High dim. geometry, homework assignment no. 4

You are asked to solve at least 4 questions. Please submit your solution in pdf format by Wednesday, February 10 at the link:

`https://www.dropbox.com/request/n9P7IhbaVpSscPbaKzRC`

1. Let X be a random vector in $\mathbb{R}^n$ with log-concave density f. Prove that

$$\left(\int_{\mathbb{R}^n} f^2\right)^{1/n} \sim \int_{\mathbb{R}^n} f^{1+1/n} \sim e^{-\mathrm{Ent}(X)/n}$$

where $A \sim B$ means that $cA \leq B \leq CA$ for universal constants $c, C > 0$. [Hint: Recall the body $K(f)$ defined in class]

2. Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body, n even. Let $E \in G_{n,n/2}$ be a random $n/2$-dimensional subspace.

   (a) Prove that
   $$\mathbb{E} v.rad.(K \cap E) \lesssim v.rad.(K),$$
   where $v.rad.(K)$ is the radius of the Euclidean ball (of the same dimension) whose volume equals that of K, and $A \lesssim B$ means that $A \leq CB$ for a universal constant $C > 0$. [Actually, there is a direct proof without the volume-diameter balance theorem showing that $\mathbb{E} v.rad.(K \cap E) \leq v.rad.(K)$]

   (b) Recall that $M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_{n-1}(x)$. Assume that $v.rad.(K) = 1$. Prove that $M(K) \geq 1$ and that
   $$\mathbb{E}\mathrm{Diam}(K \cap E) \lesssim M(K).$$

3. (i) Let $f : \mathbb{R} \to [0, \infty)$ be a log-concave probability density with $\int_{-\infty}^{\infty} t f(t) dt = 0$. Prove that
   $$\int_0^{\infty} f(t) dt \geq c$$
   for a universal constant $c > 0$. [Bonus: prove with $c = 1/e$, the optimal constant].

   (ii) Grünbaum inequality: Let $K \subseteq \mathbb{R}^n$ be a convex body of volume one. Prove that any hyperplane H through its barycenter splits K into two convex parts, each of volume at least c.

4. Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body of volume one.

   (a) Prove that there exists a hyperplane $H \subseteq \mathbb{R}^n$ with $Vol_{n-1}(K \cap H) \gtrsim 1/\sqrt{n}$. [Hint: Find a direction in which the width is at most $C\sqrt{n}$].

   (b) Conclude that $L_K \lesssim \sqrt{n}$.

5. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be convex, set $f = e^{-\psi}$ and assume that f is integrable.

   (a) Prove that $\varphi(x,t) = t\psi(x/t)$ is a convex function on $\mathbb{R}^n \times (0,\infty)$.

   (b) Assume that $e^{-\psi}$ is integrable. Deduce from Prekopa-Leindler that

   $$t \mapsto \int_{\mathbb{R}^n} e^{-t\psi(x/t)} dx$$

   is log-concave.

   (c) Conclude that the function

   $$G(p) = p^n \int_{\mathbb{R}^n} f(x)^p dx \tag{1}$$

   is log-concave in $(0,\infty)$.

   (d) Let X be a random vector in $\mathbb{R}^n$ with log-concave density $f = e^{-\psi}$. Prove that $Var(\psi(X)) - n = (\log G)''(1)$ where G is given in (1).

   (e) Recall that $\mathbb{E}\psi(X) = \text{Ent}(X)$ and conclude Nguyen's varentropy bound

   $$Var(\psi(X)) \leq n.$$

6. (a) Improve the bound obtained in class, and establish the Rogers-Shepherd inequality: For any centrally-symmetric, convex body $K \subseteq \mathbb{R}^n$ and an ℓ-dimensional subspace $E \subseteq \mathbb{R}^n$,

   $$Vol_\ell(K \cap E) \cdot Vol_{n-\ell}(Proj_E^\perp K) \leq \binom{n}{\ell} Vol_n(K).$$

   (b) Prove the Spingarn inequality: If the barycenter of K lies at the origin, then,

   $$Vol_\ell(K \cap E) \cdot Vol_{n-\ell}(Proj_E^\perp K) \geq Vol_n(K)$$

   [Hint: $f(x) = Vol_n(K \cap (E + x))$ is log-concave + Jensen].

7. Some time ago we proved in class that for a 1-Lipschitz function $f : S^{n-1} \to \mathbb{R}$, and a random $k-$dimensional subspace $E \subseteq \mathbb{R}^n$,

   $$\mathbb{E} \sup_{x \in E} |f(x) - m| \leq C\sqrt{\frac{k}{n}},$$

   where $m = \int f d\sigma_{n-1}$. Prove that

   $$\mathbb{P}\left(\sup_{x \in E} |f(x) - m| \leq C\sqrt{\frac{k}{n}}\right) \geq 1 - Ce^{-ck},$$

   where $c, C > 0$ are universal constants.

8. A probability density f in $\mathbb{R}^n$ is *more peaked* than a probability density g in $\mathbb{R}^n$ if for any centrally-symmetric convex set $K \subseteq \mathbb{R}^n$,

$$\int_K f \geq \int_K g.$$

(a) Prove that if f is more peaked than g, then for any even, log-concave function $\rho : \mathbb{R}^n \to [0, \infty)$,

$$\int_{\mathbb{R}^n} \rho f \geq \int_{\mathbb{R}^n} \rho g.$$

(b) In one dimension, show that $1_{[-1/2,1/2]}(t)$ is more peaked than $\exp(-\pi t^2)$.

(c) Prove that if a log-concave $f_i(t)$ is more peaked than a log-concave $g_i(t)$ for $i = 1, \ldots, n$, for $t \in \mathbb{R}$, then $\prod_{i=1}^n f_i(x_i)$ is more peaked than $\prod_{i=1}^n g_i(x_i)$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

[Hint: Enough to prove for $n = 2$, and use Prékopa-Leindler]

(d) Conclude that uniform probability density on $Q = [-1/2, 1/2]^n$ is more peaked than the Gaussian density $\exp(-\pi|x|^2)$ in $\mathbb{R}^n$.

(e) Use the comparison with the Gaussian to prove Vaaler's theorem: For any subspace $E \subseteq \mathbb{R}^n$,

$$Vol_n(Q \cap E) \geq 1.$$