phi_a(X_1, X_1)| = \frac{10}{9} \cdot |\Phi_a(X_1, Y_1)| = \frac{10}{9} \cdot d(x_1, y_1) = \frac{10}{9} \cdot \varepsilon,$$

where  $\Phi_a(X_1, X_1) = 0$  by (2). From (2), (5) and the fact that  $X_0 = 0$ ,

$$(10) \quad 2\sigma \leq d(x_0, y_2) = |\Phi_a(X_0, Y_2)| = |Y_2| = |(Y_2 - X_2) + (X_2 - X_0)|.$$

Note that  $|X_2 - X_0| = |\Phi_a(X_0, X_2)| = 2\sigma$  from (1), (2) and (5). Hence, by squaring the inequality (10),

$$(11) \quad (2\sigma)^2 \leq |Y_2 - X_2|^2 + 2\langle Y_2 - X_2, X_2 - X_0 \rangle + (2\sigma)^2.$$

According to (5), the point  $x_1$  is the midpoint of the geodesic between  $x_0 = a$  and  $x_2$ . Therefore  $x_2 = \exp_a(X_2) = \exp_a(2X_1)$  and by strong-convexity  $2X_1 = X_2$ . Consequently  $X_2 - X_0 = 2(X_2 - X_1)$ , and from (11) we deduce that

$$(12) \quad \langle Y_2 - X_2, X_2 - X_1 \rangle = \frac{1}{2} \langle Y_2 - X_2, X_2 - X_0 \rangle \geq -\frac{1}{4} |Y_2 - X_2|^2.$$

Our next goal, like in [22, Lemma 16], is to prove that

$$(13) \quad \langle Y_2 - X_2, Y_1 - Y_2 \rangle \geq -\frac{1}{3} |Y_2 - X_2|^2.$$

Begin by applying (2) and (5), in order to obtain

$$(14) \quad 2\sigma \leq d(y_0, x_2) = |\Phi_a(Y_0, X_2)| = |(\Phi_a(Y_0, X_2) - \Phi_a(Y_0, Y_2)) + \Phi_a(Y_0, Y_2)|.$$

From (5), the point  $y_1$  is the midpoint of the geodesic between  $y_0$  and  $y_2$ . This implies that  $F_a(Y_0, Y_2) = 2F_a(Y_0, Y_1)$  and therefore  $\Phi_a(Y_0, Y_2) = 2\Phi_a(Y_0, Y_1)$ . Recall that  $|\Phi_a(Y_0, Y_2)| = d(y_0, y_2) = 2\sigma$ , according to (5). Thus, by squaring (14) and rearranging,

$$(15) \quad \begin{aligned} -|\Phi_a(Y_0, X_2) - \Phi_a(Y_0, Y_2)|^2 &\leq 2\langle \Phi_a(Y_0, X_2) - \Phi_a(Y_0, Y_2), \Phi_a(Y_0, Y_2) \rangle \\ &= 4\langle \Phi_a(Y_0, X_2) - \Phi_a(Y_0, Y_2), \Phi_a(Y_0, Y_2) - \Phi_a(Y_0, Y_1) \rangle. \end{aligned}$$

The deduction of (13) from (15) involves several approximations. Begin by using (15) and also (8) with  $R = Y_0, Z = X_2, W = Y_2$ , to obtain

$$(16) \quad -(11/10)^2 \cdot |X_2 - Y_2|^2 \leq 4\langle \Phi_a(Y_0, X_2) - \Phi_a(Y_0, Y_2), \Phi_a(Y_0, Y_2) - \Phi_a(Y_0, Y_1) \rangle.$$

Applying (4), together with (8) for  $R = Z = X_2, W = Y_2$ , we obtain (17)

$$|Y_0| = |\Phi_a(X_0, Y_0)| \leq |\Phi_a(X_2, Y_2)| = |\Phi_a(X_2, Y_2) - \Phi_a(X_2, X_2)| \leq \frac{11}{10} \cdot |Y_2 - X_2|.$$

According to Lemma A.1 and (17), for any  $Z, W \in \{0 = X_0, X_1, X_2, Y_0, Y_1, Y_2\}$ , (18)

$$|\Phi_a(Y_0, Z) - \Phi_a(Y_0, W) - (Z - W)| \leq \bar{C}_p \cdot |Y_0| \cdot |Z - W| \leq 2\bar{C}_p \cdot |Y_2 - X_2| \cdot |Z - W|.$$

It follows from (16) and from the case  $Z = X_2, W = Y_2$  in (18) that

$$(19) \quad \begin{aligned} -(11/10)^2 \cdot |X_2 - Y_2|^2 &\leq 4\langle X_2 - Y_2, \Phi_a(Y_0, Y_2) - \Phi_a(Y_0, Y_1) \rangle \\ &\quad + 8\bar{C}_p |X_2 - Y_2|^2 \cdot |\Phi_a(Y_0, Y_2) - \Phi_a(Y_0, Y_1)|. \end{aligned}$$

Note that  $|\Phi_a(Y_0, Y_2) - \Phi_a(Y_0, Y_1)| \leq 2|Y_2 - Y_1|$ , as follows from an application of (8) with  $R = Y_0, Z = Y_2, W = Y_1$ . We now use (18) with  $Z = Y_2$  and  $W = Y_1$ , and upgrade (19) to

$$(20) \quad -(11/10)^2 |X_2 - Y_2|^2 \leq 4\langle X_2 - Y_2, Y_2 - Y_1 \rangle + 30 \cdot \bar{C}_p |X_2 - Y_2|^2 \cdot |Y_2 - Y_1|.$$

The next step is to use that  $|Y_2 - Y_1| \leq |Y_2| + |Y_1| \leq 4\delta_1 \leq 1/(300\bar{C}_p)$  according to (3) and (7). Thus (20) implies

$$-(11/10)^2 \cdot |X_2 - Y_2|^2 \leq 4\langle X_2 - Y_2, Y_2 - Y_1 \rangle + \frac{|X_2 - Y_2|^2}{10},$$

and (13) follows. From (12) and (13),

$$(21) \quad \begin{aligned} \langle Y_2 - X_2, Y_1 - X_1 \rangle &= \langle Y_2 - X_2, (Y_1 - Y_2) + (Y_2 - X_2) + (X_2 - X_1) \rangle \\ &\geq -\frac{|X_2 - Y_2|^2}{3} + |X_2 - Y_2|^2 - \frac{|X_2 - Y_2|^2}{4} \geq \frac{1}{3} \cdot |Y_2 - X_2|^2. \end{aligned}$$

According to (9), (21) and the Cauchy-Schwartz inequality,

$$(22) \quad \frac{10}{9} \cdot \varepsilon \cdot |Y_2 - X_2| \geq |Y_2 - X_2| \cdot |Y_1 - X_1| \geq \langle Y_2 - X_2, Y_1 - X_1 \rangle \geq \frac{1}{3} \cdot |Y_2 - X_2|^2.$$

From (22),

$$(23) \quad |Y_2 - X_2| \leq 4\varepsilon.$$

We may summarize (9), (17) and (23) by

$$(24) \quad |Y_i - X_i| \leq 5\varepsilon \quad (i = 0, 1, 2).$$

For  $i = 0, 1, 2$ , we use (1), (24) and also (8) with  $R = Z = X_i$  and  $W = Y_i$ . This yields

$$(25) \quad d(x_i, y_i) = |\Phi_a(X_i, Y_i)| = |\Phi_a(X_i, Y_i) - \Phi_a(X_i, X_i)| \leq (11/10) \cdot |Y_i - X_i| \leq 6\varepsilon,$$

where  $\Phi_a(X_i, X_i) = 0$  according to (2). The lemma follows from (6) and (25).  $\square$



## Bibliography

- [1] Ambrosio, L., *Lecture notes on optimal transport problems*. Mathematical aspects of evolving interfaces (Funchal, 2000). Lecture Notes in Math., Vol. 1812, Springer, Berlin, (2003), 1–52.
- [2] Bakry, D., Émery, M., *Diffusions hypercontractives*. Séminaire de probabilités XIX, 1983/84, Lecture Notes in Math., Vol. 1123, Springer, Berlin, (1985), 177-206.
- [3] Bakry, D., Qian, Z., *Some new results on eigenvectors via dimension, diameter, and Ricci curvature*. Adv. Math., Vol. 155, No. 1, (2000), 98-153.
- [4] Bakry, D., Gentil, I., Ledoux, M., *Analysis and geometry of Markov diffusion operators*. Grundlehren der Mathematischen Wissenschaften, Vol. 348, Springer, 2014.
- [5] Bayle, V., *Propriétés de Concavité du Profil Isopérimétrique et Applications*. Ph.D. thesis, Institut Joseph Fourier, Grenoble, 2004. Available at <http://www.youscribe.com>
- [6] Bobkov, S. G., *Extremal properties of half-spaces for log-concave distributions*. Ann. Probab., Vol. 24, No. 1, (1996), 35-48.
- [7] Bobkov, S. G., *On concentration of distributions of random weighted sums*. Ann. Prob., Vol. 31, No. 1, (2003), 195-215.
- [8] Bobkov, S. G., Nazarov, F. L., *Sharp dilation-type inequalities with fixed parameter of convexity*. J. Math. Sci. (N. Y.), Vol. 152, No. 6, (2008), 826-839.
- [9] Borell, C., *Convex set functions in  $d$ -space*. Period. Math. Hungar., Vol. 6, No. 2, (1975), 111-136.
- [10] Brascamp, H. J., Lieb, E. H., *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*. J. Funct. Anal., Vol. 22, No. 4, (1976), 366-389.
- [11] Burns, K., Gidea, M., *Differential geometry and topology. With a view to dynamical systems*. Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2005.
- [12] Buser, P., *A note on the isoperimetric constant*. Ann. Sci. École Norm. Sup. (4), Vol. 15, No. 2, (1982), 213-230.
- [13] Caffarelli, L., Feldman, M., McCann, R. J., *Constructing optimal maps for Monge’s transport problem as a limit of strictly convex costs*. J. Amer. Math. Soc., Vol. 15, No. 1, (2002), 1-26.
- [14] Cayley, A., *On Monge’s “Mémoire sur la Théorie des Déblais et des Remblais.”* Proc. London Math. Soc., Vol. s1-14, Issue 1, (1882), 139-143.  
Available at <http://dx.doi.org/10.1112/plms/s1-14.1.139>
- [15] Chavel, I., *Riemannian geometry. A modern introduction*. Cambridge Studies in Advanced Mathematics, Vol. 98. Cambridge University Press, Cambridge, 2006.
- [16] Cheeger, J., Ebin, D. G., *Comparison theorems in Riemannian geometry*. AMS Chelsea Publishing, Providence, RI, 2008.
- [17] Cordero-Erausquin, D., McCann, R. J., Schmuckenschläger, M., *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*. Invent. Math., Vol. 146, No. 2, (2001), 219-257.
- [18] Cordero-Erausquin, D., McCann, R. J., Schmuckenschläger, M., *Prékopa-Leindler type inequalities on Riemannian manifolds, Jacobi fields and optimal transport*. Ann. Fac. Sci. Toulouse Math. (6), Vol. 15, No. 4, (2006), 613–635.
- [19] Edwards, C. H., *Advanced calculus of several variables*. Dover Publications, Inc., New York, 1994.
- [20] Evans, L. C., Gangbo, W., *Differential equations methods for the Monge-Kantorovich mass transfer problem*. Mem. Amer. Math. Soc., Vol. 137, No. 65, (1999), 1–66.
- [21] Evans, L. C., Gariepy, R. F., *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
- [22] Feldman, M., McCann, R. J., *Monge’s transport problem on a Riemannian manifold*. Trans. Amer. Math. Soc., Vol. 354, No. 4, (2002), 1667-1697.

- [23] Fradelizi, M., *Concentration inequalities for  $s$ -concave measures of dilations of Borel sets and applications*. Electron. J. Probab., Vol. 14, No. 71, (2009), 2068-2090.
- [24] Fradelizi, M., Guédon, O., *The extreme points of subsets of  $s$ -concave probabilities and a geometric localization theorem*. Discrete Comput. Geom., Vol. 31, No. 2, (2004), 327-335.
- [25] Gangbo, W., *The Monge mass transfer problem and its applications*. Monge-Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997). Contemp. Math., Vol. 226, Amer. Math. Soc., Providence, RI, (1999), 79-104.
- [26] Gromov, M., *Paul Levy's isoperimetric inequality*. Appendix C in the book *Metric structures for Riemannian and non-Riemannian spaces* by M. Gromov. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007.
- [27] Gromov, M., *Isoperimetric inequalities in Riemannian manifolds*. Appendix I in the book *Asymptotic Theory of Finite Dimensional Normed Spaces* by V. D. Milman and G. Schechtman. Lecture notes in Math., Vol. 1200, Springer-Verlag, Berlin, 1986.
- [28] Gromov, M., *Isoperimetry of waists and concentration of maps*. Geom. Funct. Anal. (GAFA), Vol. 13, No. 1, (2003), 178-215.
- [29] Gromov, M., Milman, V. D., *Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces*. Compositio Math., Vol. 62, No. 3, (1987), 263-282.
- [30] Heintze, E., Karcher, H., *A general comparison theorem with applications to volume estimates for submanifolds*. Ann. Sci. École Norm. Sup. (4), Vol. 11, No. 4, (1978), 451-470.
- [31] Kannan, R., Lovász, L., Simonovits, M., *Isoperimetric problems for convex bodies and a localization lemma*. Discrete Comput. Geom., Vol. 13, No. 3-4, (1995), 541-559.
- [32] Kantorovich, L. V., Akilov, G. P., *Functional analysis*. Second edition. Pergamon Press, Oxford-Elmsford, NY, 1982.
- [33] Ledoux, M., *Spectral gap, logarithmic Sobolev constant, and geometric bounds*. Surveys in differential geometry. Vol. IX, Int. Press, Somerville, MA, (2004), 219-240.
- [34] Li, P., Yau, S. T., *Eigenvalues of a compact Riemannian manifold*. Amer. Math. Soc., Proc. Symp. Pure Math., Vol. 36, (1980), 205-239.
- [35] Lovász, L., *Hit-and-run mixes fast*. Math. Program., Vol. 86, No. 3, (1999), 443-461.
- [36] Lovász, L., Simonovits, M., *Random walks in a convex body and an improved volume algorithm*. Random Structures Algorithms, Vol. 4, No. 4, (1993), 359-412.
- [37] Milman, E., *Sharp isoperimetric inequalities and model spaces for curvature-dimension-diameter condition*. J. Eur. Math. Soc., Vol. 17, No. 5, (2015), 1041-1078.
- [38] Milman, E., *On the role of convexity in isoperimetry, spectral gap and concentration*. Invent. Math., Vol. 177, No. 1, (2009), 1-43.
- [39] Morgan, F., *Manifolds with density*. Notices Amer. Math. Soc., Vol. 52, No. 8, (2005), 853-858.
- [40] Nazarov, F., Sodin, M., Volberg, A., *The geometric Kannan-Lovsz-Simonovits lemma, dimension-free estimates for the distribution of the values of polynomials, and the distribution of the zeros of random analytic functions*. St. Petersburg Math. J., Vol. 14, No. 2, (2003), 351-366.
- [41] Payne, L. E., Weinberger, H. F., *An optimal Poincaré inequality for convex domains*. Arch. Rational Mech. Anal., Vol. 5, (1960), 286-292.
- [42] Stein, E. M., *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, 1970.
- [43] Stein, E. M., Shakarchi, R., *Real analysis. Measure theory, integration, and Hilbert spaces*. Princeton Lectures in Analysis, III. Princeton University Press, Princeton, NJ, 2005.
- [44] Trudinger, N. S., Wang, X.-J., *On the Monge mass transfer problem*. Calc. Var. Partial Differential Equations, Vol. 13, No. 1, (2001), 19-31.
- [45] Whitney, H., *Analytic extensions of functions defined in closed sets*. Trans. Amer. Math. Soc., Vol. 36, No. 1, (1934), 63-89.
- [46] Yang, D., *Lower bound estimates of the first eigenvalue for compact manifolds with positive Ricci curvature*. Pacific J. Math., Vol. 190, no. 2, (1999), 383-398.
- [47] Yang, H. C., Zhong, J. Q., *On the estimate of the first eigenvalue of a compact Riemannian manifold*. Sci. Sinica Ser., Vol. 27, No. 12, (1984), 1265-1273.