High dim. geometry, homework assignment no. 1

You are asked to solve at least 3 questions. Please submit your solution in pdf format by Wednesday, November 18 at 2PM at the link:

`https://www.dropbox.com/request/b77IfQLxLts6ffpeDrZu`

1. Let X and Y be independent random vectors supported in the sphere S^{n-1}. Assume that Y is distributed uniformly in the sphere. Prove that

$$\langle X, Y \rangle \qquad \text{and} \qquad Y_1$$

have the same distribution.

2. Let $n \geq 100$. Let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ be a unit vector such that

$$\forall i, \qquad |\theta_i| \leq \frac{5}{\sqrt{n}}.$$

Let X be a random vector in $\mathbb{R}^n$, distributed uniformly in $[-\sqrt{3}, \sqrt{3}]^n$. Denote by $f_\theta(t)$ the continuous density of $\langle X, \theta \rangle$. Prove that for $t \in \mathbb{R}$,

$$\left| f_\theta(t) - \frac{\exp(-t^2/2)}{\sqrt{2\pi}} \right| \leq \frac{C}{n} \tag{1}$$

for a universal constant $C > 0$.

Bonus: Assume only that $|\theta_i| \leq 1/10$ for all i and set $\varepsilon = \sum_i \theta_i^4$. Prove that for any $|t| \leq 1/(10\varepsilon^{1/4})$, the left-hand side of (1) is bounded by $C\varepsilon\,(1 + t^4)\, e^{-t^2/2}$.

3. Let (X, d) be a metric space and μ a Borel probability measure on X. For $t, \varepsilon \in (0, 1)$ consider the isoperimetric profile

$$I_\varepsilon(t) = \inf \{\mu(A_\varepsilon)\,;\, A \subseteq X,\ \mu(A) = t\}$$

where $A_\varepsilon = \{x \in X\,;\, \inf_{y \in A} d(x, y) < \varepsilon\}$ is the ε-neighborhood.

Suppose that (X, d_X, μ_X) and (Y, d_Y, μ_Y) are two such *metric probability spaces*. Assume that there exists a measure preserving contraction $T : X \to Y$, that is, T pushes forward μ_X to μ_Y while $d_Y(T(p), T(q)) \leq d_X(p, q)$ for $p, q \in X$. Prove that

$$I_\varepsilon^{(X)}(t) \leq I_\varepsilon^{(Y)}(t) \qquad \text{for all } t, \varepsilon \in (0, 1).$$

4. Consider the metric probability space $X = (\mathbb{R}^n, d, \gamma_n)$ where $d(x,y) = |x-y|$ and where γ_n is the standard Gaussian measure on $\mathbb{R}^n$, whose density equals

$$(2\pi)^{-n/2} e^{-|x|^2/2}.$$

Consider the metric probability space $Y = (Q, d, \lambda)$ where $Q = [0, \sqrt{2\pi}]^n \subseteq \mathbb{R}^n$ is a cube, and where λ is the uniform probability measure (Lebesgue) on Q. Prove that the map

$$T(x_1, \ldots, x_n) = \sqrt{2\pi}\,(\Phi(x_1), \ldots, \Phi(x_n))$$

is a measure preserving contraction from X to Y, where $\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-s^2/2} ds$. [Hint: Compute the operator norm of T', and use a mean value theorem to show that T is a contraction, and the Jacobian $\det T'$ to show measure preservation].

5. Let $p \geq 1$ be an integer, and consider the unit ball of ℓ_p^n, namely,

$$B_p^n = \left\{ x \in \mathbb{R}^n \,;\, \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \leq 1 \right\}.$$

The cone measure μ on its boundary is defined, for a Borel set $A \subseteq \partial B_p^n$ via

$$\mu(A) = \frac{Vol_n(\{tx \,;\, x \in A, 0 \leq t \leq 1\})}{Vol_n(B_p^n)}.$$

(a) Let $X_1, \ldots, X_n$ be i.i.d random variables, whose density is proportional to $\exp(-|t|^p)$. Prove that

$$(X_1, \ldots, X_n)/\|X\|_p$$

is distributed according to the cone measure on ∂B_p^n.

(b) Prove that $(X_1, \ldots, X_n)/\|X\|_p$ and $\|X\|_p$ are independent. What is the density of $\|X\|_p^p$?

(c) Prove that the density of $\sum_{i=1}^{p} |X_i|^p$ is an exponential random variable of parameter one (i.e., density e^{-t} on $[0, \infty)$).

(d) Let E be an exponential random variable of parameter one, independent of the X_i's. Prove that $\|X\|_p/(\|X\|_p^p + E)^{1/p}$ has density nt^{n-1} in $[0,1]$.

(e) Prove that

$$(X_1, \ldots, X_n)/(\|X\|_p^p + E)^{1/p}$$

is distributed uniformly in B_p^n.

(f) Conclude the generalized Archimedes principle: If $(X_1, \ldots, X_n)$ is distributed according to the cone measure on ∂B_p^n, then $(X_1, \ldots, X_{n-p})$ is distributed uniformly in B_p^{n-p}.