Logarithmically-concave moment measures I

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Abstract We discuss a certain Riemannian metric, related to the toric Kähler-Einstein equation, that is associated in a linearly-invariant manner with a given log-concave measure in $\mathbb{R}^n$. We use this metric in order to bound the second derivatives of the solution to the toric Kähler-Einstein equation, and in order to obtain spectral-gap estimates similar to those of Payne and Weinberger.

1 Introduction

In this paper we explore a certain geometric structure related to the *moment measure* of a convex function. This geometric structure is well-known in the community of complex geometers, see, e.g., Donaldson [13] for a discussion from the perspective of Kähler geometry.

Our motivation stems from the Kannan-Lovasź-Simonovits conjecture [17, Section 5], which is concerned with the isoperimetric problem for high-dimensional convex bodies. Essentially, our idea is to replace the standard Euclidean metric by a special Riemannian metric on the given convex body K. This Riemannian metric has many favorable properties, such as a Poincaré inequality with constant one, a positive Ricci tensor, the linear functions are eigenfunctions of the Laplacian, etc. Perhaps this alternative geometry does not deviate too much from the standard Euclidean geometry on K, and it is conceivable that the study of this Riemannian metric will turn out to be relevant to the Kannan-Lovasź-Simonovits conjecture.

Let μ be an arbitrary Borel probability measure on $\mathbb{R}^n$ whose barycenter is at the origin. Assume furthermore that μ is not supported in a hyperplane. It was proven in [12] that there exists an essentially-continuous convex function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, uniquely determined up to translations, such that μ is the *moment measure*

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of ψ, i.e.,

$$\int_{\mathbb{R}^n} b(y)d\mu(y) = \int_{\mathbb{R}^n} b(\nabla\psi(x))e^{-\psi(x)}dx \tag{1}$$

for any μ-integrable function $b : \mathbb{R}^n \to \mathbb{R}$. In other words, the gradient map $x \mapsto \nabla\psi(x)$ pushes the probability measure $e^{-\psi(x)}dx$ forward to μ. The argument in [12] closely follows the variational approach of Berman and Berndtsson [5], which succeeded the continuity methods of Wang and Zhu [29] and Donaldson [13].

Even in the case where μ is absolutely-continuous with a C^∞-smooth density, it is not guaranteed that ψ is differentiable. From the regularity theory of the Brenier map, developed by Caffarelli [9] and Urbas [28], we learn that in order to conclude that ψ is sufficiently smooth, one has to assume that the support of μ is convex.

An absolutely-continuous probability measure on $\mathbb{R}^n$ is called *log-concave* if it is supported on an open, convex set $K \subset \mathbb{R}^n$, and its density takes the form $\exp(-\rho)$ where the function $\rho : K \to \mathbb{R}$ is convex. An important example of a log-concave measure is the uniform probability measure on a convex body in $\mathbb{R}^n$. Here we assume that μ is log-concave and furthermore, we require that the following conditions are met:

(2) The convex set $K \subset \mathbb{R}^n$ is bounded, the function ρ is C^∞-smooth, and ρ and its derivatives of all orders are bounded in K.

Under these regularity assumptions, we can assert that

(3) The convex function ψ is finite and C^∞-smooth in the entire $\mathbb{R}^n$.

The validity of (3) under the assumption (2) was proven by Wang and Zhu [29] and by Donaldson [13] via the continuity method. Berman and Berndtsson [5] explained how to deduce (3) from (2) by using Caffarelli's regularity theory [9]. In fact, the argument in [5] requires only the boundness of ρ, and not of its derivatives, see also the Appendix in Alesker, Dar and Milman [2]. Since the function ψ is smooth, it follows from (1) that the transport equation

$$e^{-\rho(\nabla\psi(x))}\det\nabla^2\psi(x) = e^{-\psi(x)} \tag{4}$$

holds everywhere in $\mathbb{R}^n$, where $\nabla^2\psi(x)$ is the Hessian matrix of ψ. In the case where $\rho \equiv Const$, equation (4) is called the *toric Kähler-Einstein equation*. We write $x \cdot y$ for the standard scalar product of $x, y \in \mathbb{R}^n$, and $|x| = \sqrt{x \cdot x}$.

Theorem 1. *Let μ be a log-concave probability measure on $\mathbb{R}^n$ with barycenter at the origin that satisfies the regularity conditions (2). Then, with the above notation, for any $x \in \mathbb{R}^n$,*

$$\Delta\psi(x) \leq 2R^2(K)$$

where $R(K) = \sup_{x\in K}|x|$ is the outer radius of K, and $\Delta\psi = \sum_i \partial^2\psi/\partial x_i^2$ is the Laplacian of ψ.

Theorem 1 is proven by analyzing a certain *weighted Riemannian manifold*. A weighted Riemannian manifold, sometimes called a *Riemannian metric-measure* *space*, is a triple

$$X = (\Omega, g, \mu)$$

where Ω is a smooth manifold (usually an open set in $\mathbb{R}^n$), where g is a Riemannian metric on Ω, and μ is a measure on Ω with a smooth density with respect to the Riemannian volume measure. In this paper we study the weighted Riemannian manifold

$$M_\mu^* = \left(\mathbb{R}^n, \nabla^2 \psi, e^{-\psi(x)} dx \right). \tag{5}$$

That is, the measure associated with M_μ^* has density $e^{-\psi}$ with respect to the Lebesgue measure on $\mathbb{R}^n$, and the Riemannian tensor on $\mathbb{R}^n$ which is induced by the Hessian of ψ is

$$\sum_{i,j=1}^n \psi_{ij} dx^i dx^j, \tag{6}$$

where we abbreviate $\psi_{ij} = \partial^2 \psi / \partial x^i \partial x^j$. There is also a dual description of M_μ^*. Recall that the Legendre transform of $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the convex function

$$f^*(x) = \sup_{\substack{y \in \mathbb{R}^n \\ f(y) < +\infty}} [x \cdot y - f(y)] \qquad\qquad (x \in \mathbb{R}^n).$$

We refer the reader to Rockafellar [26] for the basic properties of the Legendre transform. Denote $\varphi = \psi^*$. From (4) we see that the Hessian matrix of the convex function ψ is always invertible, hence it is positive-definite. Therefore φ is a smooth function in K whose Hessian is always positive-definite. Consequently, the map $\nabla \varphi : K \to \mathbb{R}^n$ is a diffeomorphism, and $\nabla \psi$ is its inverse map. One may directly verify that the weighted Riemannian manifold M_μ^* is canonically isomorphic to

$$M_\mu = \left(K, \nabla^2 \varphi, \mu \right),$$

with $x \mapsto \nabla \psi(x)$ being the isomorphism map. In differential geometry, the isomorphism between M_μ and M_μ^* is the passage from complex coordinates to action/angle coordinates, see, e.g., Abreu [1]. Here are some basic properties of our weighted Riemannian manifold:

(i) The space M_μ is stochastically complete. That is, the diffusion process associated with M_μ is well-defined, it has μ as a stationary measure and "it never reaches the boundary of K".
(ii) The Bakry-Émery-Ricci tensor of M_μ is positive. In fact, it is at least half of the Riemannian metric tensor.
(iii) The Laplacian associated with M_μ has an interesting spectrum: The first nonzero eigenvalue is -1, and the corresponding eigenspace contains all linear functions.

Property (ii) is a particular case of the results of Kolesnikov [23, Theorem 4.3] (the notation of Kolesnikov is related to ours via $V = \Phi = \psi$), and properties (i) and (iii) are discussed below.

It is important to note that the construction of M_μ does not rely on the Euclidean structure: Suppose that V is a real n-dimensional linear space and μ is a probability measure on V satisfying the assumptions of Theorem 1. Then the convex function $\psi : V^* \to \mathbb{R}$ whose moment measure is μ is well-defined up to translations, and it induces the weighted Riemannian manifolds M_μ and M_μ^* via the procedure described above. The fact that M_μ is well-defined without any reference to a Euclidean structure is in sharp contrast with the Riemannian metric-measure space $(\mathbb{R}^n, |\cdot|, \mu)$ that is frequently used for the analysis of the log-concave measure μ.

In the following sections we prove the assertions made in the Introduction, and as a sample of possible applications, we explain below how to recover the classical Payne-Weinberger spectral gap inequality [25], up to a constant factor:

Corollary 1. *Let μ be a log-concave probability measure on $\mathbb{R}^n$ with barycenter at the origin that satisfies the regularity conditions (2). Then, for any μ-integrable, smooth function $f : K \to \mathbb{R}$,*

$$\int_K f^2 d\mu - \left(\int_K f d\mu \right)^2 \leq 2R^2(K) \int_K |\nabla f|^2 d\mu. \tag{7}$$

The constant $2R^2(K)$ on the right-hand side of (7) is not optimal. In the case where μ is the uniform probability measure on a convex body $K \subset \mathbb{R}^n$ with a central symmetry (i.e., $K = -K$), the best possible constant is $4R^2(K)/\pi^2$, see Payne and Weinberger [25].

Throughout this note, a convex body in $\mathbb{R}^n$ is a bounded, open, convex set. We write $\log$ for the natural logarithm. A smooth function or a smooth manifold are C^∞-smooth. The unit sphere is $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$. The five sections below use a variety of techniques, from Itô calculus to maximum principles. We tried to make each section as independent of the others as possible.

Acknowledgements. The author would like to thank Bo Berndtsson, Dario Cordero-Erausquin, Ronen Eldan, Alexander Kolesnikov, Eveline Legendre, Emanuel Milman, Ron Peled, Yanir Rubinstein and Boris Tsirelson for interesting discussions related to this work. Supported by a grant from the European Research Council (ERC).

2 Continuity of the moment measure

This section is concerned with the continuity of the correspondence between convex functions and their moment measures. Our main result here is Proposition 1 below. We say that a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ is *centered* if

$$\int_{\mathbb{R}^n} e^{-\psi(x)} dx = 1, \qquad \int_{\mathbb{R}^n} x_i e^{-\psi(x)} dx = 0, \ i = 1, \ldots, n. \tag{8}$$

The role of the barycenter condition in (8) is to prevent translations of ψ which result in the same moment measure. It is well-known that any convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\int e^{-\psi} = 1$ must tend to $+\infty$ at infinity. More precisely, for any such convex function ψ there exist $A, B > 0$ with

$$\psi(x) \geq A|x| - B \qquad (x \in \mathbb{R}^n), \tag{9}$$

see, e.g., [19, Lemma 2.1]).

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be a compact set, and let $\psi, \psi_1, \psi_2, \ldots : \mathbb{R}^n \to \mathbb{R}$ be centered, convex functions. Denote by $\mu, \mu_1, \mu_2, \ldots$ the corresponding moment measures, which are assumed to be supported in Ω. Then the following are equivalent:*

(i) $\psi_\ell \longrightarrow \psi$ pointwise in $\mathbb{R}^n$.
(ii) $\mu_\ell \longrightarrow \mu$ weakly (i.e., $\int b d\mu_\ell \to \int b d\mu$ for any continuous function $b : \Omega \to \mathbb{R}$).

Several lemmas are required for the proof of Proposition 1. For a centered, convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ we define

$$K(\psi) = \left\{ x \in \mathbb{R}^n \, ; \, \psi(x) \leq 2n + \inf_{y \in \mathbb{R}^n} \psi(y) \right\},$$

a convex set in $\mathbb{R}^n$. Since the barycenter of $e^{-\psi(x)}dx$ lies at the origin, then $\psi(0) \leq n + \inf_{x \in \mathbb{R}^n} \psi(x)$, according to Fradelizi [14]. Hence the origin is necessarily in the interior of $K(\psi)$. For $x \in \mathbb{R}^n$ consider the Minkowski functional

$$\|x\|_\psi = \inf \left\{ \lambda > 0; x/\lambda \in K(\psi) \right\}.$$

Since a convex function is continuous, then $\psi(x/\|x\|_\psi) = 2n + \inf \psi$ for $0 \neq x \in \mathbb{R}^n$.

Lemma 1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a centered, convex function. Then,*

$$\psi(x) \geq n\|x\|_\psi + \psi(0) - 2n \qquad (x \in \mathbb{R}^n). \tag{10}$$

Proof. Since the barycenter of $e^{-\psi(x)}dx$ lies at the origin, from Fradelizi [14],

$$\psi(0) \leq n + \inf_{x \in \mathbb{R}^n} \psi(x). \tag{11}$$

Whenever $x \in K(\psi)$ we have $\|x\|_\psi \leq 1$. Therefore (10) follows from (11) for $x \in K(\psi)$. In order to prove (10) for $x \notin K(\psi)$, we observe that for such x we have $\|x\|_\psi \geq 1$ and hence

$$\psi(0) + n \leq \inf_{y \in \mathbb{R}^n} \psi(y) + 2n = \psi\left(\frac{x}{\|x\|_\psi} \right) \leq \left(1 - \frac{1}{\|x\|_\psi} \right) \cdot \psi(0) + \frac{1}{\|x\|_\psi} \cdot \psi(x),$$

due to the convexity of ψ. We conclude that $\psi(x) \geq \psi(0) + n\|x\|_\psi$ for any $x \notin K(\psi)$, and (10) is proven in all cases. □

Proof of the direction (i) ⇒ (ii) in Proposition 1. Denote

$$K = \{x \in \mathbb{R}^n; \psi(x) \leq 2n + 1 + \psi(0)\},$$

a convex set containing a neighborhood of the origin. Since $e^{-\psi}$ is integrable, then K must be of finite volume, hence bounded. According to Rockafellar [26, Theorem 10.8], the convergence of ψ_ℓ to ψ is locally uniform in $\mathbb{R}^n$. In particular, the convergence is uniform on K, and there exists $\ell_0 \geq 1$ such that $\psi_\ell(x) > 2n + \psi_\ell(0)$ for any $x \in \partial K$ and $\ell \geq \ell_0$. Setting $M = \psi(0) - 1$ we conclude that

$$K(\psi_\ell) \subseteq K, \quad \psi_\ell(0) \geq M \qquad \text{for all } \ell \geq \ell_0. \tag{12}$$

Denote $R = \sup_{x \in K} |x|$. From (12) and Lemma 1, for any $\ell \geq \ell_0$,

$$\psi_\ell(x) \geq n\|x\|_{\psi_\ell} + \psi_\ell(0) - 2n \geq \frac{n}{R}|x| + (M - 2n) \qquad (x \in \mathbb{R}^n). \tag{13}$$

According to our assumption (i) and [26, Theorem 24.5] we have that

$$\nabla\psi_\ell(x) \stackrel{\ell\to\infty}{\longrightarrow} \nabla\psi(x)$$

for any $x \in \mathbb{R}^n$ in which $\psi, \psi_1, \psi_2, \ldots$ are differentiable. Let $b : \Omega \to \mathbb{R}$ be a continuous function. Since a convex function is differentiable almost everywhere, we conclude that

$$b(\nabla\psi_\ell(x))e^{-\psi_\ell(x)} \stackrel{\ell\to\infty}{\longrightarrow} b(\nabla\psi(x))e^{-\psi(x)} \quad \text{for almost any } x \in \mathbb{R}^n.$$

The function b is bounded because Ω is compact. We may use the dominated convergence theorem, thanks to (13), and conclude that

$$\int_\Omega b d\mu_\ell = \int_{\mathbb{R}^n} b(\nabla\psi_\ell(x))e^{-\psi_\ell(x)}dx \stackrel{\ell\to\infty}{\longrightarrow} \int_{\mathbb{R}^n} b(\nabla\psi(x))e^{-\psi(x)}dx = \int_\Omega b d\mu.$$

Thus (ii) is proven. □

It still remains to prove the direction (ii) ⇒ (i) in Proposition 1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is L-Lipschitz if $|f(x) - f(y)| \leq L|x - y|$ for any $x, y \in \mathbb{R}^n$.

Lemma 2. *Let $L, \varepsilon > 0$. Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a centered, L-Lipschitz, convex function, such that*

$$\int_{\mathbb{R}^n} |\nabla\psi(x) \cdot \theta| e^{-\psi(x)}dx \geq \varepsilon \qquad \textit{for all } \theta \in S^{n-1}. \tag{14}$$

Then,

$$\alpha|x| - \beta \leq \psi(x) \leq L|x| + \gamma \qquad (x \in \mathbb{R}^n), \tag{15}$$

where $\alpha, \beta, \gamma > 0$ are constants depending only on L, ε and n.

Proof. Fix $\theta \in S^{n-1}$ and set $H = \theta^{\perp}$, the hyperplane orthogonal to θ. The function

$$m_\theta(y) = \inf_{t \in \mathbb{R}} \psi(y + t\theta) \qquad\qquad (y \in H)$$

is convex. Furthermore, for any fixed $y \in H$, the function $t \mapsto \psi(y + t\theta)$ is convex, L-Lipschitz and tends to $+\infty$ as $t \to \pm\infty$. Hence the one-dimensional convex function $t \mapsto \psi(y + t\theta)$ attains its minimum at a certain point $t_0 \in \mathbb{R}$, is non-decreasing on $[t_0, +\infty)$ and non-increasing on $(-\infty, t_0]$. Therefore, for any $y \in H$,

$$\int_{-\infty}^{\infty} \left| \frac{\partial \psi(y + t\theta)}{\partial t} \right| e^{-\psi(y+t\theta)} dt = \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial t} e^{-\psi(y+t\theta)} \right| dt = 2 e^{-m_\theta(y)}.$$

We now integrate over $y \in H$ and use Fubini's theorem to conclude that

$$\int_{\mathbb{R}^n} |\nabla \psi(x) \cdot \theta| e^{-\psi(x)} dx = 2 \int_H e^{-m_\theta(y)} dy. \tag{16}$$

Consider the interval

$$I_\theta = \{ t \in \mathbb{R} \, ; \, t\theta \in K(\psi) \} \, . \tag{17}$$

Then,

$$\int_{-\infty}^{\infty} e^{-\psi(t\theta)/2} dt \geq \int_{I_\theta} e^{-\psi(t\theta)/2} dt \geq e^{-n - \frac{m_\theta(0)}{2}} |I_\theta| \tag{18}$$

where $|I_\theta|$ is the length of the interval I_θ. Fix a point $y \in H$. Then there exists $t_0 \in \mathbb{R}$ for which $m_\theta(y) = \psi(y + t_0 \theta)$. From (18) and from the convexity of ψ,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\psi\left(\frac{y}{2} + t\theta\right)} dt = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\psi\left(\frac{y + t_0\theta}{2} + \frac{t\theta}{2}\right)} dt \ &\geq \ \frac{1}{2} e^{-\frac{m_\theta(y)}{2}} \int_{-\infty}^{\infty} e^{-\frac{\psi(t\theta)}{2}} dt \\ &\geq \frac{1}{2} e^{-\frac{m_\theta(y) + m_\theta(0)}{2}} e^{-n} |I_\theta| \ \geq \ \frac{1}{2} e^{-m_\theta(y)} e^{-2n} |I_\theta|, \end{aligned} \tag{19}$$

where in the last passage we used that $m_\theta(0) \leq \psi(0) \leq n + \inf \psi \leq n + m_\theta(y)$, because the barycenter of $e^{-\psi(x)}dx$ lies at the origin. Integrating (19) over $y \in H$, we see that

$$\int_H e^{-m_\theta(y)} dy \leq \frac{2e^{2n}}{|I_\theta|} \int_H \int_{-\infty}^{\infty} e^{-\psi\left(\frac{y}{2} + t\theta\right)} dt dy = \frac{2^n e^{2n}}{|I_\theta|} \int_{\mathbb{R}^n} e^{-\psi} = \frac{2^n e^{2n}}{|I_\theta|}.$$

Combine the last inequality with (14) and (16). This leads to the bound

$$|I_\theta| \leq C_n \left(\int_{\mathbb{R}^n} |\nabla \psi(x) \cdot \theta| e^{-\psi(x)} dx \right)^{-1} \leq \frac{C_n}{\varepsilon}, \tag{20}$$

for some constant C_n depending only on n. Recall that the origin belongs to $K(\psi)$ and hence $0 \in I_\theta$. By letting θ range over all of S^{n-1} and glancing at (17) and (20), we see that

$$K(\psi) \subseteq B\left(0, C_n/\varepsilon\right) \tag{21}$$

where $B(x,r) = \{y \in \mathbb{R}^n; |y-x| \leq r\}$. From (21) and from Lemma 1,

$$\psi(x) \geq \psi(0) - 2n + n\|x\|_\psi \geq \psi(0) - 2n + \frac{\varepsilon}{\tilde{C}_n}|x| \qquad (x \in \mathbb{R}^n), \tag{22}$$

for $\tilde{C}_n = C_n/n$. By integrating (22) we obtain

$$1 = \int_{\mathbb{R}^n} e^{-\psi} \leq e^{-(\psi(0)-2n)} \int_{\mathbb{R}^n} e^{-\varepsilon|x|/\tilde{C}_n} dx.$$

Therefore, $\psi(0) \leq \gamma$ for $\gamma = 2n + \log(\int_{\mathbb{R}^n} e^{-\varepsilon|x|/\tilde{C}_n} dx)$. Since ψ is L-Lipschitz, then the right-hand side inequality of (15) follows. Next, observe that

$$1 = \int_{\mathbb{R}^n} e^{-\psi(x)} dx \geq \int_{\mathbb{R}^n} e^{-\psi(0) - L|x|} dx = e^{-\psi(0)} \int_{\mathbb{R}^n} e^{-L|x|} dx.$$

Hence $\psi(0) \geq \log(\int_{\mathbb{R}^n} e^{-L|x|} dx)$, and the left-hand side inequality of (15) follows from (22). □

Proof of the direction (ii) ⇒ (i) in Proposition 1.

Step 1. We claim that

$$\liminf_{\ell \to \infty} \left(\inf_{\theta \in S^{n-1}} \int_\Omega |x \cdot \theta| d\mu_\ell(x) \right) > 0. \tag{23}$$

Assume that (23) fails. Then there exist sequences $\ell_j \in \mathbb{N}$ and $\theta_j \in S^{n-1}$ such that

$$\lim_{j \to \infty} \int_\Omega |x \cdot \theta_j| d\mu_{\ell_j}(x) = 0. \tag{24}$$

Passing to a subsequence, if necessary, we may assume that $\theta_j \longrightarrow \theta_0 \in S^{n-1}$. The sequence of functions $|x \cdot \theta_j|$ tends to $|x \cdot \theta_0|$ uniformly in $x \in \Omega$. Hence, from (ii) and (24),

$$\int_\Omega |x \cdot \theta_0| d\mu(x) = \lim_{j \to \infty} \int_\Omega |x \cdot \theta_0| d\mu_{\ell_j}(x) = \lim_{j \to \infty} \int_\Omega |x \cdot \theta_j| d\mu_{\ell_j}(x) = 0.$$

Therefore μ is supported in the hyperplane $\theta_0^\perp$. However, μ is the moment measure of the convex function $\psi : \mathbb{R}^n \to \mathbb{R}$, and according to [12, Proposition 1], it cannot be supported in a hyperplane. We have thus arrived at a contradiction, and (23) is proven.

Step 2. We will prove that there exist $\alpha, \beta, \gamma > 0$ and $\ell_0 \geq 1$ such that

$$\alpha|x| - \beta \leq \psi_\ell(x) \leq L|x| + \gamma \qquad (\ell \geq \ell_0, x \in \mathbb{R}^n). \tag{25}$$

Indeed, according to Step 1, there exists $\ell_0 \geq 1$ and $\varepsilon_0 > 0$ such that

$$\int_{\mathbb{R}^n} |\nabla \psi_\ell(x) \cdot \theta| e^{-\psi_\ell(x)} dx = \int_{\Omega} |x \cdot \theta| d\mu_\ell(x) > \varepsilon_0 \qquad (\ell \geq \ell_0, \theta \in S^{n-1}). \quad (26)$$

Denote $L = \sup_{x \in \Omega} |x|$. The function ψ_ℓ is centered and convex. Furthermore, for almost any $x \in \mathbb{R}^n$ we know that $\nabla \psi_\ell(x) \in \Omega$, because the moment measure of ψ_ℓ is supported in Ω. Hence, for $\ell \geq 1$,

$$|\nabla \psi_\ell(x)| \leq L \qquad \text{for almost any } x \in \mathbb{R}^n. \quad (27)$$

Since a convex function is always locally-Lipschitz, then (27) implies that ψ_ℓ is L-Lipschitz, for any ℓ. We may now apply Lemma 2, thanks to (26), and conclude (25).

Step 3. Assume by contradiction that there exists $x_0 \in \mathbb{R}^n$ for which $\psi_\ell(x_0)$ does not converge to $\psi(x_0)$. Then there exist $\varepsilon > 0$ and a subsequence ℓ_j such that

$$|\psi_{\ell_j}(x_0) - \psi(x_0)| \geq \varepsilon \qquad (j = 1, 2, \ldots). \quad (28)$$

From (25) we know that the sequence of functions $\{\psi_{\ell_j}\}_{j=1,2,\ldots}$ is uniformly bounded on any compact subset of $\mathbb{R}^n$. Furthermore, ψ_{ℓ_j} is L-Lipschitz for any j. According to the Arzelá-Ascoli theorem, we may pass to a subsequence and assume that ψ_{ℓ_j} converges locally uniformly in $\mathbb{R}^n$, to a certain function F. The function F is convex and L-Lipschitz, as it is the limit of convex and L-Lipschitz functions. Furthermore, thanks to (25) we may apply the dominated convergence theorem and conclude that F is centered.

To summarize, the functions $F, \psi_{\ell_1}, \psi_{\ell_2}, \ldots$ are L-Lipschitz, centered and convex. We know that $\psi_{\ell_j} \longrightarrow F$ locally uniformly in $\mathbb{R}^n$. According to the implication (i) $\Rightarrow$ (ii) proven above, the sequence of measure $\{\mu_{\ell_j}\}_{j=1,2,\ldots}$ converges weakly to the moment measure of F. But we assumed that μ_{ℓ_j} converges weakly to μ, and hence μ is the moment measure of F. Thus $\psi, F : \mathbb{R}^n \to \mathbb{R}$ are two centered, convex functions with the same moment measure μ. This means that $\psi \equiv F$, according to the uniqueness part in [12]. Therefore $\psi_{\ell_j} \longrightarrow \psi$ pointwise in $\mathbb{R}^n$, in contradiction to (28), and the proof is complete. □

3 A preliminary weak bound using the maximum principle

In this section we prove a rather weak form of Theorem 1, which will be needed for the proof of the theorem later on in Section 5. Throughout this section, μ is a log-concave probability measure on $\mathbb{R}^n$ with barycenter at the origin, supported on a convex body $K \subset \mathbb{R}^n$, with density $e^{-\rho}$ satisfying the regularity conditions (2). Also, $\psi : \mathbb{R}^n \to \mathbb{R}$ is the smooth, convex function whose moment measure is μ, which is uniquely defined up to translation, and $\varphi = \psi^*$ is its Legendre transform. In this section we make the following strict-convexity assumptions:

(⋆) The convex body K has a smooth boundary and its Gauss curvature is positive everywhere. Additionally, there exists $\varepsilon_0 > 0$ with

$$\nabla^2 \rho(x) \geq \varepsilon_0 \cdot Id \qquad (x \in K), \tag{29}$$

in the sense of symmetric matrices.

Denote by $\|A\|$ the operator norm of the matrix A. Our goal in this section is to prove the following:

Proposition 2. *Under the above assumptions,*

$$\sup_{x \in \mathbb{R}^n} \|\nabla^2 \psi(x)\| < +\infty.$$

The argument we present for the demonstration of Proposition 2 closely follows the proof of Caffarelli's contraction theorem [10, Theorem 11]. An alternative approach to Proposition 2 is outlined in Kolesnikov [22, Section 6]. We begin the proof of Proposition 2 with the following lemma, which is due to Berman and Berndtsson [5]. Their proof is reproduced here for completeness.

Lemma 3. $\sup_{x \in K} \varphi(x) < +\infty$.

Proof. Since K is bounded, it suffices to show that φ is α-Hölder for some $\alpha > 0$. According to the Sobolev inequality in the convex domain $K \subset \mathbb{R}^n$ (see, e.g., [27, Chapter 1]), it is sufficient to prove that

$$\int_K |\nabla \varphi(x)|^p dx < +\infty, \tag{30}$$

for some $p > n$. Fix $p > n$. The map $x \mapsto \nabla \varphi(x)$ pushes the measure μ forward to $\exp(-\psi(x))dx$. Hence,

$$\int_K |\nabla \varphi|^p d\mu = \int_{\mathbb{R}^n} |x|^p e^{-\psi(x)} dx < +\infty, \tag{31}$$

where we used the fact that $e^{-\psi}$ decays exponentially at infinity (see, e.g., (9) above or [19, Lemma 2.1]). Since ρ is a bounded function on K and $e^{-\rho}$ is the density of μ, then (30) follows from (31). □

For $x \in \mathbb{R}^n$ denote $h_K(x) = \sup_{y \in K} x \cdot y$, the supporting functional of K. The following lemma is analogous to [10, Lemma 4].

Lemma 4. $\lim_{R \to \infty} \sup_{|x| \geq R} |\nabla \psi(x) - \nabla h_K(x)| = 0$.

Proof. The function $\varphi : K \to \mathbb{R}$ is convex, hence bounded from below by some affine function, which in turn is greater than some constant on the bounded set K. According to Lemma 3, the function φ is also bounded from above. Set $M = \sup_{x \in K} |\varphi(x)|$. By elementary properties of the Legendre transform, for any $x \in \mathbb{R}^n$,

$$\psi(x) = x \cdot \nabla\psi(x) - \varphi(\nabla\psi(x)) \leq x \cdot \nabla\psi(x) + M. \tag{32}$$

Recall that $x/|x|$ is the outer unit normal to K at the boundary point $\nabla h_K(x)$ whenever $0 \neq x \in \mathbb{R}^n$, and that $\sup_{y \in K} x \cdot y = x \cdot \nabla h_K(x)$. Therefore, for any $x \in \mathbb{R}^n$,

$$\psi(x) = \sup_{y \in K} [x \cdot y - \varphi(y)] \geq -M + \sup_{y \in K} x \cdot y = -M + x \cdot \nabla h_K(x). \tag{33}$$

Using (32) and (33),

$$(\nabla h_K(x) - \nabla\psi(x)) \cdot \frac{x}{|x|} \leq \frac{2M}{|x|} \qquad (0 \neq x \in \mathbb{R}^n). \tag{34}$$

Recall that $\nabla\psi(x) \in K$ for any $x \in \mathbb{R}^n$. Since ∂K is smooth with positive Gauss curvature, inequality (34) implies that there exist $R_K, \alpha_K > 0$, depending only on K, with

$$|\nabla h_K(x) - \nabla\psi(x)| \leq \alpha_K \sqrt{\frac{2M}{|x|}} \qquad \text{for } |x| \geq R_K. \tag{35}$$

The lemma follows from (35). □

For $\varepsilon > 0, \theta \in \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ denote

$$\delta_{\theta\theta}^{(\varepsilon)} f(x) = f(x + \varepsilon\theta) + f(x - \varepsilon\theta) - 2f(x) \qquad (x \in \mathbb{R}^n).$$

For a smooth f and a small ε, the quantity $\delta_{\theta\theta}^{(\varepsilon)} f(x)/\varepsilon^2$ approximates the pure second derivative $f_{\theta\theta}(x)$. We would like to use the maximum principle for the function $\psi_{\theta\theta}(x)$, but we do not know whether or not it attains its supremum. This is the reason for using the approximate second derivative $\delta_{\theta\theta}^{(\varepsilon)}\psi(x)$ as a substitute.

Corollary 2. *Fix $0 < \varepsilon < 1$. Then the supremum of $\delta_{\theta\theta}^{(\varepsilon)}\psi(x)$ over all $x \in \mathbb{R}^n$ and $\theta \in S^{n-1}$ is attained.*

Proof. According to Lemma 4 and the continuity and 0-homogeneity of $\nabla h_K(x)$,

$$\lim_{R\to\infty} \sup_{\substack{|x|\geq R \\ x_1,x_2 \in B(x,1)}} |\nabla\psi(x_1) - \nabla\psi(x_2)| = \lim_{R\to\infty} \sup_{\substack{|x|\geq R \\ x_1,x_2 \in B(x,1)}} |\nabla h_K(x_1) - \nabla h_K(x_2)|$$

$$= \lim_{R\to\infty} \sup_{\substack{|x|=1 \\ x_1,x_2 \in B(x,1/R)}} |\nabla h_K(x_1) - \nabla h_K(x_2)| = 0, \tag{36}$$

where $B(x,r) = \{y \in \mathbb{R}^n; |x - y| < r\}$. From Lagrange's mean value theorem,

$$\delta_{\theta\theta}^{(\varepsilon)}\psi(x) = (\psi(x+\varepsilon\theta) - \psi(x)) \ - \ (\psi(x) - \psi(x-\varepsilon\theta)) \\ \leq \varepsilon \sup_{x_1, x_2 \in B(x,\varepsilon)} |\nabla\psi(x_1) - \nabla\psi(x_2)|. \tag{37}$$

According to (36) and (37),

$$\lim_{R\to\infty} \sup_{\substack{|x|\geq R \\ \theta \in S^{n-1}}} \delta_{\theta\theta}^{(\varepsilon)}\psi(x) \leq \varepsilon \lim_{R\to\infty} \sup_{\substack{|x|\geq R \\ x_1,x_2 \in B(x,\varepsilon)}} |\nabla\psi(x_1) - \nabla\psi(x_2)| = 0. \tag{38}$$

Since ψ is convex and smooth, then the function $\delta_{\theta\theta}^{(\varepsilon)}\psi$ is non-negative and continuous in $(x,\theta) \in \mathbb{R}^n \times S^{n-1}$. It thus follows from (38) that its supremum is attained. □

We shall apply the well-known matrix inequality, which states that when A and B are symmetric, positive-definite $n \times n$ matrices, then

$$\log\det B \leq \log\det A + Tr\left[A^{-1}(B-A)\right] = \log\det A + Tr\left[A^{-1}B\right] - n, \tag{39}$$

where $Tr(A)$ stands for the trace of the matrix A. Recall that the transport equation (4) is valid, hence,

$$\log\det\nabla^2\psi(x) = -\psi(x) + (\rho\circ\nabla\psi)(x) \qquad (x\in\mathbb{R}^n). \tag{40}$$

In particular, $\nabla^2\psi(x)$ is always an invertible matrix which is in fact positive-definite. We denote its inverse by $\left(\nabla^2\psi(x)\right)^{-1} = (\psi^{ij}(x))_{i,j=1,\dots,n}$. For a smooth function $u:\mathbb{R}^n\to\mathbb{R}$ denote

$$Au(x) = Tr\left[\left(\nabla^2\psi(x)\right)^{-1}\nabla^2 u(x)\right] = \psi^{ij}(x)u_{ij}(x) \qquad (x\in\mathbb{R}^n), \tag{41}$$

where we adhere to the Einstein convention: When an index is repeated twice in an expression, once as a subscript and once as a superscript, then we sum over this index from 1 to n. According to (39) for any $\theta\in\mathbb{R}^n$,

$$\log\det\nabla^2\psi(x+\theta) \leq \log\det\nabla^2\psi(x) + \psi^{ij}(x)\psi_{ij}(x+\theta) - n \qquad (x\in\mathbb{R}^n), \tag{42}$$

with an equality for $\theta = 0$.

Proof of Proposition 2. We follow Caffarelli's argument [10, Theorem 11]. Our assumption (29) yields that the function $\rho(x) - \varepsilon_0|x|^2/2$ is convex. Hence, for any x,y such that $x-y, x+y, x \in K$,

$$\rho(x+y) + \rho(x-y) - 2\rho(x) \geq \frac{\varepsilon_0}{2}\left(|x+y|^2 + |x-y|^2 - 2|x|^2\right) = \varepsilon_0|y|^2. \tag{43}$$

Fix $0 < \varepsilon < 1$ and abbreviate $\delta_{\theta\theta}f = \delta_{\theta\theta}^{(\varepsilon)}f$. From (40) and (42) as well as some simple algebraic manipulations, for any $\theta\in\mathbb{R}^n$,

$$A(\delta_{\theta\theta}\psi) \geq \delta_{\theta\theta}\left(\log\det\nabla^2\psi\right) = -\delta_{\theta\theta}\psi + \delta_{\theta\theta}(\rho\circ\nabla\psi). \tag{44}$$

According to Corollary 2, the maximum of $(x, \theta) \mapsto \delta_{\theta\theta}\psi(x)$ over $\mathbb{R}^n \times S^{n-1}$ is attained at some $(x_0, e) \in \mathbb{R}^n \times S^{n-1}$. Since ψ is smooth, then at the point x_0,

$$0 = \nabla(\delta_{ee}\psi)(x_0) = \nabla\psi(x_0 + \varepsilon e) + \nabla\psi(x_0 + \varepsilon e) - 2\nabla\psi(x_0).$$

In other words, there exists a vector $u \in \mathbb{R}^n$ such that

$$\nabla\psi(x_0 + \varepsilon e) = \nabla\psi(x_0) + u, \qquad \nabla\psi(x_0 - \varepsilon e) = \nabla\psi(x_0) - u.$$

Setting $v = \nabla\psi(x_0)$ and using (43), we obtain

$$\delta_{ee}(\rho \circ \nabla\psi)(x_0) = \rho(v + u) + \rho(v - u) - 2\rho(v) \geq \varepsilon_0|u|^2. \tag{45}$$

The smooth function $x \mapsto \delta_{ee}\psi(x)$ reaches a maximum at x_0, hence the matrix $\nabla^2(\delta_{ee}\psi)(x_0)$ is negative semi-definite. Since the matrix $(\nabla^2\psi)^{-1}(x_0)$ is positive-definite, then from the definition (41),

$$0 \geq A(\delta_{ee}\psi)(x_0). \tag{46}$$

Now, (44), (45) and (46) yield

$$\delta_{ee}\psi(x_0) \geq \delta_{ee}\left(\rho \circ \nabla\psi\right)(x_0) \geq \varepsilon_0|u|^2. \tag{47}$$

By the convexity of ψ,

$$\psi(x_0 + \varepsilon e) - \psi(x_0) \leq \nabla\psi(x_0 + \varepsilon e) \cdot (\varepsilon e) = (v + u) \cdot (\varepsilon e)$$

and

$$\psi(x_0 - \varepsilon e) - \psi(x_0) \leq \nabla\psi(x_0 - \varepsilon e) \cdot (-\varepsilon e) = (v - u) \cdot (-\varepsilon e).$$

Summing the last two inequalities yields

$$\delta_{ee}\psi(x_0) \leq (v + u) \cdot (\varepsilon e) + (v - u) \cdot (-\varepsilon e) = 2\varepsilon(u \cdot e) \leq 2|u|\varepsilon. \tag{48}$$

The inequalities (47) and (48) imply that $|u| \leq 2\varepsilon/\varepsilon_0$ and hence from (48),

$$\delta_{ee}(\psi)(x_0) \leq 4\varepsilon^2/\varepsilon_0.$$

Consequently, for any $x \in \mathbb{R}^n$ and $\theta \in S^{n-1}$ we have $\delta_{\theta\theta}^{(\varepsilon)}\psi(x) \leq 4\varepsilon^2/\varepsilon_0$, and hence

$$\psi_{\theta\theta}(x) = \lim_{\varepsilon \to 0^+} \frac{\delta_{\theta\theta}^{(\varepsilon)}\psi(x)}{\varepsilon^2} \leq \frac{4}{\varepsilon_0}.$$

Therefore $\|\nabla^2\psi(x)\| \leq 4/\varepsilon_0$ for any $x \in \mathbb{R}^n$, and the proof is complete. □

Remark 1. Our proof of Proposition 2 provides the explicit bound

$$\sup_{x \in \mathbb{R}^n} \|\nabla^2\psi(x)\| \leq 4/\varepsilon_0. \tag{49}$$

By arguing as in [11], one may improve the right-hand side of (49) to just $1/\varepsilon_0$. We omit the straightforward details.

4 Diffusion processes and stochastic completeness

In this section we consider a diffusion process associated with transportation of measure. Our point of view owes much to the article by Kolesnikov [23], and we make an effort to maintain a discussion as general as the one in Kolesnikov's work.

Let μ be a probability measure supported on an open set $K \subseteq \mathbb{R}^n$, with density $e^{-\rho}$ where $\rho : K \to \mathbb{R}$ is a smooth function. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, convex function with

$$\lim_{R\to\infty} \left(\inf_{|x|\geq R} \psi(x) \right) = +\infty. \tag{50}$$

Condition (50) holds automatically when $\int e^{-\psi} < \infty$, see (9) above. Rather than requiring that the transport equation (4) hold true, in this section we make the more general assumption that

$$e^{-\rho(\nabla\psi(x))} \det \nabla^2\psi(x) = e^{-V(x)} \qquad\qquad (x \in \mathbb{R}^n) \tag{51}$$

for a certain smooth function $V : \mathbb{R}^n \to \mathbb{R}$. Clearly, when μ is the moment measure of ψ, equation (51) holds true with $V = \psi$ and condition (50) holds as well. The transport equation (51) means that the map $x \mapsto \nabla\psi(x)$ pushes the probability measure $e^{-V(x)}dx$ forward to μ. In this section we explain and prove the following:

Proposition 3. *Let $K \subseteq \mathbb{R}^n$ be an open set, and let $V, \psi : \mathbb{R}^n \to \mathbb{R}$ and $\rho : K \to \mathbb{R}$ be smooth functions with ψ being convex. Assume (50) and (51), and furthermore, that*

$$\inf_{x\in K} \nabla\rho(x) \cdot x > -\infty. \tag{52}$$

Then the weighted Riemannian manifold $M = \left(\mathbb{R}^n, \nabla^2\psi, e^{-V(x)}dx\right)$ is stochastically complete.

Remark 2. Note that in the most interesting case where $V = \psi$, the weighted Riemannian manifold M from Proposition 3 coincides with M_μ^* as defined in (5) and (6) above. Additionally, in the case where μ is log-concave with barycenter at the origin, condition (52) does hold true: In this case, according to Fradelizi [14], we know that $\rho(0) \leq n + \inf_{x\in K} \rho(x)$. By convexity,

$$\nabla\rho(x) \cdot x \geq \rho(x) - \rho(0) \geq -n \qquad\qquad (x \in K),$$

and (52) follows. Thus Proposition 3 implies the stochastic completeness of M_μ^* when μ is a log-concave probability measure with barycenter at the origin, which satisfies the regularity conditions (2).

We now turn to a detailed explanation of *stochastic completeness* of a weighted Riemannian manifold. See, e.g., Grigor'yan [15] for more information. The *Dirichlet form* associated with the weighted Riemannian manifold $M = (\Omega, g, \nu)$ is defined as

$$\Gamma(u,v) = \int_\Omega g\left(\nabla_g u, \nabla_g v\right) d\nu, \tag{53}$$

where $u, v : \Omega \to \mathbb{R}$ are smooth functions for which the integral in (53) exists. Here, $\nabla_g u$ stands for the Riemannian gradient of u. The *Laplacian* associated with M is the unique operator L, acting on smooth functions $u : \Omega \to \mathbb{R}$, for which

$$\int_\Omega (Lu) v d\nu = -\Gamma(u,v) \tag{54}$$

for any compactly-supported, smooth function $v : \Omega \to \mathbb{R}$. In the case of the weighted manifold $M = \left(\mathbb{R}^n, \nabla^2\psi, e^{-V(x)}dx\right)$ from Proposition 3, we may express the Dirichlet form as follows:

$$\Gamma(u,v) = \int_{\mathbb{R}^n} \left(\psi^{ij} u_i v_j\right) e^{-V} \tag{55}$$

where $\nabla^2\psi(x)^{-1} = (\psi^{ij}(x))_{i,j=1,\ldots,n}$ and $u_i = \partial u / \partial x^i$. Note that the matrix $\nabla^2\psi(x)$ is invertible, thanks to (51). As in Section 3 above, we use the Einstein summation convention; thus in (55) we sum over i, j from 1 to n. We will also make use of abbreviations such as $\psi_{ijk} = \partial^3\psi / (\partial x^i \partial x^j \partial x^k)$, and also $\psi^i_{j\ell} = \psi^{ik}\psi_{jk\ell}$ and $\psi^{ij}_k = \psi^{i\ell}\psi^{jm}\psi_{\ell m k}$. Therefore, for example,

$$(\psi^{ij})_k = \frac{\partial \psi^{ij}(x)}{\partial x^k} = -\psi^{i\ell}\psi^{jm}\psi_{\ell m k} = -\psi^{ij}_k.$$

We may now express the Laplacian L associated with $M = \left(\mathbb{R}^n, \nabla^2\psi, e^{-V(x)}dx\right)$ by

$$Lu = \psi^{ij} u_{ij} - (\psi^{ij}_j + \psi^{ij} V_j) u_i \tag{56}$$

as may be directly verified from (55) by integration by parts.

Lemma 5. *For any smooth function* $u : \mathbb{R}^n \to \mathbb{R}$*,*

$$Lu = \psi^{ij} u_{ij} - \sum_{j=1}^n \rho_j(\nabla\psi(x)) u_j. \tag{57}$$

Proof. We take the logarithmic derivative of (51) and obtain that for $\ell = 1, \ldots, n$,

$$\psi^i_{i\ell}(x) = -V_\ell(x) + \sum_{i=1}^n \rho_i(\nabla\psi(x))\psi_{i\ell}(x) \qquad (x \in \mathbb{R}^n). \tag{58}$$

Multiplying (58) by $\psi^{j\ell}$ and summing over ℓ we see that for $j = 1, \ldots, n$,

$$\psi_i^{ij}(x) = -\psi^{j\ell}(x)V_\ell(x) + \rho_j(\nabla\psi(x)) \qquad (x \in \mathbb{R}^n). \tag{59}$$

Now (57) follows from (56) and (59). □

Lemma 6. *Under the assumptions of Proposition 3, there exists $A \geq 0$ such that for all $x \in \mathbb{R}^n$,*

$$(L\psi)(x) \leq A.$$

Proof. Set $A = \max\{0, n - \inf_{y \in K} \nabla\rho(y) \cdot y\}$, which is a finite number according to our assumption (52). From Lemma 5,

$$L\psi(x) = \psi^{ij}\psi_{ij} - \sum_{j=1}^n \rho_j(\nabla\psi(x))\psi_j(x) = n - \sum_{j=1}^n \rho_j(\nabla\psi(x))\psi_j(x).$$

It remains to prove that $n - \sum_j \rho_j(\nabla\psi(x))\psi_j(x) \leq A$, or equivalently, we need to show that

$$\nabla\rho(y) \cdot y \geq n - A \qquad \text{for all } y \in K. \tag{60}$$

However, (60) holds true in view of the definition of A above. Therefore $L\psi \leq A$ pointwise in $\mathbb{R}^n$. □

The Laplacian L associated with a weighted Riemannian manifold M is a second-order, elliptic operator with smooth coefficients. We say that M is *stochastically complete* if the Itô diffusion process whose generator is L is well-defined at all times $t \in [0, \infty)$. In the particular case of Proposition 3, this means the following: Let $(B_t)_{t \geq 0}$ be the standard n-dimensional Brownian motion. The diffusion equation with generator L as in (57) is the stochastic differential equation:

$$dY_t = \sqrt{2}\left(\nabla^2\psi(Y_t)\right)^{-1/2} dB_t - \nabla\rho(\nabla\psi(Y_t))dt, \tag{61}$$

where $(\nabla^2\psi(x))^{-1/2}$ is the positive-definite square root of $(\nabla^2\psi(x))^{-1}$. For background on stochastic calculus, the reader may consult sources such as Kallenberg [16] or Øksendal [24]. The *stochastic completeness* of M is equivalent to the existence of a solution $(Y_t)_{t \geq 0}$ to the equation (61), with an initial condition $Y_0 = z$ for a fixed $z \in \mathbb{R}^n$, that does not explode in finite time. Proposition 3 therefore follows from the next proposition:

Proposition 4. *Let ψ, V and ρ be as in Proposition 3. Fix $z \in \mathbb{R}^n$. Then there exists a unique stochastic process $(Y_t)_{t \geq 0}$, adapted to the filtration induced by the Brownian motion, such that for all $t \geq 0$,*

$$Y_t = z + \int_0^t \sqrt{2}\left(\nabla^2\psi\left(Y_t\right)\right)^{-1/2} dB_t - \int_0^t \nabla\rho(\nabla\psi(Y_t))dt, \tag{62}$$

and such that almost-surely, the map $t \mapsto Y_t$ $(t \geq 0)$ is continuous in $[0, +\infty)$.

Proof. Since $\psi(x)$ tends to $+\infty$ when $x \to \infty$, then the convex set $\{\psi \leq R\} = \{x \in \mathbb{R}^n; \psi(x) \leq R\}$ is compact for any $R \in \mathbb{R}$. We use Theorem 21.3 in Kallenberg [16] and the remark following it. We deduce that there exists a unique continuous stochastic process $(Y_t)_{t \geq 0}$ and stopping times $T_k = \inf\{t \geq 0; \psi(Y_t) \geq k\}$ such that for any $k > \psi(z), t \geq 0$,

$$Y_{\min\{t,T_k\}} = z + \int_0^{\min\{t,T_k\}} \sqrt{2} \left(\nabla^2 \psi(Y_t)\right)^{-1/2} dB_t - \int_0^{\min\{t,T_k\}} \nabla \rho(\nabla \psi(Y_t)) dt. \tag{63}$$

Denote $T = \sup_k T_k$. We would like to prove that $T = +\infty$ almost-surely. According to Dynkin's formula and Lemma 6, for any $k > \psi(z)$ and $t \geq 0$,

$$\mathbb{E}\psi(Y_{\min\{t,T_k\}}) = \psi(z) + \mathbb{E} \int_0^{\min\{t,T_k\}} (L\psi)(Y_t) dt \leq \psi(z) + 2At,$$

where A is the parameter from Lemma 6. Set $\alpha = -\inf_{x \in \mathbb{R}^n} \psi(x)$, a finite number in view of (50). Then $\psi(x) + \alpha$ is non-negative. By Markov-Chebyshev's inequality, for any $t \geq 0$ and $k > \psi(z)$,

$$\mathbb{P}(T_k \leq t) = \mathbb{P}\left(\psi(Y_{\min\{t,T_k\}}) \geq k\right) \leq \frac{\mathbb{E}\psi(Y_{\min\{t,T_k\}}) + \alpha}{k + \alpha} \leq \frac{2At + \psi(z) + \alpha}{k + \alpha}.$$

Hence, for any $t \geq 0$,

$$\mathbb{P}(T \leq t) \leq \inf_k \mathbb{P}(T_k \leq t) \leq \liminf_{k \to \infty} \frac{2At + \psi(z) + \alpha}{k + \alpha} = 0.$$

Therefore $T = +\infty$ almost surely. We may let k tend to infinity in (63) and deduce (62). The uniqueness of the continuous stochastic process $(Y_t)_{t \geq 0}$ that satisfies (62) follows from the uniqueness of the solution to (63). □

For $z \in \mathbb{R}^n$ write $(Y_t^{(z)})_{t \geq 0}$ for the stochastic process from Proposition 4 with $Y_0 = z$. Denote by ν the probability measure on $\mathbb{R}^n$ whose density is $e^{-V(x)} dx$. The lemma below is certainly part of the standard theory of diffusion processes. We were not able to find a precise reference, hence we provide a proof which relies on the existence of the heat kernel.

Lemma 7. *There exists a smooth function $p_t(x, y)$ $(x, y \in \mathbb{R}^n, t > 0)$ which is symmetric in x and y, such that for any $y \in \mathbb{R}^n$ and $t > 0$, the random vector*

$$Y_t^{(y)}$$

has density $x \mapsto p_t(x, y)$ with respect to ν.

Proof. We appeal to Theorem 7.13 and Theorem 7.20 in Grigor'yan [15], which deal with heat kernels on weighted Riemannian manifolds. According to these theorems, there exists a heat kernel, that is, a non-negative function $p_t(x, y)$ $(x, y \in$

$\mathbb{R}^n, t > 0)$ symmetric in x and y and smooth jointly in (t, x, y), that satisfies the following two properties:

(i) For any $y \in \mathbb{R}^n$, the function $u(t, x) = p_t(x, y)$ satisfies

$$\frac{\partial u(t,x)}{\partial t} = L_x u(t,x) \qquad\qquad (x \in \mathbb{R}^n, t > 0)$$

where by $L_x u(t, x)$ we mean that the operator L is acting on the x-variables.

(ii) For any smooth, compactly-supported function $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} p_t(x,y) f(y) d\nu(y) \stackrel{t \to 0^+}{\longrightarrow} f(x), \tag{64}$$

and the convergence in (64) is locally uniform in $x \in \mathbb{R}^n$.

Theorem 7.13 in Grigor'yan [15] also guarantees that $\int p_t(x, y) d\nu(x) \leq 1$ for any y. It remains to prove that the random vector $Y_t^{(y)}$ has density $x \mapsto p_t(x, y)$ with respect to ν. Equivalently, we need to show that for any smooth, compactly-supported function $f : \mathbb{R}^n \to \mathbb{R}$ and $y \in \mathbb{R}^n, t > 0$,

$$\mathbb{E} f\left(Y_t^{(y)}\right) = \int_{\mathbb{R}^n} f(x) p_t(x,y) d\nu(x). \tag{65}$$

Denote by $v(t, y) \ \ (t > 0, y \in \mathbb{R}^n)$ the right-hand side of (65), a smooth, bounded function. We also set $v(0, y) = f(y) \ \ (y \in \mathbb{R}^n)$ by continuity, according to (ii). Then the function $v(t, y)$ is continuous and bounded in $(t, y) \in [0, +\infty) \times \mathbb{R}^n$. Since f is compactly-supported then we may safely differentiate under the integral sign with respect to y and t, and obtain

$$\frac{\partial v(t,y)}{\partial t} = \int_{\mathbb{R}^n} f(x) \frac{\partial p_t(x,y)}{\partial t} d\nu(y), \quad L_y v(t,y) = \int_{\mathbb{R}^n} f(x) \left(L_y p_t(x,y)\right) d\nu(y).$$

From (i) we learn that

$$\frac{\partial v(t,y)}{\partial t} = L_y v(t,y) \qquad\qquad (y \in \mathbb{R}^n, t > 0). \tag{66}$$

Fix $t_0 > 0$ and $y \in \mathbb{R}^n$. Denote $Z_t = v\left(t_0 - t, Y_t^{(y)}\right)$ for $0 \leq t \leq t_0$. Then $(Z_t)_{0 \leq t \leq t_0}$ is a continuous stochastic process. From Itô's formula and (66), for $0 \leq t \leq t_0$,

$$Z_t = Z_0 + R_t + \int_0^t \left[L_y v\left(t_0 - t, Y_t^{(y)}\right) - \frac{\partial v}{\partial t}\left(t_0 - t, Y_t^{(y)}\right) \right] dt = Z_0 + R_t$$

where $(R_t)_{0 \leq t \leq t_0}$ is a local martingale with $R_0 = 0$. Since v is bounded, then $(Z_t)_{0 \leq t \leq t_0}$ is a bounded process, and $(R_t)_{0 \leq t \leq t_0}$ is in fact a martingale. In particular $\mathbb{E} R_{t_0} = \mathbb{E} R_0 = 0$. Thus,

$$\mathbb{E}f\left(Y_{t_0}^{(y)}\right) = \mathbb{E}Z_{t_0} = \mathbb{E}Z_0 = v(t_0, y) = \int_{\mathbb{R}^n} f(x) p_{t_0}(x, y) d\nu(x),$$

and (65) is proven. □

Corollary 3. *Suppose that Z is a random vector in $\mathbb{R}^n$, distributed according to ν, independent of the Brownian motion $(B_t)_{t \geq 0}$ used for the construction of $(Y_t^{(z)})_{t \geq 0, z \in \mathbb{R}^n}$.*

Then, for any $t \geq 0$, the random vector $Y_t^{(Z)}$ is also distributed according to ν.

Proof. According to Lemma 7, for any measurable set $A \subset \mathbb{R}^n$,

$$\begin{aligned}\mathbb{P}\left(Y_t^{(Z)} \in A\right) &= \int_{\mathbb{R}^n} \mathbb{P}\left(Y_t^{(z)} \in A\right) d\nu(z) = \int_{\mathbb{R}^n} \left(\int_A p_t(z, x) d\nu(x)\right) d\nu(z) \\ &= \int_A \left(\int_{\mathbb{R}^n} p_t(x, z) d\nu(z)\right) d\nu(x) = \nu(A).\end{aligned}$$

□

Remark 3. Our choice to use stochastic processes in this paper is just a matter of personal taste. All of the arguments here can be easily rephrased in analytic terminology. For instance, the proof of Proposition 4 relies on the fact that $L\psi$ is bounded from above, similarly to the analytic approach in Grigor'yan [15, Section 8.4]. Another example is the use of local martingales towards the end of Lemma 7, which may be replaced by analytic arguments as in [15, Section 7.4].

5 Bakry-Émery technique

In this section we prove Theorem 1. While the viewpoint and ideas of Bakry and Émery [4] are certainly the main source of inspiration for our analysis, we are not sure whether the abstract framework in [3, 4] entirely encompasses the subtlety of our specific weighted Riemannian manifold. For instance, Lemma 9 below seems related to the positivity of the *carré du champ* Γ_2 and to property (ii) in Section 1 above. In the case $\varepsilon \geq 1/2$, Lemma 9 actually follows from an application of [3, Lemma 2.4] with $f(x) = x^1$ and $\rho = 1/2$. Yet, in general, it appears to us advantageous to proceed by analyzing our model for itself, rather than viewing it as an abstract diffusion semigroup satisfying a curvature-dimension bound.

Let μ be a log-concave probability measure on $\mathbb{R}^n$ satisfying the regularity assumptions (2), whose barycenter lies at the origin. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be convex and smooth, such that the transport equation (4) holds true. In Section 4 we proved that M_μ^* is stochastically complete. Since M_{μ^*} is isomorphic to M_μ, then M_μ is also stochastically complete.

Let us describe in greater detail the diffusion process associated with $M_\mu = (K, \nabla^2\varphi, \mu)$. Recall that the Legendre transform $\varphi = \psi^*$ is smooth and convex on K, and that

$$\varphi(x) + \psi(\nabla\varphi(x)) = x \cdot \nabla\varphi(x) \qquad (x \in K).$$

We may rephrase (4) in terms of $\varphi = \psi^*$, and using $(\nabla^2\varphi(x))^{-1} = \nabla^2\psi(\nabla\varphi(x))$, we arrive at the equation

$$\det \nabla^2\varphi(x) = e^{x\cdot\nabla\varphi(x) - \varphi(x) - \rho(x)} \qquad (x \in K). \tag{67}$$

The Hessian matrix $\nabla^2\varphi$ is invertible everywhere, so we write $\left(\nabla^2\varphi(x)\right)^{-1} = (\varphi^{ij}(x))_{i,j=1,\ldots,n}$, and as before we use abbreviations such as $\varphi_i^{jk} = \varphi^{j\ell}\varphi^{km}\varphi_{i\ell m}$. In this section, for a smooth function $u : K \to \mathbb{R}$, denote

$$Lu(x) = \varphi^{ij}u_{ij} - x^i u_i \qquad \text{for } x = (x^1, \ldots, x^n) \in K. \tag{68}$$

The following lemma is "dual" to Lemma 5.

Lemma 8. *The operator L from (68) is the Laplacian associated with the weighted Riemannian manifold M_μ.*

Proof. By taking the logarithmic derivative of (67) and arguing as in the proof of Lemma 5, we obtain that for any $x \in K, i = 1, \ldots, n$,

$$\varphi_j^{ij} = x^i - \varphi^{ij}\rho_j. \tag{69}$$

Integrating by parts and using (69), we see that for any two smooth functions $u, v : K \to \mathbb{R}$ with one of them compactly-supported,

$$\int_K \varphi^{ij}u_i v_j d\mu = -\int_K v(\varphi^{ij}u_{ij} - (\varphi_j^{ij} + \varphi^{ij}\rho_j)u_i)e^{-\rho} = -\int_K v(Lu)d\mu. \qquad \square$$

Lemma 9. *Fix $\varepsilon > 0$. For $x \in K$ set $f(x) = \varphi^{11}(x)$. Then, for the function $f^\varepsilon(x) = f(x)^\varepsilon$ we have*

$$L\left(f^\varepsilon\right) + \varepsilon f^\varepsilon \geq 0.$$

Proof. For $i, j = 1, \ldots, n$,

$$f_i = (\varphi^{11})_i = -\varphi^{1k}\varphi^{1\ell}\varphi_{ik\ell}, \qquad f_{ij} = -\varphi_{ij}^{11} + 2\varphi_j^{1k}\varphi_{ik}^{1}.$$

Therefore,

$$Lf = \varphi^{ij}f_{ij} - x^i f_i = -\varphi_j^{11j} + 2\varphi_i^{1j}\varphi_j^{1i} + x^j\varphi_j^{11}. \tag{70}$$

Taking the logarithm of (67) and differentiating with respect to x^i and x^ℓ, we see that

$$\varphi^j_{ji\ell} - \varphi^{jk}_i \varphi_{jk\ell} = -\rho_{i\ell} + \varphi_{i\ell} + x^j \varphi_{i\ell j} \qquad (i, \ell = 1, \dots, n).$$

Multiplying by $\varphi^{1i}\varphi^{1\ell}$ and summing yields

$$\varphi^{j11}_j - \varphi^{1j}_k \varphi^{1k}_j = -\varphi^{1i}\varphi^{1\ell}\rho_{i\ell} + \varphi^{11} + x^j \varphi^{11}_j. \tag{71}$$

Since ρ is convex then its Hessian matrix is non-negative definite and $\rho_{i\ell}\varphi^{1i}\varphi^{1\ell} \geq 0$. From (70) and (71),

$$Lf = \varphi^{1j}_k \varphi^{1k}_j - \varphi^{11} + \rho_{i\ell}\varphi^{1i}\varphi^{1\ell} \geq \varphi^{1j}_k \varphi^{1k}_j - \varphi^{11} = \varphi^{1j}_k \varphi^{1k}_j - f. \tag{72}$$

The chain rule of the Laplacian is $L(\lambda(f)) = \lambda'(f)Lf + \lambda''(f)\varphi^{ij} f_j f_j$, as may be verified directly. Using the chain rule with $\lambda(t) = t^\varepsilon$ we see that (72) leads to

$$\begin{aligned} L\left(f^\varepsilon\right) &= \varepsilon f^{\varepsilon-1} Lf + \varepsilon(\varepsilon-1) f^{\varepsilon-2} \varphi^{11j}\varphi^{11}_j \\ &\geq \varepsilon f^{\varepsilon-1} \varphi^{1j}_k \varphi^{1k}_j - \varepsilon f^\varepsilon + \varepsilon(\varepsilon-1) f^{\varepsilon-2}\varphi^{11j}\varphi^{11}_j. \end{aligned}$$

That is,

$$\begin{aligned} L\left(f^\varepsilon\right) + \varepsilon f^\varepsilon &\geq \varepsilon f^{\varepsilon-1}\left[\varphi^{1j}_k \varphi^{1k}_j + (\varepsilon - 1)\frac{\varphi^{11j}\varphi^{11}_j}{\varphi^{11}}\right] \\ &\geq \varepsilon f^{\varepsilon-1}\left[\varphi^{1j}_k \varphi^{1k}_j - \frac{\varphi^{11j}\varphi^{11}_j}{\varphi^{11}}\right], \end{aligned} \tag{73}$$

where we used the fact that $\varphi^{11j}\varphi^{11}_j \geq 0$ in the last passage (or more generally, $\varphi^{ij} h_i h_j \geq 0$ for any smooth function h). It remains to show that the right-hand side of (73) is non-negative. Denote $A = (\varphi^{1j}_k)_{j,k=1,\dots,n}$. The matrix $B = (\varphi^{1jk})_{j,k=1,\dots,n}$ is a symmetric matrix, since $\varphi^{1jk} = \varphi^{1\ell}\varphi^{jm}\varphi^{kr}\varphi_{\ell m r}$. We have $A = (\nabla^2\varphi)B$, and hence

$$\begin{aligned} \varphi^{1j}_k \varphi^{1k}_j = Tr(A^2) &= Tr\left[\left((\nabla^2\varphi)^{1/2} B (\nabla^2\varphi)^{1/2}\right)^2\right] \\ &= \left\|(\nabla^2\varphi)^{1/2} B (\nabla^2\varphi)^{1/2}\right\|^2_{HS}, \end{aligned}$$

since the matrix $(\nabla^2\varphi)^{1/2} B (\nabla^2\varphi)^{1/2}$ is symmetric, where $\|T\|_{HS}$ stands for the Hilbert-Schmidt norm of the matrix T. We will use the fact that the Hilbert-Schmidt norm is at least as large as the operator norm, that is, $\|T\|^2_{HS} \geq |Tx|^2/|x|^2$ for any $0 \neq x \in \mathbb{R}^n$. Setting $e_1 = (1, 0, \dots, 0)$, we conclude that

$$\varphi^{1j}_k \varphi^{1k}_j \geq \frac{\left|(\nabla^2\varphi)^{1/2} B (\nabla^2\varphi)^{1/2} (\nabla^2\varphi)^{-1/2} e_1\right|^2}{\left|(\nabla^2\varphi)^{-1/2} e_1\right|^2} = \frac{\varphi^{11i}\varphi_{ij}\varphi^{11j}}{\varphi^{11}} = \frac{\varphi^{11}_j \varphi^{11j}}{\varphi^{11}}.$$

Therefore the right-hand side of (73) is non-negative, and the lemma follows. $\square$

Let $(B_t)_{t\geq 0}$ be the standard n-dimensional Brownian motion. From the results of Section 4, the diffusion process whose generator is L from (68) is well-defined. That is, there exists a unique stochastic process $(X_t^{(z)})_{t\geq 0, z\in K}$, continuous in t and adapted to the filtration induced by the Brownian motion, such that for all $t \geq 0$,

$$X_t^{(z)} = z + \int_0^t \sqrt{2}\left(\nabla^2\varphi\left(X_t^{(z)}\right)\right)^{-1/2} dB_t - \int_0^t X_t^{(z)} dt. \tag{74}$$

Our proof of Theorem 1 relies on a few lemmas in which the main technical obstacle is to prove the integrability of certain local martingales.

Lemma 10. *Fix $z \in K$ and set $X_t = X_t^{(z)}$ $(t \geq 0)$. Then for any $t \geq 0$,*

$$\mathbb{E}X_t = e^{-t} z, \tag{75}$$

and for any $\theta \in S^{n-1}$,

$$e^{2t}\mathbb{E}(X_t\cdot\theta)^2 \geq (z\cdot\theta)^2 + 2\int_0^t e^{2s}\mathbb{E}\left[(\nabla^2\varphi)^{-1}(X_s)\theta\cdot\theta\right] ds. \tag{76}$$

Proof. From Itô's formula and (74),

$$d(e^t X_t) = e^t dX_t + e^t X_t dt = \sqrt{2} e^t \left(\nabla^2\varphi(X_t)\right)^{-1/2} dB_t.$$

Therefore $(e^t X_t)_{0\leq t\leq T}$ is a local martingale, for any fixed number $T > 0$. However, $e^t X_t \in e^T K$ for $0 \leq t \leq T$, and $K \subset \mathbb{R}^n$ is a bounded set. Therefore $(e^t X_t)_{0\leq t\leq T}$ is a bounded process, and hence it is a martingale. We conclude that

$$\mathbb{E}e^t X_t = \mathbb{E}e^0 X_0 = z \qquad (t \geq 0),$$

and (75) is proven. It remains to prove (76). Without loss of generality we may assume that $\theta = e_1 = (1, 0, \ldots, 0)$. Denoting $Y_t = X_t \cdot e_1$, we obtain from (74) that

$$dY_t = \sqrt{2}\left(\nabla^2\varphi(X_t)\right)^{-1/2} e_1 \cdot dB_t - Y_t dt.$$

Set $Z_t = e^{2t} Y_t^2 = e^{2t}(X_t\cdot e_1)^2$. According to Itô's formula,

$$dZ_t = 2e^{2t} Y_t^2 dt + 2e^{2t} Y_t dY_t + \frac{1}{2}\cdot(2e^{2t})\cdot 2\varphi^{11}(X_t)dt = 2e^{2t}\varphi^{11}(X_t)dt + dM_t$$

where $(M_t)_{t\geq 0}$ is a local martingale with $M_0 = 0$. This implies that for any $t \geq 0$,

$$Z_t = (z\cdot e_1)^2 + M_t + \int_0^t \left(2e^{2s}\varphi^{11}(X_s)\right) ds. \tag{77}$$

Since φ^{11} is positive, then for any $t \geq 0$,

$$Z_t - (z\cdot e_1)^2 \geq M_t. \tag{78}$$

The convex body K is bounded, and hence $(Z_t)_{0\leq t\leq T}$ is a bounded process for any number $T>0$. According to (78), the local martingale $(M_t)_{0\leq t\leq T}$ is bounded from above, and by Fatou's lemma it is a sub-martingale. In particular $\mathbb{E}M_t \geq \mathbb{E}M_0 = 0$ for any t. From (77),

$$\mathbb{E}Z_t \geq (z\cdot e_1)^2 + 2\mathbb{E}\int_0^t e^{2s}\varphi^{11}(X_s)ds \qquad (t\geq 0).$$

Since $\mathbb{E}Z_t < +\infty$ and φ^{11} is positive, we may use Fubini's theorem to conclude that for any $t\geq 0$,

$$\mathbb{E}Z_t \geq (z\cdot e_1)^2 + 2\int_0^t e^{2s}\mathbb{E}\varphi^{11}(X_s)ds. \qquad \square$$

Remark 4. Once Theorem 1 is established, we can prove that equality holds in (76). Indeed, it follows from Theorem 1 and (77) that $(M_t)_{0\leq t\leq T}$ is a bounded process and hence a martingale.

Lemma 11. *Assume that the convex body K has a smooth boundary and that its Gauss curvature is positive everywhere. Assume also that there exists $\varepsilon_0>0$ with*

$$\nabla^2\rho(x) \geq \varepsilon_0\cdot Id \qquad (x\in K) \tag{79}$$

in the sense of symmetric matrices. Fix $z\in K$ and set $X_t = X_t^{(z)}$ $(t\geq 0)$. Denote $f(x)=\varphi^{11}(x)$ for $x\in K$. Then, for any $t,\varepsilon>0$,

$$f(z) \leq e^t\left(\mathbb{E}f^{\varepsilon}(X_t)\right)^{1/\varepsilon}. \tag{80}$$

Proof. Our assumptions enable the application of Proposition 2. According to the conclusion of Proposition 2, there exists $M>0$ such that

$$\nabla^2\psi(y) \leq M\cdot Id \qquad (y\in\mathbb{R}^n).$$

Since $(\nabla^2\varphi)^{-1}(x) = \nabla^2\psi(\nabla\varphi(x))$, then,

$$f(x) = \varphi^{11}(x) \leq M \qquad (x\in K). \tag{81}$$

From Itô's formula and (74),

$$e^{\varepsilon t}f^{\varepsilon}(X_t) = f^{\varepsilon}(z) + M_t + \int_0^t e^{\varepsilon s}\left[(Lf^{\varepsilon})(X_s) + \varepsilon f^{\varepsilon}(X_s)\right]ds, \tag{82}$$

where M_t is a local martingale with $M_0=0$. According to (82) and Lemma 9, for any $t\geq 0$,

$$e^{\varepsilon t} f^{\varepsilon}(X_t) \geq f^{\varepsilon}(z) + M_t. \tag{83}$$

We may now use (81) and (83) in order to conclude that the local martingale $(M_t)_{0 \leq t \leq T}$ is bounded from above, for any number $T > 0$. Hence it is a sub-martingale, and $\mathbb{E}M_t \geq \mathbb{E}M_0 = 0$ for any $t \geq 0$. Now (80) follows by taking the expectation of (83). □

Remark 5. We will only use (80) for $\varepsilon = 1$, even though the statement for a small ε is much stronger. In the limit where ε tends to zero, it is not too difficult to prove that the right-hand side of (80) approaches $\exp(t + \mathbb{E} \log f(X_t))$.

The covariance matrix of a square-integrable random vector $Z = (Z_1, \ldots, Z_n) \in \mathbb{R}^n$ is defined to be

$$Cov(Z) = \left(\mathbb{E} Z_i Z_j - \mathbb{E} Z_i \cdot \mathbb{E} Z_j \right)_{i,j=1,\ldots,n}.$$

Corollary 4. *Assume that the convex body K has a smooth boundary and that its Gauss curvature is positive everywhere. Assume also that there exists $\varepsilon_0 > 0$ with*

$$\nabla^2 \rho(x) \geq \varepsilon_0 \cdot Id \qquad (x \in K). \tag{84}$$

Then for any $z \in K$ and $t > 0$,

$$(\nabla^2 \varphi)^{-1}(z) \leq \frac{e^{2t}}{2(e^t - 1)} \cdot Cov\left(X_t^{(z)} \right)$$

in the sense of symmetric matrices.

Proof. Fix $z \in K, t > 0$ and $\theta \in S^{n-1}$. We need to prove that

$$\left(\nabla^2 \varphi(z) \right)^{-1} \theta \cdot \theta \leq \frac{e^{2t}}{2(e^t - 1)} Var(X_t^{(z)} \cdot \theta). \tag{85}$$

Without loss of generality we may assume that $\theta = e_1 = (1, 0, \ldots, 0)$. We use Lemma 10 and also Lemma 11 with $\varepsilon = 1$, and obtain

$$e^{2t} \mathbb{E}(X_t^{(z)} \cdot e_1)^2 \geq (z \cdot e_1)^2 + 2 \int_0^t e^{2s} \mathbb{E} \varphi^{11}(X_s^{(z)}) ds \geq (z \cdot e_1)^2 + 2 \varphi^{11}(z) \int_0^t e^s ds.$$

Recall that $\mathbb{E} X_t^{(z)} = e^{-t} z$, according to Lemma 10. Consequently,

$$\varphi^{11}(z) \leq \frac{e^{2t}}{2(e^t - 1)} \left(\mathbb{E}(X_t^{(z)} \cdot e_1)^2 - (e^{-t} z \cdot e_1)^2 \right) = \frac{e^{2t}}{2(e^t - 1)} Var(X_t^{(z)} \cdot e_1),$$

and (85) is proven for $\theta = e_1$. □

Proof of Theorem 1. Assume first that the convex body K has a smooth boundary, that its Gauss curvature is positive everywhere, and that there exists ε_0 for which

(84) holds true. We apply Corollary 4 with $t = \log 2$, and conclude that for any $z \in K$,

$$Tr\left[(\nabla^2\varphi)^{-1}(z)\right] \leq 2Tr\left[Cov(X_t^{(z)})\right] \leq 2\mathbb{E}\left|X_t^{(z)}\right|^2 \leq 2R^2(K)$$

as $X_t^{(z)} \in K$ almost surely. Therefore, for any $x \in \mathbb{R}^n$, setting $z = \nabla\psi(x)$ we have

$$\Delta\psi(x) = Tr\left[\nabla^2\psi(x)\right] = Tr\left[(\nabla^2\varphi)^{-1}(z)\right] \leq 2R^2(K). \tag{86}$$

It still remains to eliminate the extra strict-convexity assumptions. To that end, we select a sequence of smooth convex bodies $K_\ell \subset \mathbb{R}^n$, each with a positive Gauss curvature, that converge in the Hausdorff metric to K. We then consider a sequence of log-concave probability measures μ_ℓ with barycenter at the origin that converge weakly to μ, such that μ_ℓ is supported on K_ℓ and such that the smooth density of μ_ℓ satisfies (84) with, say, $\varepsilon_0 = 1/\ell$. We also assume that μ_ℓ and K_ℓ satisfy the regularity conditions (2).

It is not very difficult to construct the μ_ℓ's: For instance, convolve μ with a tiny Gaussian (this preserves log-concavity), multiply the density by $\exp(-|x|^2/\ell)$, truncate with K_ℓ and translate a little so that the barycenter would lie at the origin. This way we obtain a sequence of smooth, convex functions $\psi_\ell : \mathbb{R}^n \to \mathbb{R}$ such that μ_ℓ is the moment measure of ψ_ℓ. We may translate, and assume that ψ and each of the $\psi_\ell' s$ are *centered*, in the terminology of Section 2. According to (86), we know that

$$\Delta\psi_\ell(x) \leq 2R^2(K_\ell) \qquad\qquad (x \in \mathbb{R}^n, \ell \geq 1). \tag{87}$$

Furthermore, $\mu_\ell \longrightarrow \mu$ weakly, and by Proposition 1, also $\psi_\ell \longrightarrow \psi$ pointwise in $\mathbb{R}^n$. Since ψ_ℓ and ψ are smooth, then [26, Theorem 24.5] implies that

$$\nabla\psi_\ell(x) \stackrel{\ell\to\infty}{\longrightarrow} \nabla\psi(x) \qquad\qquad (x \in \mathbb{R}^n).$$

The function ψ_ℓ is $R(K_\ell)$-Lipschitz, and $R(K_\ell) \longrightarrow R(K)$. Hence $\sup_{\ell,x}|\nabla\psi_\ell(x)|$ is finite. By the bounded convergence theorem, for any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$\int_{B(x_0,\varepsilon)} \Delta\psi_\ell = \int_{\partial B(x_0,\varepsilon)} \nabla\psi_\ell \cdot N \stackrel{\ell\to\infty}{\longrightarrow} \int_{\partial B(x_0,\varepsilon)} \nabla\psi \cdot N = \int_{B(x_0,\varepsilon)} \Delta\psi, \tag{88}$$

where N is the outer unit normal. From (87) and (88) we conclude that for any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$\int_{B(x_0,\varepsilon)} \Delta\psi \leq Vol_n(B(x_0,\varepsilon)) \cdot \limsup_{\ell\to\infty} 2R^2(K_\ell) = 2Vol_n(B(x_0,\varepsilon))R^2(K),$$

where Vol_n is the Lebesgue measure in $\mathbb{R}^n$. Since ψ is smooth, then we may let ε tend to zero and conclude that $\Delta\psi(x_0) \leq 2R^2(K)$, for any $x_0 \in \mathbb{R}^n$. □

Posteriori, we may strengthen Corollary 4 and eliminate the strict-convexity assumptions. These assumptions were used only in the proof of Lemma 11, to deduce the existence of some number $M > 0$ for which $\nabla^2\psi(x) \leq M \cdot Id$, for all $x \in \mathbb{R}^n$. Theorem 1 provides such a number $M = 2R^2(K)$, without any strict-convexity assumptions on ρ or K. We may therefore upgrade Corollary 4, and conclude that

Corollary 5. *Suppose that μ is a log-concave probability measure in $\mathbb{R}^n$ with barycenter at the origin, satisfying the regularity conditions (2). Let $(X_t^{(z)})_{t\geq 0, z\in K}$ be the stochastic process given by (74). Then this process is well-defined and bounded, and for any $z \in K$ and $t > 0$,*

$$(\nabla^2\varphi)^{-1}(z) \leq \frac{e^{2t}}{2(e^t - 1)} \cdot Cov\left(X_t^{(z)}\right)$$

in the sense of symmetric matrices.

6 The Brascamp-Lieb inequality as a Poincaré inequality

We retain the assumptions and notation of the previous section. That is, μ is a log-concave probability measure on $\mathbb{R}^n$, with barycenter at the origin, that satisfies the regularity assumptions (2). The measure μ is the moment-measure of the smooth and convex function $\psi : \mathbb{R}^n \to \mathbb{R}$. Equation (4) holds true, and we denote $\varphi = \psi^*$. According to the Brascamp-Lieb inequality [8], for any smooth function $u : \mathbb{R}^n \to \mathbb{R}$ such that $ue^{-\psi}$ is integrable,

$$\int_{\mathbb{R}^n} ue^{-\psi} = 0 \qquad \Longrightarrow \qquad \int_{\mathbb{R}^n} u^2 e^{-\psi} \leq \int_{\mathbb{R}^n} \left[(\nabla^2\psi)^{-1}\nabla u \cdot \nabla u\right] e^{-\psi}. \quad (89)$$

Equality in (89) holds when $u(x) = \nabla\psi(x) \cdot \theta$ for some $\theta \in \mathbb{R}^n$. Note that (89) is precisely the Poincaré inequality with the best constant of the weighted Riemannian manifold M_μ^*. By using the isomorphism between M_μ and M_μ^*, we translate (89) as follows: For any smooth function $f : K \to \mathbb{R}$ which is μ-integrable,

$$Var_\mu(f) \leq \int_K \left(\varphi^{ij} f_i f_j\right) d\mu, \quad (90)$$

where $Var_\mu(f) = \int f^2 d\mu - (\int f d\mu)^2$. Equality in (90) holds when $f(x) = A + x\cdot\theta$ for some $\theta \in \mathbb{R}^n$ and $A \in \mathbb{R}$. This is in accordance with the fact that linear functions are eigenfunctions, i.e.,

$$Lx^i = -x^i \qquad (i = 1, \dots, n)$$

where $Lu = \varphi^{ij}u_{ij} - x^i u_i$ is the Laplacian of the weighted Riemannian manifold M_μ. In fact, (90) means that the spectrum of the (Friedrich extension of the) operator L cannot intersect the interval $(-1, 0)$, and that the restriction of $-L$ to the subspace of mean-zero functions is at least the identity operator, in the sense of symmetric operators.

Theorem 1 states that $\Delta\psi(x) \leq 2R^2(K)$ everywhere in $\mathbb{R}^n$. A weak conclusion is that $\nabla^2\psi(x) \leq 2R^2(K) \cdot Id$, or rather, that $(\nabla^2\varphi(x))^{-1} \leq 2R^2(K) \cdot Id$. By substituting this information into (90), we see that for any smooth function $f \in L^1(\mu)$,

$$Var_\mu(f) \leq 2R^2(K) \int_K |\nabla f|^2 d\mu. \tag{91}$$

This completes the proof of Corollary 1. See [20, 21] for more Poincaré-type inequalities that are obtained by imposing a Riemannian structure on the convex body K. The Kannan-Lovasź-Simonovits conjecture speculates that $R^2(K)$ in (91) may be replaced by a universal constant times $\|Cov(\mu)\|$, where $Cov(\mu)$ is the covariance matrix of the random vector that is distributed according to μ, and $\|\cdot\|$ is the operator norm.

A potential way to make progress towards the Kannan-Lovasź-Simonovits conjecture is to try to bound the matrices $(\nabla^2\varphi)^{-1}(x)$ $(x \in K)$ in terms of $Cov(\mu)$. The following proposition provides a modest step in this direction:

Proposition 5. *Fix $\theta \in S^{n-1}$ and denote*

$$V = \int_{\mathbb{R}^n} (x \cdot \theta)^2 d\mu(x).$$

Then, for any $p \geq 1$,

$$\left(\int_K \left| \frac{(\nabla^2\varphi)^{-1}\theta \cdot \theta}{V} \right|^p d\mu \right)^{1/p} \leq 4p^2.$$

Proof. Without loss of generality, assume that $\theta = e_1 = (1, 0, \ldots, 0)$. According to Corollary 5, for any $z \in K$ and $t > 0$,

$$\varphi^{11}(z) \leq \frac{e^{2t}}{2(e^t - 1)} Var\left(X_t^{(z)} \cdot e_1\right) \leq \frac{e^{2t}}{2(e^t - 1)} \mathbb{E}\left(X_t^{(z)} \cdot e_1\right)^2. \tag{92}$$

Let Z be a random vector that is distributed according to μ, independent of the Brownian motion used in the construction of the process $(X_t^{(z)})_{t \geq 0, z \in K}$. It follows from Corollary 3 that for any fixed $t \geq 0$ the random vector $X_t^{(Z)}$ is also distributed according to μ. By setting $t = \log 2$ in (92) and applying Hölder's inequality, we see that for any $p \geq 1$,

$$\mathbb{E}\left|\varphi^{11}(Z)\right|^p \leq 2^p \mathbb{E}\left|X_t^{(Z)} \cdot e_1\right|^{2p} = 2^p \mathbb{E}\left|Z \cdot e_1\right|^{2p}. \tag{93}$$

The random vector Z has a log-concave density. According to the Berwald inequality [6, 7],

$$\left(\mathbb{E}\left|Z\cdot e_1\right|^{2p}\right)^{1/(2p)}\leq\frac{\Gamma(2p+1)^{1/(2p)}}{\Gamma(3)^{1/2}}\sqrt{\mathbb{E}\left|Z\cdot e_1\right|^2}\leq\frac{2p}{\sqrt{2}}\sqrt{V}. \tag{94}$$

(The Berwald inequality is formulated in [6, 7] for the uniform measure on a convex body, but it is well-known that is applies for all log-concave probability measures. For instance, one may deduce the log-concave version from the convex-body version by using a marginal argument as in [18]). The proposition follows from (93) and (94). □

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