High dim. geometry, homework assignment no. 2

You are asked to solve at least 3 questions. Please submit your solution in pdf format by Wednesday, December 16 at 2PM at the link:

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https://www.dropbox.com/request/eHYH6Z9eRw9Q8SfBtc53
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1. (a) Let $(\Omega, \mathcal{G}, \mu)$ be a measure space with $\mu(\Omega) = 1$, and let $f_0, f_1, \dots, f_n \in L^2(\mu)$ be an orthonormal basis with $f_0 \equiv 1$ and $\sum_{j=1}^n f_j^2 \equiv n$. Prove that $\exists \mathcal{F} \subseteq S^{n-1}$ with $\sigma_{n-1}(\mathcal{F}) \geq 1 - 1/n$ such that for all $\theta \in \mathcal{F}$ we have

$$\left| \mu \left\{ x \in \Omega;\ \sum_{j=1}^n \theta_j f_j \leq t \right\} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-s^2/2} ds \right| \leq \frac{C}{n^\alpha}, \quad \forall t \in \mathbb{R},$$

where $C, \alpha > 0$ are universal constants.

   (b) For any even n find trigonometric polynomials $f_1, \dots, f_n \in L^2([0,1])$ satisfying the above requirements. Bonus: Can take any orthonormal basis in an irreducible representation of $SO(k)$ in $L^2(S^{k-1})$, a.k.a spherical harmonics.

2. (a) Prove that for any positively 1-homogenous measurable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{S^{n-1}} f d\sigma_{n-1} = \frac{1}{\mathbb{E}|Z|} \mathbb{E} f(Z)$$

where Z is a standard Gaussian random vector in $\mathbb{R}^n$, and prove also that $c\sqrt{n} \leq \mathbb{E}|Z| \leq C\sqrt{n}$ for universal constants $c, C > 0$.

   (b) Show that for $2 \leq k \leq n$,

$$c\sqrt{\frac{\log k}{n}} \leq \int_{S^{n-1}} \max_{i=1,\dots,k} |x_i| d\sigma_{n-1}(x) \leq C\sqrt{\frac{\log k}{n}}$$

for universal constants $c, C > 0$.

3. (a) Let $\theta \in S^{n-1}, 0 < r < 1$ and consider the cap

$$C(\theta, r) = \{x \in S^{n-1};\ \langle x, \theta \rangle > r\}.$$

Prove that $\sigma_{n-1}(C(\theta, r)) \leq Ce^{-cr^2 n}$ for universal constants $c, C > 0$.

   (b) Suppose that $P \subseteq \mathbb{R}^n$ is a polytope with N facets such that

$$\frac{1}{d} B^n \subseteq P \subseteq B^n. \tag{1}$$

Show that there exists caps $C(\theta_i, r_i)$ for $i = 1, \dots, N$ with $r_i \geq 1/d$ whose union covers the sphere.

(c) Prove that any polytope satisfying (1) has at least $\exp(cn/d^2)$ facets (and by duality, also at least that many vertices).

(d) Conclude that the Dvoretzky dimension of a polytope with N facets is at most $C \log N$ for a universal constant $C > 0$.

4. For $K \subseteq \mathbb{R}^n$ convex and centrally-symmetric denote $\rho(K) = \text{Vol}\,(K)\,\text{Vol}\,(K^\circ)$.

   (a) Let A be an invertible matrix. Show that $\rho(AK) = \rho(K)$.

   (b) Let $u \in S^{n-1}$ and define $T = S_u K$, where S_u is the Steiner symmetrization with respect to $u^\perp$. For $A \subseteq \mathbb{R}^n$ and $y \in \mathbb{R}$ denote $A_y = \{x \in \mathbb{R}^{n-1};\ (x, y) \in A\}$. Assuming $u = e_n$, show that

$$\frac{(K^\circ)_y + (K^\circ)_{-y}}{2} \subseteq (T^\circ)_y\ , \quad \forall y \in \mathbb{R}.$$

   (c) Conclude from the Brunn-Minkowski inequality that $\text{Vol}\,(K^\circ) \le \text{Vol}\,(T^\circ)$.

   (d) Prove the Santaló inequality: For any centrally symmetric convex body $K \subseteq \mathbb{R}^n$,

$$\rho(K) \le \rho(B_2^n).$$

5. Khintchine's inequality: Let $p \ge 1$ and let $\varepsilon_1, \ldots, \varepsilon_n$ be independent Bernoulli random variables with $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$. Prove that

$$c\,\|x\|_2 \le \left(\mathbb{E}\left|\sum_{i=1}^n \varepsilon_i x_i\right|^p\right)^{1/p} \le C\sqrt{p}\,\|x\|_2\,, \quad \forall x \in \mathbb{R}^n, \tag{2}$$

where $c, C > 0$ are universal constants. Note that the case $p = 2$ is an equality (for suitable constants). A possible proof: Fix $x \in S^{n-1}$.

   (a) Show that $\cosh t \le e^{t^2/2}$ and hence for any $t > 0$,

$$\mathbb{E}\exp\left(t\sum_{i=1}^n \varepsilon_i x_i\right) \le \exp\left(\frac{t^2}{2}\right).$$

   (b) Show that $\mathbb{E}\exp\left(t\left|\sum_{i=1}^n \varepsilon_i x_i\right|\right) \le 2\exp(t^2/2)$.

   (c) Assume that $p \ge 2$ is an integer, and use the inequality $t^p/p! \le e^t$ for appropriate $t > 0$ to conclude that

$$\mathbb{E}\left|\sum_{i=1}^n \varepsilon_i x_i\right|^p \le (C\sqrt{p})^p$$

   for some universal constant $C > 0$.

   (d) Prove inequality (2) for all $p \ge 2$.

   (e) Prove inequality (2) for $1 \le p \le 2$. (Hint: Hölder's inequality)