The Logarithmic Laplace Transform in Convex Geometry

Bo'az Klartag

Tel-Aviv University

Workshop at Kibbutz Hagoshrim, Israel, April 2011

Joint works with Ronen Eldan and with Emanuel Milman.

Many open problems

We would like to explain some of the **tools** used in the study of the distribution of volume in high dimensional convex bodies.

- Despite recent progress, even the simplest questions remain unsolved:

Question [Bourgain, 1980s]

Suppose $K \subset \mathbb{R}^n$ is a convex body of volume one. Does there exist an $(n-1)$-dimensional hyperplane $H \subset \mathbb{R}^n$ such that

$$Vol_{n-1}(K \cap H) > c$$

where $c > 0$ is a universal constant?

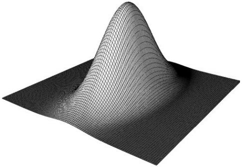
- Known: $Vol_{n-1}(K \cap H) > cn^{-1/4}$ (Bourgain '91, K. '06).
- Affirmative answer for: unconditional convex bodies, zonoids, their duals, random convex bodies, outer finite volume ratio, few vertices/facets, subspaces/quotients of L^p, Schatten class, . . .

Logarithmically-Concave densities

As was observed by K. Ball, the hyperplane conjecture is most naturally formulated in the class of **log-concave densities.**

- A probability density on $\mathbb{R}^n$ is log-concave if it takes the form $\exp(-H)$ for a **convex** function $H : \mathbb{R}^n \to (-\infty, \infty]$

Examples of log-concave densities:
The Gaussian density, the uniform density on a convex body.



1. Pointwise product of log-concave densities is (proportional to) a log-concave density.
2. Prékopa-Leindler: If X is a log-concave random vector, so is the random vector $T(X)$ for any linear map T.

Isotropic Constant

For a log-concave probability density $\rho : \mathbb{R}^n \to [0, \infty)$ set

$$L_\rho = \sup_{x \in \mathbb{R}^n} \rho^{\frac{1}{n}}(x) \det Cov(\rho)^{\frac{1}{2n}}$$

the **isotropic constant** of ρ. The isotropic constant is affinely invariant. What's its meaning?

- Normalization: Suppose X is a random vector in $\mathbb{R}^n$ with density ρ. It is **isotropic** if

  $$\mathbb{E}X = 0, \qquad Cov(X) = Id$$

  That is, all marginals have mean zero and var. one.

- For an isotropic, log-concave density ρ in $\mathbb{R}^n$,

  $$L_\rho \simeq \rho(0)^{1/n} \simeq \int_{\mathbb{R}^n} \rho^{1+\frac{1}{n}} \simeq \exp\left(\frac{1}{n} \int_{\mathbb{R}^n} \rho \log \rho \right) > c$$

An equivalent formulation of the slicing problem

The hyperplane conjecture is *directly* equivalent to the following:

Slicing problem, again:

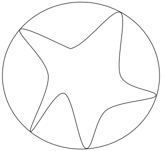
Is it true that for any n and an isotropic, log-concave $\rho : \mathbb{R}^n \to [0, \infty)$,

$$L_\rho < C$$

where $C > 0$ is a universal constant?

(this simple equivalence follows from Ball, Bourgain, Fradelizi, Hensley, Milman, Pajor and others, using Brunn-Minkowski).

- For a uniform density on $K \subset \mathbb{R}^n$, $L_K = Vol_n(K)^{-1/n}$. Can we have the same covariance as the Euclidean ball, in a substantially smaller convex set?



Remarks

1. It is straightforward to show that $L_\rho > c$, for a universal constant $c > 0$.
2. To summarize, define

$$L_n = \sup_{\rho:\mathbb{R}^n \to [0,\infty)} L_\rho.$$

It is currently known that

$$L_n \leq Cn^{1/4}.$$

3. It is enough to consider the uniform measure on centrally-symmetric convex bodies (Ball '88, K. '05):

$$L_n \leq C \sup_{K \subset \mathbb{R}^n} L_K$$

where $K \subset \mathbb{R}^n$ is convex, with $K = -K$.

Our tool: Logarithmic Laplace Transform

- The **Laplace Transform** is used in almost all known bounds for L_n.

Suppose X is a log-concave random vector, $\mathbb{E}X = 0$.

The **logarithmic Laplace transform** is the convex function

$$\Lambda(\xi) = \Lambda_X(\xi) = \log \mathbb{E} \exp(X \cdot \xi) \qquad (\xi \in \mathbb{R}^n).$$

It is non-negative with $\Lambda(0) = 0$.

The logarithmic Laplace transform helps relate:

1. Covariance matrix of X (and its "tilts"), and
2. The entropy of X (and volumes of $Z_p(X)$).

Differentiating the logarithmic Laplace transform

Recall that $\Lambda(\xi) = \log \mathbb{E} \exp(X \cdot \xi)$

- For $\xi \in \mathbb{R}^n$, denote by $\tilde{X}_\xi$ the "tilted" log-concave random vector in $\mathbb{R}^n$ whose density is proportional to

$$x \mapsto \rho(x) \exp(\xi \cdot x)$$

where ρ is the density of X.

Then,

1. $\nabla \Lambda(\xi) = \mathbb{E}\tilde{X}_\xi$.
2. The hessian $\nabla^2 \Lambda(\xi) = Cov(\tilde{X}_\xi)$.

Third derivatives? A bit complicated. With $b_\xi = \mathbb{E}\tilde{X}_\xi$,

$$\partial^i \log \det \nabla^2 \Lambda(\xi) = Tr\left[Cov(X_\xi)^{-1} \mathbb{E}(\tilde{X}_\xi^i - b_\xi^i)(\tilde{X}_\xi - b_\xi) \otimes (\tilde{X}_\xi - b_\xi)\right].$$

To tilt, or not to tilt?

- We prefer the "centered tilt",

$$X_\xi \stackrel{d}{=} \tilde{X}_\xi - b_\xi$$

where $b_\xi = \mathbb{E}\tilde{X}_\xi$. Note that $\mathbb{E}X_\xi = 0$, and X_ξ is log-concave.

Focusing on the Log. Laplace transform near $\xi \in \mathbb{R}^n$

Note that

$$\Lambda_{X_\xi}(z) = \Lambda_X(\xi + z) - [\Lambda_X(\xi) + z \cdot \nabla\Lambda_X(\xi)] \, .$$

i.e., we subtract the tangent plane at ξ, and translate so that ξ is the new origin.

- A marvelous usage of tilts appears in Cramér's theorem on moderate deviations from 1938.

Transportation of measure

Suppose X is uniform in a convex body K.
Recall that $\nabla\Lambda(\xi) = b_\xi \in K$ for all ξ, by convexity.
From the change of variables formula,

$$Vol_n(K) = Vol_n(\nabla\Lambda(\mathbb{R}^n)) = \int_{\mathbb{R}^n} \det \nabla^2\Lambda(\xi) d\xi \geq \int_{nK^\circ} \det \nabla^2\Lambda$$

- In particular, there exists $\xi \in nK^\circ$ with

$$\det \nabla^2\Lambda(\xi) = \det Cov(X_\xi) \leq \frac{Vol_n(K)}{Vol_n(nK^\circ)}.$$

Since $e^{-n} \leq \exp(\xi \cdot x) \leq e^n$ for $x \in K$, then for such $\xi \in nK^\circ$,

$$L_{X_\xi} \leq \frac{C}{Vol_n(K)^{1/n}} \left(\frac{Vol_n(K)}{Vol_n(nK^\circ)} \right)^{1/(2n)} \simeq \left(\frac{1}{Vol_n(K)\, Vol_n(nK^\circ)} \right)^{1/(2n)}$$

Log-concave densities and convex bodies

Theorem (Bourgain-Milman '87)

$$Vol_n(K) Vol_n(nK^\circ) \geq c^n$$

where $c > 0$ is a universal constant.

- Therefore $L_{X_\xi} < Const$ for **most** $\xi \in nK^\circ$.

There is a correspondence between centered log-concave densities and convex bodies due to K. Ball:

- Suppose $f : \mathbb{R}^n \to [0, \infty)$ is log-concave. Denote

$$K(f) = \left\{ x \in \mathbb{R}^n ; (n+1) \int_0^\infty f(rx) r^n dr \geq 1 \right\},$$

the convex body associated with f.
(Convexity of $K(f)$ is related to Busemann's inequality)

Applying this construction to X_ξ with $L_{X_\xi} < C$:

- Obtain a convex body T, with $L_T \simeq L_{X_\xi} < Const$.
- Direct analysis: The convex body T is geometrically close to K, the support of X.

We deduce:

Theorem (K. '06)

For any convex body $K \subset \mathbb{R}^n$ and $0 < \varepsilon < 1$, there exists another convex body $T \subset \mathbb{R}^n$ with

1. $(1-\varepsilon)K \subseteq T \subseteq (1+\varepsilon)K$,
2. $L_T \leq C/\sqrt{\varepsilon}$, *where $C > 0$ is a universal constant.*

There are a few other useful tricks with Log. Laplace.

Many properties of Log. Laplace are proven via 1D arguments

1. For a subspace E (perhaps of dimension one), use the fact that $\Lambda_X|_E = \Lambda_{Proj_E(X)}$, where $Proj_E(X)$ is again l.c.
2. Compute Λ_X via integration in polar coordinates, and use Laplace method for the integrals $\int_0^\infty t^{n-1}(l.c.)dt$.

Suppose X is log-concave, $\mathbb{E}X = 0, t > 0$. Denote

$$\{\Lambda \leq t\} = \{\xi \in \mathbb{R}^n; \Lambda(\xi) \leq t\}$$

and

$$\{\Lambda \leq t\}_{symm} = \{\Lambda \leq t\} \bigcap (-\{\Lambda \leq t\}) .$$

Direct Properties: (proven via 1D considerations)

Using these methods, one shows:

Suppose X is uniform in K and $\mathbb{E}X = 0$. Then,

$$\{\Lambda \leq n\} \simeq nK^\circ.$$

- For a general log-concave X with $\mathbb{E}X = 0$,

$$L_X \simeq \left(\det Cov(X)\right)^{1/(2n)} \cdot Vol_n\left(\{\Lambda \leq n\}\right)^{1/n}.$$

A rather technical, but useful fact, which follows from $L_X \geq c$:

For any k-dim. subspace $E \subset \mathbb{R}^n$, there exists $\theta \in E$ with $|\theta| = 1$ and

$$\sqrt{\mathbb{E}(X \cdot \theta)^2} \gtrsim Vol_k\left(\{\Lambda \leq k\} \cap E\right)^{-1/k}.$$

Relation to L^p-centroid bodies of Paouris

Suppose X is log-concave, $\mathbb{E}X = 0$. For $p \geq 1$ consider the norm

$$h_{Z_p(X)}(\theta) = \left(\mathbb{E}|X \cdot \theta|^p\right)^{1/p} \qquad (\theta \in \mathbb{R}^n).$$

- This is the supporting functional of a centrally-symmetric convex body, denoted by

$$Z_p(X)$$

(introduced by Lutwak, Zhang '97, theory developed mostly by Paouris).

From 1D considerations:

Theorem (essentially from Latała, Wojtaszczyk '08)

For any $p \geq 1$,

$$Z_p(X) \simeq p\{\Lambda_X \leq p\}^{\circ}_{symm}.$$

Borrowing from Paouris' theory of L^p-centroid bodies

- One advantage of $Z_p(X)$ over the dual $\{\Lambda_X \leq p\}$:

$$W_p(Z_p(X)) := \left(\int_{S^{n-1}} h_{Z_p}(\theta)^p \right)^{1/p} \simeq \frac{\sqrt{p}}{\sqrt{n+p}} \left(\mathbb{E}|X|^p\right)^{1/p} \gtrsim \sqrt{p}$$

when $2 \leq p \leq n$ and $\mathbb{E}|X|^2 = n$.
This is very hard to prove directly for log. Laplace.

Paouris $q^*(X)$ parameter

This is the maximal $p \geq 1$ such that **random** p-dimensional sections of $\{\Lambda_X \leq p\}_{symm}$ are approximately Euclidean.

- It is important to have the same p, because we know something about the geometry of $\{\Lambda \leq n\}$ in $\mathbb{R}^n$.

The classical Dvoretzky theorem appears

V. Milman's 1971 proof of Dvoretzky's theorem has a **formula** for the dimension in which **"random sections are Euclidean"**.

Paouris remarkable theorem

Using this formula and its relations to $W_p(Z_p(X))$, he shows:

$$(\mathbb{E}|X|^p)^{1/p} \leq C \left(\mathbb{E}|X|^2\right)^{1/2}$$

for all $1 \leq p \leq q^*(X)$ and an isotropic X, with $q^*(X) \gtrsim \sqrt{n}$.

- What else follows from Paouris approach? When $\mathbb{E}X = 0$, $1 \leq k \leq q^*(X)$, and $E \in G_{n,k}$ is a *random subspace*. Then usually,

$$Vol_k\left(\{\Lambda_X \leq k\} \cap E\right)^{1/k} \lesssim \det Cov(X)^{-1/(2n)}$$

(i.e., some upper bound on $\{\Lambda \leq k\}$, for $k \leq q^*(X)$).

Relating the tilt and the original measure

Recall the tilts X_ξ and the close relation between Λ_X and Λ_{X_ξ}:

Lemma

Suppose $\xi \in \frac{1}{2}\{\Lambda_X \leq p\}_{symm}$. *Then,*

$$\{\Lambda_X \leq p\}_{symm} \simeq \{\Lambda_{X_\xi} \leq p\}_{symm}.$$

- Any parameter which depends nicely on $\{\Lambda_X \leq p\}$ is the same for X and X_ξ. For instance, $\min\{p, q^*(X)\}$.

Combining the estimates we have so far, we obtain:

Lemma

For any $1 \leq p \leq q^*(X)$ *and* $\xi \in \frac{1}{2}\{\Lambda_X \leq p\}_{symm}$,

$$\exists \theta \in S^{n-1}, \qquad \sqrt{\mathbb{E}(X_\xi \cdot \theta)^2} \gtrsim \det Cov(X)^{1/(2n)}.$$

Lower bounds on covariance determinants

We need projections to lower dimensional subspaces, but $q^*(X)$ is unstable. Use another parameter, $q^*_{GH}(X)$, which is roughly the "worst possible" $q^*(Proj_E X)$ over all subspaces E.

- These parameters $q^*(X)$ and $q^*_{GH}(X)$ are at least $\sqrt{n}$, and much larger when we have "ψ_α" information.

Corollary

Suppose X is isotropic, set $p = q^*_{GH}(X)$. Then for any $\xi \in \frac{1}{2}\{\Lambda_X \leq p\}_{symm}$,

$$\det Cov(X_\xi) \geq c^n.$$

- This reminds us of the "transportation" argument we encountered before...

Advantages of Log. Laplace

The final ingredient in our recipe is a a refined *transportation of measure* argument (we can also use what we had earlier).

- Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is **any** non-negative convex function, $F(0) = 0$. Abbreviate

$$F_p = \{F \leq p\} = \{x \in \mathbb{R}^n; F(x) \leq p\}.$$

“The gradient image of a level set is approx. *p* times its dual”

$$\frac{1}{2}\nabla F\left(\frac{F_p}{2}\right) \subseteq pF_p^{\circ} \subseteq \nabla F\left(F_p\right).$$

Consequently, (with a bit of cheating)

$$\left(\int_{F_p} \det \nabla^2 F(x)dx\right)^{1/n} = Vol_n\left(\nabla F\left(F_p\right)\right)^{1/n} \simeq Vol_n\left(pF_p^{\circ}\right)^{1/n}.$$

Transportation argument revisited

- Suppose F is an *even* convex function with $F(0) = 0$. Use the previous formula + Santaló/Bourgain-Milman:

"integral of det.-hessian determines volume of level-set"
(minor cheating, needs to take $p/2$ for one bound, and p for the other direction)

$$\left(\int_{F_p} \det \nabla^2 F(x) \frac{dx}{Vol_n(F_p)} \right)^{1/n} \simeq \frac{p}{v.rad.(F_p)^2}.$$

where the volume-radius of a convex body $T \subset \mathbb{R}^n$ is

$$v.rad.(T) = (Vol_n(T)/Vol_n(B^n))^{1/n}.$$

- An example: $F(x) = |x|^2/2$. Then, $\left(\det \nabla^2 F(x)\right)^{1/n} = 1$ and

$$v.rad.(F_p) \simeq \sqrt{p}.$$

Combining everything

To summarize:

- When X is isotropic and $p = q^*_{GH}(X)$, we have a lower bound for
$$\det \nabla^2 \Lambda(\xi)$$
for $\xi \in \frac{1}{2}\{\Lambda \le p\}_{symm}$.
- This translates to an upper bound for
$$v.rad.\left(\{\Lambda \le q^*_{GH}(X)\}_{symm}\right).$$
- By convexity, $\{\Lambda \le n\} \subseteq \frac{n}{p}\{\Lambda \le p\}$ for $p \le n$.

Therefore,

Theorem (K., E. Milman '11)

Suppose X is an isotropic, log-concave random vector in $\mathbb{R}^n$. Then,
$$L_X \le C\sqrt{\frac{n}{q^*_{GH}(X)}}.$$

But what is this strange parameter $q^*_{GH}(X)$?

- Suppose X is isotropic. Set

$$\triangle_X(p) = \sup_{\theta \in S^{n-1}} \left(\mathbb{E}|X \cdot \theta|^p\right)^{1/p}.$$

Then,

$$q^*_{GH}(X) \simeq \left(\prod_{k=1}^{n} \triangle_X^{-1}(c\sqrt{k}) \right)^{1/n}.$$

- When X is a ψ_α-random vector with constant b_α,

$$\triangle_X(p) \leq b_\alpha p^{1/\alpha}$$

and from the theorem with E. Milman,

$$L_X \leq C\sqrt{b_\alpha^\alpha n^{1-\alpha/2}}.$$

Improves by logarithmic factors upon Dafnis-Paouris '10 and Bourgain '02.

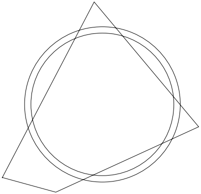
Further open problems

Theorem ("Central Limit Theorem for Convex Bodies", K. '07)

Most of the volume of a log-concave density in high dimensions, with the isotropic normalization, is concentrated near a sphere of radius $\sqrt{n}$.

Define

$$\sigma_n^2 = \sup_X Var(|X|) \sim \sup_X \mathbb{E}\left(|X| - \sqrt{n}\right)^2,$$



where the supremum runs over all log-concave, isotropic random vectors X in $\mathbb{R}^n$.

- The theorem states that

$$\sigma_n \ll \sqrt{n},$$

which implies that most marginals are approx. gaussian.

How thin is the shell?

- Current best bound, due to Guédon-E. Milman '10:

$$\sigma_n \leq Cn^{1/3}$$

(improving a previous bound of $\sigma_n \leq Cn^{3/8}$ due to Fleury '10, which improved upon $\sigma_n \leq Cn^{0.401}$, K. '07, which improved upon $\sigma_n \leq \sqrt{n/\log n}$, K. '07).

Theorem (Eldan, K. '10)

There is a universal constant C such that

$$L_n \leq C\sigma_n.$$

A Riemannian metric

Are we tired already? If not, there are other amusing games one may play with Log. Laplace.

Definition

For $\xi \in \mathbb{R}^n$, consider the positive-definite quadratic form

$$g_\xi(u, v) = Cov(X_\xi) u \cdot v \qquad (u, v \in \mathbb{R}^n)$$

- This Riemannian metric lets X_ξ "feel isotropic".
- This metric does not depend on the Euclidean structure:

$$g_\xi(u, v) = \mathbb{E} u(X_\xi - b_\xi) \cdot v(X_\xi - b_\xi) \qquad (u, v \in \mathbb{R}^{n*})$$

  where $b_\xi = \mathbb{E} X_\xi$ and u, v are viewed as linear functionals.
- The absolute values of the sectional curvatures are bounded by a universal constant. They vanish when $X_1, \ldots, X_n$ are independent r.v.'s.

A Riemannian metric

One computes that for any $\xi \in \mathbb{R}^n$,

$$|\nabla_g \log \det Cov(X_\xi)|_g \leq C\sqrt{n}\sigma_n$$

(due to affine invariance, the computation simplifies considerably: We can always "pretend that X_ξ is istotropic").

- Assume that X is isotropic. Then for $\xi \in \mathbb{R}^n$ with $d_g(0, \xi) \leq \sqrt{n}/\sigma_n$,

$$e^{-n} \leq \det Cov(X_\xi) \leq e^n.$$

Recall: we need a lower bound for an integral of $\det Cov(X_\xi)$.

- How big is the Riemannian ball of radius $\sqrt{n}/\sigma_n$ around the origin?

Level sets of Log. Laplace, again

Lemma

$$d_g(0,\xi) \le \sqrt{\Lambda_X(2\xi)}$$

Proved by inspecting the Riemannian length of the (Euclidean) segment $[0,\xi]$: By convexity,

$$d_g(0,\xi) \le \int_0^1 \sqrt{\frac{\partial^2}{\partial \xi^2}\Lambda_X(r\xi)dr} \le \sqrt{\Lambda_X(2\xi)}$$

- Therefore,

$$\left(\frac{1}{L_K}\right)^n \ge c^n Vol_n\left(\left[\Lambda \le n/\sigma_n^2\right]\right).$$

Analysis of $\{\Lambda \le p\}$ as described earlier completes the proof.

Thank you!