High dim. geometry, homework assignment no. 3

You are asked to solve at least 3 questions. Please submit your solution in pdf format by Wednesday, January 6 at 2PM at the link:

`https://www.dropbox.com/request/S9Io2HC7XisEN0LUeAJk`

1. *Decay of diameter under random projections.* Let $K \subseteq \mathbb{R}^n$ be convex, $K = -K$. Let $1 \leq \ell \leq n$ and $E \in G_{n,\ell}$ a random subspace, distributed uniformly. Prove (maybe using the 3 steps below) that with probability at least $1 - Ce^{-c\ell}$,

$$\mathrm{Diam}(Proj_E K) \leq C \max\left\{ M^*(K), \sqrt{\frac{\ell}{n}} \cdot \mathrm{Diam}(K) \right\} \tag{1}$$

where $c, C > 0$ are universal constants, where Diam is diameter and M^* is the mean width.

Step a) If $\ell \leq d_* = n(M^*/diam)^2$, then this follows from Dvoretzky's theorem.

Step b) Assume $\ell \geq d_*$. Fix a subspace $E_0 \in G_{n,\ell}$ and a $(1/2)$-net $\mathcal{F}$ in $E_0 \cap S^{n-1}$. Prove that for a random rotation $U \in O(n)$, with probability at least $1 - Ce^{-c\ell}$,

$$\max_{z \in U(\mathcal{F})} \|z\|_K^* \leq \sqrt{\frac{\ell}{n}} \cdot \mathrm{Diam}(K)$$

where $\|z\|_K^* = h_K(z) = \sup_{x \in K} z \cdot x$ is the dual norm (or supporting functional).

Step c) Use successive approximation: Write any $x \in S^{n-1} \cap U(E_0)$ as $x = \sum_{i=0}^{\infty} \delta_i y_i$ with $|\delta_i| \leq 2^{-i}$ and $y_i \in U(\mathcal{F})$, and conclude (1).

2. *Computing the Dvoretzky dimension of ℓ_p^n.* Recall that for a norm $\|\cdot\|$ on $\mathbb{R}^n$ we write $b = \sup_{x \in S^{n-1}} \|x\|$, $M = \int_{S^{n-1}} \|x\| d\sigma_{n-1}(x)$ and $d = n(M/b)^2$ is the Dvoretzky dimension.

   (a) For $1 \leq p \leq 2$, show that $cn \leq d(\ell_p^n) \leq Cn$ for universal constants $c, C > 0$.

   (b) For $2 < p < \infty$, show that $B_2^n \subseteq B_p^n \subseteq n^{1/2-1/p} B_2^n$. Conclude that for ℓ_p^n we have $b = 1$ and $M \geq n^{1/p-1/2}$ and hence

$$d(\ell_p^n) \geq cn^{2/p}.$$

(c) Assuming the existence of k-dimensional subspace of ℓ_p^n that is 5-isomorphic to Euclidean, there are vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ such that

$$\|a\|_2 \leq \left\| \sum_{i=1}^k a_i u_i \right\|_p \leq 5 \|a\|_2 , \quad \forall a \in \mathbb{R}^k. \tag{2}$$

Apply for a vector a of random signs, and use Khintchine's inequality to obtain

$$k^{p/2} \leq c_p^p \sum_{i=1}^n \left(\sum_{j=1}^k u_{j,i}^2 \right)^{p/2} ,$$

where $u_j = (u_{j,1}, \ldots, u_{j,n})$ and $c_p \leq C\sqrt{p}$.

(d) Apply for a vector $a = (u_{j,i})_{j=1,\ldots,k}$ and prove that for all i,

$$\sqrt{\sum_{j=1}^k u_{j,i}^2} \leq 5.$$

Conclude the bound

$$d(\ell_p^n) \leq c_p n^{2/p}$$

for some constant c_p depending solely on p.

3. Define a sub-exponential process, and formulate and prove an analog of *Dudley's bound* for sub-exponential processes.

4. Let $\rho : \mathbb{R}^n \to [0, \infty)$ be a log-concave probability density. Prove (in steps) that it decays exponentially at infinity, i.e., there exist $A, B > 0$ with

$$\rho(x) \leq Ae^{-B|x|} \qquad \text{for all } x \in \mathbb{R}^n. \tag{3}$$

Step a) Find $\varepsilon > 0$ such that set $K = \{x \in \mathbb{R}^n \, , \, \rho(x) > \varepsilon\}$ is convex and bounded, with non-empty interior.

Step b) Translating, we may assume that 0 is in the interior of K. Prove that there exists $R > 0$ such that

$$\rho(x) \leq \rho(0) \exp(-|x|/R) \qquad \text{for all } |x| \geq R.$$

Step c) Prove that ρ is bounded in RB^n, and conclude (3).

5. *Convergence of Steiner Symmetrization.* Let $K \subseteq \mathbb{R}^n$ be a compact set, set $R(K) = \max_{x \in K} |x|$ and assume that $R(K) > v.rad.(K)$.

   (a) Prove that there exists a finite sequence of Steiner symmetrizations, with respect to hyperplanes through the origin, that arrive at another compact set $T \subseteq \mathbb{R}^n$ with $R(T) < R(K)$. [Hint: The set $K \cap RS^{n-1}$ can only decrease, and we can "empty" a cap after cap]

   (b) Write $\mathcal{F}$ for the collection of all compacts obtained from K by applying a finite sequence of Steiner symmetrizations. Argue that $\mathcal{F}$ contains elements that are arbitrarily close to a Euclidean ball, in the Hausdorff metric.