1. Metric Measure Spaces

A metric measure space is a triplet (X, d, μ) where X is a complete and separable space (e.g. a metric space, $\mathbb{R}^n$, S^n, a Riemanninan manfiold, $\{0,1\}^n$, etc.), d is a distance function and μ is a Borel measure.

1.1. **Examples.**

(1) $K \subseteq \mathbb{R}^n$ a convex set, d is Euclidean and μ is the Lebesque measure on K

(2) S^n with geodesic distance and uniform measure.

(3) $\mathbb{R}^n$ with Euclidean distance and the Gaussian measure γ_n where

$$\frac{d\gamma_n}{dx} = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$$

.

(4) A weighted graph with weight function $w : V \cup E \to \mathbb{R}$ on vertices and edges and distance

$$d(u, v) = \min_{\text{path from } u \text{ to } v} \text{length(path)}$$

measure $\mu(v) = w(v)$ or for paths $\mu(\gamma) = \sum_{e \in \gamma} w(e)$.

1.2. **Poincaré Inequality.**

Theorem 1 (Poincaré, 1892). *Let $K \subseteq \mathbb{R}^n$ be a convex, open, bounded set. Let $f : K \to \mathbb{R}$ be a C^1-smooth, $\int_K f = 0$. Then*

$$\mathrm{Var}_K(f) = \int_K f^2 d\mu \leq C_p(K) \int_K \|\nabla f\|^2$$

when $C_p(K) \leq 2^{n-1} \cdot \mathrm{diam}^2(K)$ and $C_p(K)$ is called the Poincaré constant.

We could think of the Poincaré as having units meters squared.

Proof. Normalize $K \mapsto \lambda K$ so $\mathrm{vol}(K) = 1$

$$\int_K f^2 = \frac{1}{2} \int_K \int_K |f(x) - f(y)|^2 dx dy \tag{1.1}$$

which follows from expanding the expression and noting that $\int_K f = 0$. We choose a path from x to y and by the Fundamental Theorem of Calculus

$$f(y) - f(x) = \int_0^1 \frac{d}{dt} f((1-t)x + ty) dt = \int_0^1 \nabla f((1-t)x + ty)(y - x) dt \tag{1.2}$$

since $|x - y| \leq \mathrm{diam}(K)$ we can bound $|f(x) - f(y)|$:

$$|f(x) - f(y)| \leq \int_0^1 \|\nabla f((1-t)x + ty)\| dt \cdot \mathrm{diam}(K) \tag{1.3}$$

by Caucy Schwartz inequality

$$|f(y) - f(x)|^2 \leq \mathrm{diam}^2(K) \int_0^1 |\nabla f((1-t)x + ty)|^2 \, dt \tag{1.4}$$

now we take an integral to obtain the inequality

$$\int_K f^2 \le \frac{1}{2}\mathrm{diam}^2 \int_K\int_K\int_0^1 \|\nabla f((1-t)x+ty)\|^2 dxdydt \tag{1.5}$$

$$= \mathrm{diam}^2(K)\int_{\frac{1}{2}}^1\int_K\int_K \|\nabla f((1-t)x+ty)\|^2 dydxdt \tag{1.6}$$

fix x, change variable $z=(1-t)x+ty$ for $t\ge\frac{1}{2}$

$$\int_K \|\nabla f((1-t)x+ty)\|^2 dy = \int_{(1-t)x+tK}\|\nabla f(z)\|^2\frac{dz}{t^n} \tag{1.7}$$

$$= 2^n\int_K \|\nabla f(z)\|^2 dz \tag{1.8}$$

Hence
(1.9)

$$\int_K f^2 \le \int_{\frac{1}{2}}^1 dt\int_K dx\int_K \|\nabla f(z)\|^2 dz\cdot 2^n\cdot \mathrm{diam}^2(K) = 2^{n-1}\cdot\mathrm{diam}^2(K)\cdot\int_K\|\nabla f(z)\|^2 dz$$

□

2. Introduction to the heat equation

Let $K\subseteq\mathbb{R}^n$ be a set ∂K is smooth but open. Write $u_t(x)$ for $t\ge 0$ and $x\in K$ for the temperature at the point $x\in K$ at time t. The heat equation is

$$\begin{cases}\frac{du_t}{dt}=\triangle u_t & x\in K\\ \langle\nabla u_t,\nu\rangle = 0 & x\in\partial K\end{cases}$$

where ν is a unit normal. These conditions are called Neumanns Boundary Conditions. These means that the heat is insulated by Fourier's Law. Here $\triangle u=\sum_i \partial^{ii}u$ is the Laplacian operator.

For every smooth $u_0 : K\mapsto\mathbb{R}$ there exists a solution to the heat equation, starting from u_0, and it is smooth in all variables.

Lemma (Preservation of Total Heat)**.**

$$\int_K u_t$$

is constant in t

Proof.

$$\begin{aligned}\frac{d}{dt}\int_K u_t &= \int_k \frac{du}{dt}\\ &= \int_K \nabla u_t\\ &= \int_K \mathrm{div}\,(\nabla u_t)\\ &= \int_{\partial K}\nabla u_t\cdot\nu = 0\end{aligned}$$

□

Proposition. *Suppose* $\mathrm{Vol}_n(K) = 1$. *Suppose that* $\int_K u_0 = 1$. *Then*

$$\|u_t - 1\|_{L^2(K)}^2 \le e^{-\frac{t}{c_p(K)}} \|u_0 - 1\|_{L^2(K)}^2$$

where $c_p(K)$ *the Poincaré coefficient, is the relaxation time.*

Proof. There exists an orthonormal basis for $\triangle$,

$$1 \equiv \varphi_0, \varphi_1, \dots \in L^2(K)$$

Define $\varphi_0 \equiv 1$, and φ_i are the minimizers of the Rayleigh quotient

$$\lambda_i = \inf_{u \perp \varphi_0, \dots, \varphi_{i-1}} \frac{\int_K \|\nabla u\|^2}{\int_K u^2} = R_k(u)$$

Then φ_i are eigenfunctions of $\triangle$ with eigenvalues

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots \to \infty$$

and satisify

$$\begin{cases} \triangle \varphi_i = -\lambda_i \varphi_i & \in K \setminus \partial K \\ \nabla \varphi_i \cdot \nu = 0 & \in \partial K \end{cases}$$

(Exercise: Prove this).
An example for a solution to the heat equation

$$u_t(x) = e^{-\lambda_i t} \varphi_i(x)$$

$$-\lambda_i u_t = \frac{d}{dt} u_t(x) = e^{-\lambda_i t} \triangle \varphi_i$$

The heat operator is $P_t(u_0) = u_t$ is diagonal in the basis $\varphi_0, \varphi_1, \dots$

$$P_t(\varphi_i) = e^{-\lambda_i t} \varphi_i$$

Notice that

$$\lambda_1 = \inf \left\{ \frac{\int_K |\nabla f|^2}{\int_K u^2} ; \int_K u = 0 \right\}$$

Thus

$$\lambda_1 \int_K u^2 \le \int_K |\nabla u|^2$$

for all $u : K \to \mathbb{R}$ such that $\int_K f = 0$. Given $u_0 : K \to \mathbb{R}$ such that $\int_K u_0 = 1$ we expand u_0 in the orthogonal basis

$$u_0 = \sum_{i=0}^{\infty} a_i \varphi_i = 1 + \sum_{i=1}^{\infty} a_i \varphi_i$$

when $a_i = \langle u_i, \varphi_i \rangle$. For example for $i = 0$, we have $a_0 = \int u_0 = 1$. Thus,

$$u_t = P_t(u_0) = 1 + \sum_{i=1}^{\infty} a_i e^{-\lambda_i t} \varphi_i$$

since $\lambda_i \ge \lambda_1 = \frac{1}{c_p(K)}$.

$$\|u_t - 1\|_{L^2(K)}^2 = \sum_{i=1}^{\infty} |a_i|^2 e^{-2\lambda_i t} \le e^{-\frac{2t}{c_p(K)}} \cdot \sum_{i=1}^{\infty} |a_i|^2 = e^{-\frac{2t}{c_p(K)}} \|u_0 - 1\|_{L^2(K)}^2$$

as desired □

We could think of the Poincaré coefficient as a measure of conductance of a set.

2.1. **Other ways to measure connnectivity / conductance.** Isoperimetric / Cheeger constant of $K \subseteq \mathbb{R}^n$ open

$$h_K = \inf_{A \subseteq K} \frac{\mathrm{Vol}_{n-1}(\partial A \cap K)}{\min \{\mathrm{Vol}_n(A), \mathrm{Vol}_n(K \setminus A)\}}$$

when ∂A is smooth.

Fact 2. *Cheeger Inequality (Under general assumption)*

$$h_K^2 \cdot C_p(K) \leq 4$$

Fact 3. *If $K \subseteq \mathbb{R}^n$ is convex then*

$$\frac{1}{9} \leq h_K^2 \cdot C_p(K) \leq 4$$

Cheeger Inequality on metric-measure spaces. (X, d, μ) is a metric-measure space with $\mu(X) = 1$. Let $A \subseteq X$ be a measurable set.

Definition 4. The surface area is defined as

$$\mu^+(\partial A) = \lim_{\varepsilon \to 0} \frac{\mu(N_\varepsilon(A)) - \mu(A)}{\varepsilon}$$

where $N_\varepsilon(A) = \{x \in X \mid d(x, A) < \varepsilon\}$

Definition 5. The isoperimetric / Cheeger constant is

$$h_X = \inf_{A \subseteq X, 0 < \mu(A) < 1} \frac{\mu^+(\partial A)}{\min \{\mu(A), 1 - \mu(A)\}}$$

Remark. for $X = K \subseteq \mathbb{R}^n$ convex the infimum in the Cheeger constant is attained and satisfies $\mu(A) = \frac{1}{2}$. What's the Poincare's constant $C_p(X)$?

$$\forall f \quad \mathrm{Var}(f) \leq C_p(X) \cdot \int_X |\nabla f|^2 \, d\mu$$

Definition 6. Let $f : X \to \mathbb{R}$ and f is Lipschitz on balls that is for every ball $B(x_0, r)$ exists L such that

$$|f(x) - f(y)| \leq L \cdot d(x, y) \quad \forall x, y \in B(x_0, r)$$

We define the gradient

$$|\nabla f|(x) = \lim_{\varepsilon \to 0} \sup_{0 < d(y,x) < \varepsilon} \frac{|f(y) - f(x)|}{d(x, y)}$$

Definition 7. $C_p(X)$ is the infimal $C \geq 0$ such that $\forall f : X \to \mathbb{R}$ Lipschitz on balls such that

$$\mathrm{Var}(\mathrm{f}) \leq \mathrm{C} \cdot \int_{\mathrm{X}} |\nabla \mathrm{f}|^2$$

Theorem 8. *Cheeger's Inequality*

$$h_X^2 \cdot C_p(X) \leq 4$$

Lemma 9. *For any Lipschitz $f : X \to \mathbb{R}$,*

$$\int_{-\infty}^{\infty} \mu^+ (\partial \{f \geq t\})\, dt \leq \int_X |\nabla f|\, d\mu$$

Remark 10. The co-area formula $\forall f : \mathbb{R}^n \to \mathbb{R}$

$$\int_{-\infty}^{\infty} \mathrm{vol}_{n-1(\{f=t\})} dt = \int_{\mathbb{R}^n} |\nabla f|$$

Remark 11. $\forall f : X \to [0, \infty)$ which μ-integrable

$$\int_X f d\mu = \int_X \int_0^{\infty} 1_{\{f(x) \geq t\}} dt d\mu = \int_0^{\infty} \mu(\{f \geq t\})\, dt$$

Remark 12. Co-area inequality can be indentity if we add regularity assumptions

Proof. (In the case when f in bounded) so $\sup |f| < \infty$. We may add a constant to f, and make it non-negative. Define $f_h(x) = \sup_{d(x,y)<h} f(y) \geq f(x)$. For any t

$$\{f_h > t\} = N_h(\{f > t\})$$

So

$$\int_X f_h d\mu = \int_0^{\infty} \mu(\{f_h > t\})\, dt = \int_0^{\infty} \mu(N_h(\{f > t\}))\, dt$$

For f we have

$$\int_X f d\mu = \int_0^{\infty} \mu(\{f > t\})\, dt$$

then

$$\int_X \frac{f_h - f}{h} d\mu = \int_0^{\infty} \frac{\mu(N_h(\{f > t\})) - \mu\{f > t\}}{h} dt$$

By the bounded convergence theorem

$$\int_X \frac{f_h - f}{h} = \int_X \frac{|f_h - f|}{h} \to \int |\nabla f|\, d\mu \leq \int_X \limsup_{h \to 0^+} \frac{f_h - f}{h} d\mu$$

By Fatou's lemma

$$\int_0^{\infty} \frac{\mu(N_h(\{f > t\})) - \mu\{f > t\}}{h} dt \geq \int_0^{\infty} \lim \inf_{h \to 0^+} \frac{\mu(N_h\{f > t\}) - \mu\{f > t\}}{h} dt$$

$$\geq \int_0^{\infty} \mu^+(\{f > t\})\, dt$$

□