19/11/20

Tutorial on Brunn-Minkowski

- A geometric inequality from 1887. Useful for isoperimetry, concentration, etc.

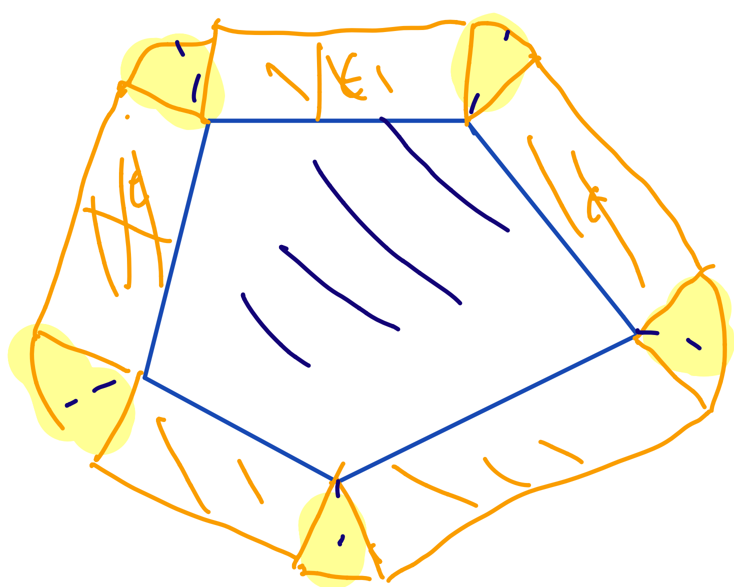
Minkowski sum: $A, B \subseteq \mathbb{R}^n$

$$A + B = \{x + y \; ; \; x \in A, \; y \in B\}$$

$$tA = \{tx \; ; \; x \in A\}$$

Example: • $B = B^n$ Euclidean unit ball,

$$A + tB^n = \text{t-nbhd of } A$$



$$\text{Area}(A + tB^n) = |A| + t|\partial A| + \pi t^2$$

A convex

- In general, in $\mathbb{R}^n$, (Minkowski's thm)

$K, T \subseteq \mathbb{R}^n$ convex, $\forall s \geq 0$

$|K + sT| =$ polynomial in s of degree n

$$= \mathrm{Vol}_n(K) + c_1 s + \cdots + c_{n-1} s^{n-1} + \mathrm{Vol}_n(T) s^n$$

- The coefficients of these polynomials are called <u>mixed volumes</u>, important in convexity. (Minkowski)

<u>Brunn-Minkowski inequality:</u> $\forall A, B \subseteq \mathbb{R}^n$, <u>non-empty</u>, Borel sets. Then,

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$

<u>Example:</u>

$$B(0,r_1) + B(0,r_2) = B(0,r_1+r_2)$$

$$|B(0,r_1+r_2)|^{\frac{1}{n}} = |B(0,r_1)|^{\frac{1}{n}} + |B(0,r_2)|^{\frac{1}{n}}$$

$$\begin{array}{ccc} \| & & \| \\ \kappa_n (r_1+r_2) & & \kappa_n r_1 + \kappa_n r_2 \end{array}$$

$$\kappa_n = \mathrm{Vol}_n(B(0,1)) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

- Equality case, when A, B are compact, is when A, B are <u>convex</u> and B is a homothetic translate of A.

<u>Application:</u> Isoperimetric inequality in $\mathbb{R}^n$.

$$\forall A \subseteq \mathbb{R}^n \quad \frac{|\partial A|}{|A|^{\frac{n-1}{n}}} \geq \frac{|\partial B|}{|B|^{\frac{n-1}{n}}} = n\,|B|^{\frac{1}{n}}.$$

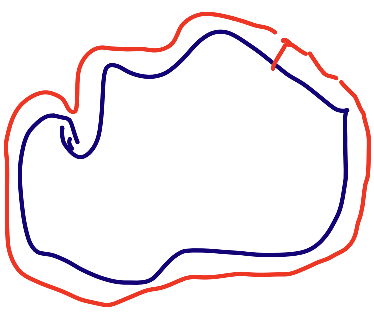
where $B = B(0,1)$ is a Euclidean unit ball

. In fact, we can say: $\forall A$

$$(*) \quad |A| = |B| \implies |A+\varepsilon B| \geq |B+\varepsilon B|$$

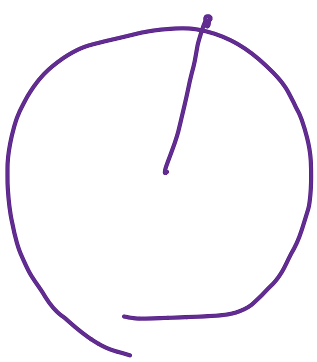
. The latter formulation implies the former:

$$|\partial A| = \lim_{\varepsilon \to 0} \frac{|A+\varepsilon B| - |A|}{\varepsilon}$$



In particular,

$$|\partial B| = \lim_{\varepsilon \to 0} \frac{|\overbrace{B+\varepsilon B}^{(1+\varepsilon)B}| - |B|}{\varepsilon}$$



$$= |B| \lim_{\varepsilon \to 0} \frac{(1+\varepsilon)^n - 1}{\varepsilon} = n|B|$$

Proof of $(*)$:

$$|A+\varepsilon B| \geq \left(|A|^{\frac{1}{n}} + |\varepsilon B|^{\frac{1}{n}}\right)^n$$
$$= \left(|B|^{\frac{1}{n}} + |\varepsilon B|^{\frac{1}{n}}\right)^n$$

However, for B we have equality in B-M, so

$$|B+\varepsilon B| = \left(|B|^{\frac{1}{n}} + |\varepsilon B|^{\frac{1}{n}}\right)^n$$

So

$$|A+\varepsilon B| \geq |B+\varepsilon B|.$$

- <u>The multiplicative form of B-M:</u>

$\forall\, K, T \subseteq \mathbb{R}^n$ Borel sets, $0 < \lambda < 1$,

$$|(1-\lambda)K + \lambda T| \geq |K|^{1-\lambda} \cdot |T|^{\lambda}$$

<u>Proof:</u> Follows from B-M and AM-GM, as

$$|(1-\lambda)K+\lambda T|^{\frac{1}{n}} \geq |(1-\lambda)K|^{\frac{1}{n}} + |\lambda T|^{\frac{1}{n}}$$
$$= (1-\lambda)|K|^{\frac{1}{n}} + \lambda |T|^{\frac{1}{n}}$$

$$\geq \left(|K|^{1-\lambda} \, |T|^{\lambda} \right)^{\frac{1}{n}}.$$

<u>Remark:</u> Easy to go back. In fact, the "main point" of B-M is

$$|K| = |T| = 1 \implies \left| \frac{K+T}{2} \right| \geq 1.$$

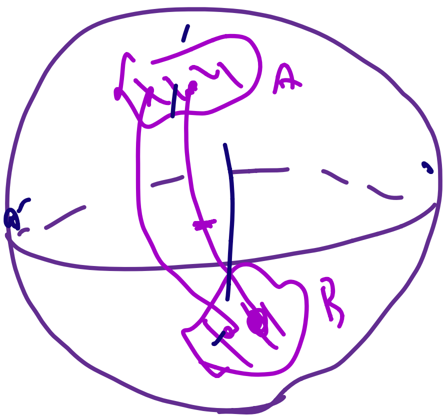
- The multiplicative version admits generalization to metric spaces, Riemannian mfld. Why

$$\frac{K+T}{2} = \left\{ \frac{x+y}{2} \,;\, x \in K,\ y \in T \right\}$$

$$= \left\{ \begin{array}{l} \text{all midpoints of geodesics connecting} \\ \text{a point in } K \text{ with a} \\ \text{point in } T \end{array} \right\}$$

B-M on the sphere: $K, T \subseteq S^{n-1}$

$$\frac{K+T}{2} = \left\{ \text{All midpoints of } \underline{\text{ALL}} \text{ minimizing geodesics connecting } K \text{ to } T \right\}$$



Indeed, B-M is true:

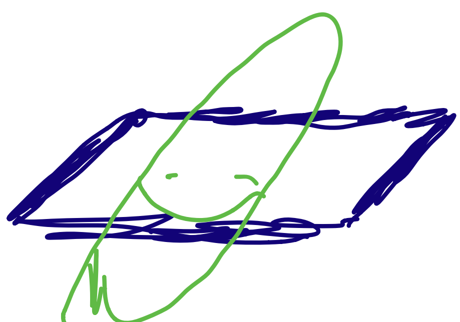
$$\left| \frac{K+T}{2} \right| \geq \sqrt{|K| \cdot |T|}$$

- In fact, true in mfld with non-negative Ricci curvature.

Proofs of Brunn-Minkowski

<u>Steiner symmetrization</u> (1838)

- Given $A \subseteq \mathbb{R}^n$ Borel set and a hyperplane $H \subseteq \mathbb{R}^n$
- We define another set

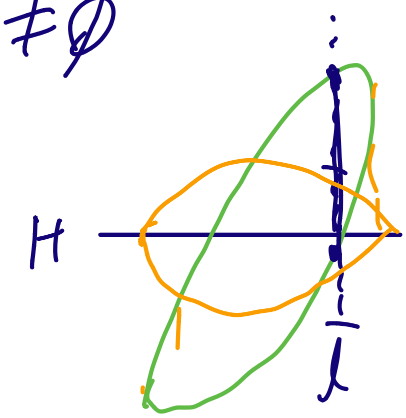


$S_H(A)$

The Steiner symmetral of A wrt H.

Def: Take any line $\ell \perp H$.

- $S_H(A) \cap \ell \neq \emptyset \iff A \cap \ell \neq \emptyset$
- The set $S_H(A) \cap \ell$ is an interval centered at H of length $|A \cap \ell|$.



Exercise: Ellipsoids go to ellipsoids.

Properties: $\forall A \subseteq \mathbb{R}^n$, $\forall$ hyperplane H |

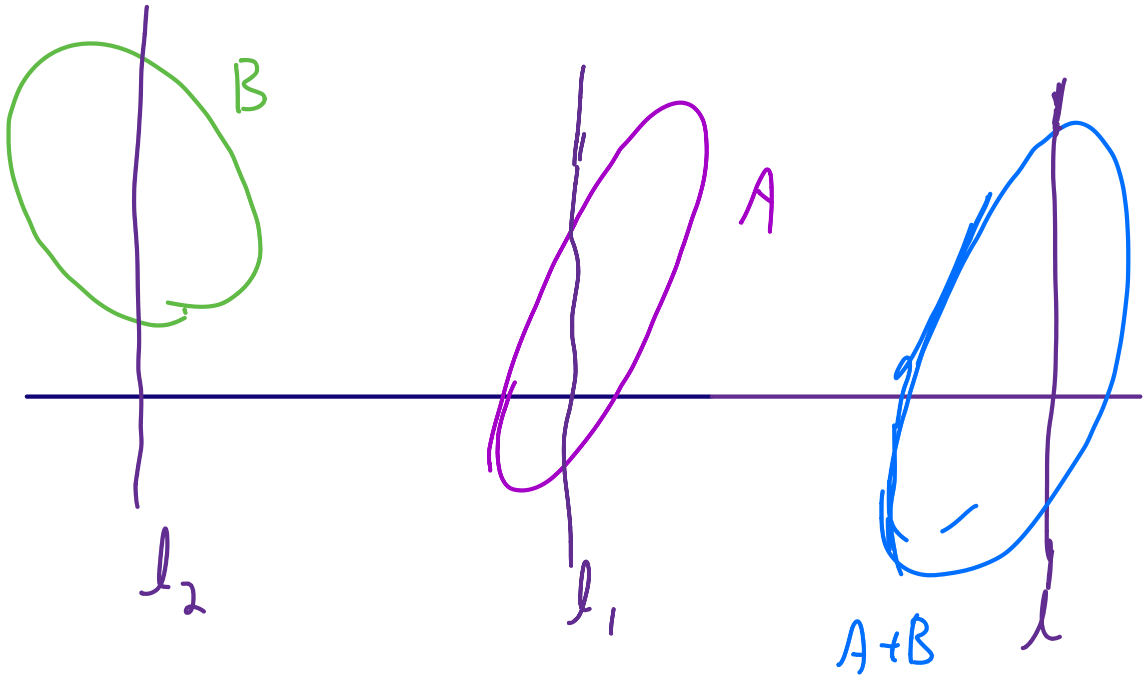
✓ 1) $|A| = |S_H(A)|$

✓ 2) $A \subseteq B \Rightarrow S_H(A) \subseteq S_H(B)$

3) A convex $\Rightarrow S_H(A)$ is convex

4) $S_H(A+B) \supseteq S_H(A)+S_H(B)$

Abort 4:



$\forall \ell$

$$(A+B)\cap \ell = \bigcup_{\ell_1+\ell_2=\ell} (A\cap \ell_1)+(B\cap \ell_2)$$

$$(S_H(A)+S_H(B))\cap \ell$$

$$= \bigcup_{\ell_1+\ell_2=\ell} \underbrace{(S_H(A)\cap \ell_1)+(S_H(B)\cap \ell_2)}_{\text{first center the intervals and then add them and take the union}}$$

Just the maximal interval

$S_H(A+B)\cap \ell$ = first add the intervals, take their union, and center

Fill in the little argument.

5) Guided Exercise: For any compact $A \subseteq \mathbb{R}^n$ $\exists$ hyperplanes $H_1, H_2, H_3, \dots \subseteq \mathbb{R}^n$ such that

$$S_{H_n}\left(\sim S_{H_2}\left(S_{H_1}(A)\right)\right) \xrightarrow[N\to\infty]{} B$$

in Hausdorff metric, where B is a Euclidean ball with $|A| = |B|$.

· Remark: Random choice of H_i's work.

Proof of Brunn-Minkowski:

Choose $H_1, H_2, \ldots$ that work simultaneously for A, B, $A+B$. Then

$$S_{H_1}(A+B) \supseteq S_{H_1}(A) + S_{H_1}(B)$$

$\vdots$

$$S_{H_N} \cdots S_{H_1}(A+B) \supseteq S_{H_N} \cdots S_{H_1}(A) + S_{H_N} \cdots S_{H_1}(B)$$

$$\downarrow \qquad\qquad \downarrow$$

$$B\left(0, c_n |A+B|^{\frac{1}{n}}\right) \supseteq \underbrace{B\left(0, c_n |A|^{\frac{1}{n}}\right) + B\left(0, c_n |B|^{\frac{1}{n}}\right)}_{\substack{\parallel \\ B\left(0,\, c_n\left(|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}\right)\right)}}$$

where $c_n = \mathrm{Vol}_n(B^n)^{-\frac{1}{n}}$.

Hence

$$c_n |A+B|^{\frac{1}{n}} \geq c_n\left(|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}\right)$$

$\blacksquare$