Isoperimetric inequalities in high-dimensional convex sets
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Lecture 10: The covariance process of stochastic localization

In this lecture we complete the proof of the thin-shell theorem. Let μ be an isotropic, log-concave probability measure in $\mathbb{R}^n$ with density p. It is an exercise to show that for proving the thin-shell theorem we may approximate μ and assume that p is continuous and compactly-supported.

Recall that for $t \geq 0$ and $y \in \mathbb{R}^n$ we consider the probability density

$$p_{t,y}(x) = e^{y \cdot x - t|x|^2/2 - \Lambda_t(y)} p(x) \qquad (x \in \mathbb{R}^n) \tag{1}$$

where

$$\Lambda_t(y) = \log \int_{\mathbb{R}^n} e^{y \cdot x - t|x|^2/2} p(x) \mathrm{d}x$$

is a normalizing factor. The barycenter and covariance of $p_{t,y}$ are given by

$$a_t(y) = \nabla \Lambda_t(y) = \int_{\mathbb{R}^n} x p_{t,y}(x) \mathrm{d}x \in \mathbb{R}^n$$

and

$$A_t(y) = \nabla^2 \Lambda_t(y) = \mathrm{Cov}(p_{t,y}) \in \mathbb{R}^{n \times n}.$$

We would also need the symmetric 3-tensor

$$\nabla^3 \Lambda_t(y) = \int_{\mathbb{R}^n} (x - a_t(y))^{\otimes 3} p_{t,y}(x) \mathrm{d}x \in \mathbb{R}^{n \times n \times n}.$$

Recall that $p_{t,y}$ is t-uniformly log-concave, i.e., $\nabla^2(-\log p_{t,y}) \geq t \cdot \mathrm{Id}$ for almost every $y \in \mathbb{R}^n$. The main advantage of t-uniform log-concavity is the possibility to apply the improved Lichnerowicz inequality. It would help us bound the tensor $\nabla^3 \Lambda_t(y)$ of third moments of p_t. In fact, one of our main proof ingredients is the following:

Lemma 10.1. *Let $t > 0$ and suppose that X is a centered, t-uniformly log-concave random vector in $\mathbb{R}^n$. Let $\lambda_1, \ldots, \lambda_n \geq 0$ be the eigenvalues of $\mathrm{Cov}(X)$ and let $u_1, \ldots, u_n \in \mathbb{R}^n$ be a corresponding orthonormal basis of eigenvectors. Abbreviate $X_i = \langle X, u_i \rangle$. Then for $1 \leq k \leq n$ and $s > 0$,*

$$\sum_{i,j=1}^{n} (\mathbb{E} X_i X_j X_k)^2 1_{\{\lambda_i \vee \lambda_j \leq s\}} \leq 4 t^{-1/2} s^{3/2} \lambda_k, \tag{2}$$

where $a \vee b = \max\{a, b\}$, i.e., in (2) we only sum over i, j with $\max\{\lambda_i, \lambda_j\} \leq s$.

Proof. Write $E \subseteq \mathbb{R}^n$ for the subspace spanned by the vectors u_i for which $\lambda_i \leq s$. Let $Proj_E$ be the orthogonal projection operator onto E in $\mathbb{R}^n$. Denote

$$Y = Proj_E X.$$

It follows from the Prékopa-Leindler inequality that Y is also t-uniformly log-concave, and

$$\|\mathrm{Cov}(Y)\|_{op} \leq s.$$

The improved log-concave Lichnerowicz inequality thus implies that the Poincaré constant of Y, denoted by $C_P(Y)$, satisfies

$$C_P(Y) \leq \sqrt{\frac{s}{t}}. \tag{3}$$

Set

$$H = \mathbb{E}\left[X_k Y \otimes Y\right] \in \mathbb{R}^{n \times n}.$$

By the definition of the subspace E,

$$\sum_{i,j=1}^{n} (\mathbb{E} X_i X_j X_k)^2 1_{\{\lambda_i \vee \lambda_j \leq s\}} = \mathrm{Tr}(H^2) \tag{4}$$

Moreover, by using (3) and the Poincaré inequalitry,

$$\begin{aligned} \mathrm{Var}(\langle HY, Y \rangle) &\leq C_P(Y) \cdot \mathbb{E}|2HY|^2 \leq 4t^{-1/2} s^{1/2} \cdot \mathrm{Tr}(H^2 \mathrm{Cov}(Y)) \\ &\leq 4t^{-1/2} s^{3/2} \cdot \mathrm{Tr} H^2. \end{aligned} \tag{5}$$

On the other hand, since $\mathbb{E} X_k = 0$, the Cauchy-Schwarz inequality shows that

$$\begin{aligned} \mathrm{Tr}(H^2) &= \mathbb{E} X_k \langle HY, Y \rangle \leq (\mathbb{E} X_k^2)^{1/2} \cdot (\mathrm{Var}\langle HY, Y \rangle)^{1/2} \\ &= \lambda_k^{1/2} \cdot \sqrt{\mathrm{Var}\langle HY, Y \rangle}. \end{aligned} \tag{6}$$

From (5) and (6),

$$\sqrt{\mathrm{Var}\langle HY, Y \rangle} \leq 4t^{-1/2} s^{3/2} \lambda_k^{1/2}. \tag{7}$$

The conclusion of the lemma follows from (4), (6) and (7). □

Let $(W_t)_{t \geq 0}$ be a standard Brownian motion in $\mathbb{R}^n$ with $W_0 = 0$. Consider the stochastic process $(\theta_t)_{t \geq 0}$ from the last lecture, for whose definition we offer two alternatives:

1. The first option is to introduce a random vector X in $\mathbb{R}^n$ with law μ, independent of the Brownian motion $(W_t)_{t\geq 0}$, and set

$$\theta_t = tX + W_t.$$

2. The second option is to uniquely define $(\theta_t)_{t\geq 0}$ via the integral equation

$$\theta_t = W_t + \int_0^t a_s(\theta_s)\mathrm{d}s.$$

The two options coincide in law, as we have seen last week. Write $\mathcal{F}_t$ for the σ-algebra generated by $(\theta_s)_{0\leq s\leq t}$. When we say that τ is a *stopping time* we mean that for any $t > 0$, the event $\{\tau \leq t\}$ is measurable with respect to $\mathcal{F}_t$. Denote

$$p_t = p_{t,\theta_t}, \qquad a_t = a_t(\theta_t), \qquad A_t = A_t(\theta_t), \qquad \Lambda_t = \Lambda_t(\theta_t)$$

and write

$$\frac{1}{t} \geq \lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_n(t) > 0 \tag{8}$$

for the eigenvalues of the covariance matrix A_t, repeated according to their multiplicity. Since μ is isotropic, at $t = 0$ we have $A_0 = \mathrm{Id}$ and hence

$$\lambda_1(0) = \lambda_2(0) = \ldots = \lambda_n(0) = 1.$$

For any k, the eigenvalue $\lambda_k(t)$ equals 1 at time $t = 0$, and it is smaller than 1 at any time $t > 1$. In the interval $(0, 1)$, however, the eigenvalue $\lambda_1(t)$ is typically very large, see the example in the exercise below. In view of Corollary 9.10 from last week, the missing ingredient in the proof of the thin-shell theorem along the lines of [4] is the following:

Proposition 10.2. *We have*

$$\sum_{k=1}^n \mathbb{E}\exp\left(2\int_0^1 \lambda_k(t)\mathrm{d}t\right) \leq Cn,$$

where $C > 0$ is a universal constant.

The proof of Proposition 10.2 relies on the following proposition, which is a straightforward variant of a recent breakthrough bound by Guan [2].

Proposition 10.3. *For any $t > 0$ and any stopping time τ,*

$$\frac{1}{n}\sum_{k=1}^n \mathbb{P}\left(\lambda_k(t\wedge\tau) \geq 3\right) \leq Ce^{-1/t^{\alpha}},$$

where $a \wedge b = \min\{a, b\}$ and where $C, \alpha > 0$ are universal constants.

It is conceivable that $\alpha = 1$ in Proposition 10.3, see [3]. Proposition 10.3 tells us that while a single eigenvalue may explode at some time $t \in (0,1)$, it is unlikely that many eigenvalues are simultaneously large.

Proof of Proposition 10.2 assuming Proposition 10.3. For $k = 1, \dots, n$ consider the stopping time

$$\tau_k = \inf\left\{t > 0\,;\ \lambda_k(t) \geq 3\right\}.$$

For any fixed $t > 0$ and $i = 1, \dots, k$, under the event $\tau_k \leq t$ we have

$$\lambda_i(t \wedge \tau_k) \geq \lambda_k(t \wedge \tau_k) \geq 3.$$

Hence, for $i = 1, \dots, k$,

$$\mathbb{P}\left(\tau_k \leq t\right) \leq \mathbb{P}\left(\lambda_i(t \wedge \tau_k) \geq 3\right).$$

By adding these k inequalities and using Proposition 10.3, for any $t > 0$,

$$\begin{aligned} \mathbb{P}\left(\tau_k \leq t\right) &\leq \frac{1}{k}\sum_{i=1}^{k} \mathbb{P}\left(\lambda_i(t \wedge \tau_k) \geq 3\right) \leq \frac{1}{k}\sum_{i=1}^{n} \mathbb{P}\left(\lambda_i(t \wedge \tau_k) \geq 3\right) \\ &\leq C\frac{n}{k}\exp(-1/t^{\alpha}). \end{aligned} \tag{9}$$

Recall that $\alpha > 0$ is a universal constant. It follows from (9) that

$$\mathbb{E}\tau_k^{-2} \leq C\left(1 + \log\frac{n}{k}\right)^{2/\alpha}. \tag{10}$$

Indeed, in view of (9) inequality (10) clearly holds if $k \geq n/2$. For $k < n/2$ we obtain from (39) that for $s \geq 2^{2/\alpha}$,

$$\mathbb{P}\left(\frac{\tau_k^{-2}}{\left(\log\frac{n}{k}\right)^{2/\alpha}} \geq s\right) \leq C\frac{n}{k}\exp(-s^{\alpha/2}\cdot\log\frac{n}{k}) = C\left(\frac{n}{k}\right)^{1-s^{\alpha/2}} \leq Ce^{-\tilde{c}s^{\alpha/2}}.$$

By integrating over $2^{2/\alpha} \leq s < \infty$ we obtain (10). Consequently, since $\lambda_k(t) \leq 1/t$,

$$\int_0^1 \lambda_k(t)\mathrm{d}t \leq 3(\tau_k \wedge 1) + \int_{\tau_k \wedge 1}^1 \frac{\mathrm{d}t}{t} \leq 3 - \log(\tau_k \wedge 1). \tag{11}$$

Therefore, by (10) and (11),

$$\begin{aligned} \mathbb{E}\sum_{k=1}^{n}\exp\left(2\int_0^1 \lambda_k(t)\mathrm{d}t\right) &\leq e^6\cdot\mathbb{E}\sum_{k=1}^{n}\mathbb{E}\left[\tau_k^{-2} \vee 1\right] \leq C\sum_{k=1}^{n}\mathbb{E}[\tau_k^{-2} + 1] \\ &\leq Cn\cdot\frac{1}{n}\sum_{k=1}^{n}\left(1 + \log\frac{n}{k}\right)^{2/\alpha} \leq \tilde{C}n, \end{aligned} \tag{12}$$

where the last passage follows from the fact that the function $(1 + \log(1/x))^{2/\alpha}$ is monotone and integrable in $[0, 1]$, and the Riemann sum in (12) may be bounded by the integral. $\square$

The proof of Proposition 10.3 requires rather elaborate analysis of the time evolution of the eigenvalues of the covariance matrix A_t. Write

$$\xi_{ij}(t) = (\xi_{ij1}(t), \xi_{ij2}(t) \ldots, \xi_{ijn}(t)) \in \mathbb{R}^n$$

where

$$\xi_{ijk}(t) = \int_{\mathbb{R}^n} \langle x - a_t, u_i \rangle \cdot \langle x - a_t, u_j \rangle \cdot \langle x - a_t, u_k \rangle p_t(x) \mathrm{d}x \in \mathbb{R},$$

with $u_1(t), \ldots, u_n(t) \in \mathbb{R}^n$ being any orthonormal basis of eigenvectors of A_t corresponding to the eigenvalues $\lambda_1(t) \geq \ldots \geq \lambda_n(t)$. Let us fix a stopping time τ.

Lemma 10.4. *For any smooth, increasing function $f : [0, \infty) \to \mathbb{R}$ and almost any $t > 0$,*

$$\frac{d}{dt} \mathbb{E} \sum_{i=1}^n f(\lambda_i(t \wedge \tau)) \leq \frac{1}{2} \sum_{i,j=1}^n \mathbb{E} \left[|\xi_{ij}(t)|^2 \frac{f'(\lambda_i(t)) - f'(\lambda_j(t))}{\lambda_i(t) - \lambda_j(t)} \cdot 1_{\{t < \tau\}} \right], \quad (13)$$

where we interpret the quotient by continuity as $f''(\lambda_i(t))$ when $\lambda_i(t) = \lambda_j(t)$. Moreover, the function that is differentiated on the left-hand side of (13) is absolutely continuous in $t \in [0, \infty)$.

The expression in the right-hand side of (13) is reminiscent of the Daleckii-Krein formula for the second derivative of matrix functions. For a function $f : \mathbb{R} \to \mathbb{R}$ and a symmetric matrix A whose spectral decomposition is

$$A = \sum_{i=1}^n \lambda_i u_i \otimes u_i$$

for numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and an orthonormal basis $u_1, \ldots, u_n \in \mathbb{R}^n$ we write

$$f(A) = \sum_{i=1}^n f(\lambda_i) u_i \otimes u_i.$$

The Daleckii-Krein formula states that for any two symmetric matrices $A, H \in \mathbb{R}^{n \times n}$, as $\varepsilon \to 0$,

$$\mathrm{Tr} f(A + \varepsilon H) = \mathrm{Tr} f(A) + \varepsilon \cdot \mathrm{Tr}[f'(A)H] + \frac{\varepsilon^2}{2} \cdot \mathrm{Tr}[(B \circ H)H] + o(\varepsilon^2)$$

where $\circ$ is the Schur product or Hadamard product (i.e., entry-wise product), and

$$B = \sum_{i,j=1}^{n} \frac{f'(\lambda_i) - f'(\lambda_j)}{\lambda_i - \lambda_j} u_i \otimes u_j.$$

For $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we write $(\nabla^3 \Lambda_t)v \in \mathbb{R}^{n \times n}$ for the symmetric matrix whose i, j entry is

$$\left[(\nabla^3 \Lambda_t)v\right]_{ij} = \sum_{k=1}^{n} \Lambda_{t,ijk} v_k$$

where $\Lambda_t = (\Lambda_{t,ijk})_{i,j,k=1,\ldots,n}$. Lemma 10.4 follows from the following identity:

Lemma 10.5. *For any smooth function $f : [0, \infty) \to \mathbb{R}$ and almost any $t > 0$,*

$$\frac{d}{dt}\mathbb{E}\sum_{i=1}^{n} f(\lambda_i(t \wedge \tau)) = \frac{1}{2}\sum_{i,j=1}^{n} \mathbb{E}\left[|\xi_{ij}(t)|^2 \frac{f'(\lambda_i(t)) - f'(\lambda_j(t))}{\lambda_i(t) - \lambda_j(t)} \cdot 1_{\{t<\tau\}}\right] - \mathbb{E}\left[\sum_{i=1}^{n} \lambda_i^2(t) f'(\lambda_i(t)) \cdot 1_{\{t<\tau\}}\right].$$

Moreover, the function that is differentiated is absolutely-continuous in $t \in [0, +\infty)$.

Proof. We will prove this lemma by using Itô calculus and the "first option" above for the definition of $(\theta_t)_{t \geq 0}$, i.e.,

$$\theta_t = tX + W_t.$$

Recall from last week that for some Brownian motion $(B_t)_{t \geq 0}$ we have

$$d\theta_t = dB_t + a_t dt \tag{14}$$

and that

$$p_t = p_{t,\theta_t}$$

is the conditional law of X given $(\theta_s)_{0 \leq s \leq t}$. Recall that $\mathcal{F}_t$ is the σ-algebra generated by $(\theta_s)_{0 \leq s \leq t}$. Hence, for any continuous test function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \varphi p_t = \mathbb{E}\left[\varphi(X) | \mathcal{F}_t\right]. \tag{15}$$

The stochastic process on the left-hand side of (15) is a martingale, since it represents conditional expectations with respect to a non-decreasing family of σ-algebras. In fact, since p is compactly-supported and continuous, it follows that for any $x \in \mathbb{R}^n$,

$$(p_t(x))_{t \geq 0} \tag{16}$$

is a martingale as well. Recalling that

$$p_t(x) = e^{\theta_t \cdot x - t|x|^2/2 - \Lambda_t(\theta_t)} p(x)$$

we may apply the Itô formula based on (14) and obtain the evolution equation of the martingale (16), namely

$$dp_t(x) = \langle x - a_t, dB_t \rangle p_t(x). \tag{17}$$

It follows from (17) that

$$da_t = d\left[\int_{\mathbb{R}^n} x p_t(x) \mathrm{d}x\right] = \int_{\mathbb{R}^n} x \langle x - a_t, dB_t \rangle p_t(x) \mathrm{d}x = A_t dB_t.$$

Thus,

$$d(a_t \otimes a_t) = (A_t dB_t \otimes a_t + a_t \otimes A_t dB_t) + A_t^2 dt$$

and consequently,

$$dA_t = d\left[\int_{\mathbb{R}^n} (x \otimes x) p_t(x) \mathrm{d}x\right] - d[a_t \otimes a_t] = (\nabla^3 \Lambda_t) dB_t - A_t^2 dt.$$

Hence, for any stopping time τ,

$$dA_{t \wedge \tau} = 1_{\{t < \tau\}} \cdot \left[(\nabla^3 \Lambda_t) dB_t - A_t^2 dt\right].$$

Consequently,

$$d\mathrm{Tr} f(A_{t \wedge \tau}) = 1_{\{t < \tau\}} \cdot \mathrm{Tr}\left[f'(A_t)(\nabla^3 \Lambda_t) dB_t - f'(A_t) A_t^2 dt + \frac{1}{2} D_t dt\right], \tag{18}$$

where the Itô term equals

$$D_t = \sum_{i,j=1}^{n} |\xi_{ij}(t)|^2 \frac{f'(\lambda_i(t)) - f'(\lambda_j(t))}{\lambda_i(t) - \lambda_j(t)},$$

thanks to the Daleckii-Krein formula. By taking expectation the dB_t term in (18) vanishes, completing the proof. □

Since the measure μ is compactly-supported, there exists $R > 0$ depending on μ such that

$$|\xi_{ij}(t)| \leq R \qquad \text{for all } i, j \text{ and } t \geq 0.$$

It is an instructive exercise to use Lemma 10.4 with $f(x) = e^{\beta x}$ in order to prove that for all $0 < t < c_\mu$,

$$\mathbb{P}(\lambda_1(t \wedge \tau) \geq 2) \leq e^{-\tilde{c}_\mu / t} \tag{19}$$

for some constants $c_\mu, \tilde{c}_\mu > 0$ depending on the compactly-supported measure μ.

Our next goal is to use Lemma 10.4 and prove a bootstrap estimate for a certain class of functions considered by Guan [2], which generalizes the class of functions $f(t) = t^q$ $(q \geq 3)$ considered in Chen [1].

Lemma 10.6. *Let $D > 1, r \in [2,3], t > 0$ and let τ be a stopping time. Suppose that $f : [0,\infty) \to [0,\infty)$ is a smooth, increasing function such that*

$$\begin{cases} f(x) = x^2, & \forall x \geq r \\ f''(x) \leq D^2 f(x), & \forall x \geq 0 \end{cases} \tag{20}$$

Then, for almost any $t > 0$,

$$\frac{d}{dt}\mathbb{E}\sum_{i=1}^n f(\lambda_i(t\wedge\tau)) \leq C\left(\frac{1}{t} + \frac{D^2}{\sqrt{t}}\right)\cdot \mathbb{E}\sum_{i=1}^n f(\lambda_i(t\wedge\tau)). \tag{21}$$

where $C > 0$ is a universal constant.

Proof. Abbreviate $\lambda_i = \lambda_i(t)$ and $\xi_{ij} = \xi_{ij}(t)$. Since f is positive, by Lemma 10.4 it suffices to prove that

$$\sum_{i,j=1}^n |\xi_{ij}|^2 \frac{f'(\lambda_i) - f'(\lambda_j)}{\lambda_i - \lambda_j} \leq C\left(\frac{1}{t} + \frac{D^2}{\sqrt{t}}\right)\cdot\sum_{i=1}^n f(\lambda_i). \tag{22}$$

Since the probability density p_t is t-uniformly log-concave, Lemma 10.1 shows that for any $s > 0$ and $k = 1,\dots,n$,

$$\sum_{i,j=1}^n \xi_{ijk}^2 1_{\{\lambda_i\vee\lambda_j\leq s\}} \leq 4t^{-1/2}s^{3/2}\lambda_k. \tag{23}$$

The bound (22) follows from several applications of (23) as well as from the bound

$$\lambda_i \leq 1/t$$

which was discussed in (8).

Step 1. Since ξ_{ijk} is symmetric in i, j and k, by using (23) with $s = \lambda_i$ we see that

$$\begin{aligned}\sum_{i,j=1}^n |\xi_{ij}|^2 1_{\{\lambda_i\geq r\}} &= \sum_{i,j,k}\xi_{ijk}^2 1_{\{\lambda_i\geq r\}} \leq 3\sum_i 1_{\{\lambda_i\geq r\}}\sum_{j,k}\xi_{ijk}^2 1_{\{\lambda_j\vee\lambda_k\leq\lambda_i\}} \\ &\leq \frac{12}{\sqrt{t}}\sum_i \lambda_i^{5/2} 1_{\{\lambda_i\geq r\}} \leq \frac{12}{t}\sum_i \lambda_i^2 1_{\{\lambda_i\geq r\}} \leq \frac{12}{t}\sum_{i=1}^n f(\lambda_i).\end{aligned} \tag{24}$$

Step 2. Consider the contribution to the left-hand side of (22) of all i, j with

$$\min\{\lambda_i, \lambda_j\} \geq r. \tag{25}$$

Since $f'(x) = 2x$ when $x \geq r$, this contribution equals

$$\begin{aligned}\sum_{i,j=1}^{n} \frac{f'(\lambda_i) - f'(\lambda_j)}{\lambda_i - \lambda_j} |\xi_{ij}|^2 1_{\{\min(\lambda_i,\lambda_j)\geq r\}} &= 2\sum_{i,j} |\xi_{ij}|^2 1_{\{\min(\lambda_i,\lambda_j)\geq r\}} \\ &\leq 2\sum_{i,j} |\xi_{ij}|^2 1_{\{\lambda_i \geq r\}} \leq \frac{24}{t},\end{aligned}$$

where we used (24) in the last passage.

Step 3. Consider the contribution to the left-hand side of (22) of all i, j with

$$\lambda_i \leq r, \lambda_j \geq r+1 \qquad \text{or} \qquad \lambda_j \leq r, \lambda_i \geq r+1. \tag{26}$$

This contribution equals

$$\begin{aligned}&2\sum_{i,j=1}^{n} \frac{f'(\lambda_i) - f'(\lambda_j)}{\lambda_i - \lambda_j} |\xi_{ij}|^2 1_{\{\lambda_i \geq r+1\}} 1_{\{\lambda_j \leq r\}} \\ &\qquad \leq \sum_{i,j} \frac{4\lambda_i}{\lambda_i - \lambda_j} |\xi_{ij}|^2 1_{\{\lambda_i \geq r+1\}} 1_{\{\lambda_j \leq r\}} \leq 16 \sum_{i,j} |\xi_{ij}|^2 1_{\{\lambda_i \geq r\}} \leq \frac{16 \cdot 12}{t}\end{aligned}$$

Here we used that $f' \geq 0$ as well as the fact that $\lambda_i/(\lambda_i - \lambda_j) \leq r+1 \leq 4$ when $\lambda_j \leq r$ and $\lambda_i \geq r+1$, and in the last passage we used (24).

Step 4. Let us show that

$$\sum_{i,j=1}^{n} |\xi_{ij}|^2 f(\lambda_i) 1_{\{\lambda_i \vee \lambda_j \leq r+1\}} \leq \frac{C}{\sqrt{t}} \sum_{k=1}^{n} f(\lambda_k). \tag{27}$$

Write $a \vee b \vee c = \max\{a, b, c\}$. By applying (23) with $s = r+1$, and recalling that $r \leq 3$,

$$\begin{aligned}&\sum_{i,j,k=1}^{n} f(\lambda_i) \xi_{ijk}^2 1_{\{\lambda_i \vee \lambda_j \vee \lambda_k \leq r+1\}} \qquad (28) \\ &\qquad \leq \frac{4}{\sqrt{t}} \sum_{i} f(\lambda_i) \cdot (r+1)^{3/2} \lambda_i \cdot 1_{\{\lambda_i \leq r+1\}} \leq \frac{4^{7/2}}{\sqrt{t}} \sum_{i=1}^{n} f(\lambda_i).\end{aligned}$$

Next, we use that if $\lambda_i \leq r+1$ then $f(\lambda_i) \leq f(r+1) = (r+1)^2 \leq 16$ while if $\lambda_i \geq r+1$ then $f(\lambda_i) = \lambda_i^2$. We again apply (23) with $s = r+1$ to obtain

$$\sum_{i,j,k=1}^{n} f(\lambda_i)\xi_{ijk}^2 1_{\{\lambda_i \vee \lambda_j \leq r+1 \leq \lambda_k\}} \leq \frac{C}{\sqrt{t}} \sum_k \lambda_k 1_{\{\lambda_k \geq r+1\}} \tag{29}$$

$$\leq \frac{C'}{\sqrt{t}} \sum_k \lambda_k^2 1_{\{\lambda_k \geq r+1\}} \leq \frac{C'}{\sqrt{t}} \sum_{k=1}^{n} f(\lambda_k).$$

By adding (28) and (29) we obtain (27).

Step 5. Consider the contribution to the left-hand side of (22) of all i, j with

$$\max\{\lambda_i, \lambda_j\} \leq r+1. \tag{30}$$

By using (20) and the fact that f is non-negative and increasing, we see that this contribution is at most

$$2\sum_{i,j=1}^{n} \frac{f'(\lambda_i) - f'(\lambda_j)}{\lambda_i - \lambda_j} |\xi_{ij}|^2 1_{\{\lambda_j \leq \lambda_i \leq r+1\}} \leq 4D^2 \sum_{i,j} f(\lambda_i)|\xi_{ij}|^2 1_{\{\lambda_j \leq \lambda_i \leq r+1\}}$$

$$\leq C\frac{D^2}{\sqrt{t}} \sum_{k=1}^{n} f(\lambda_k),$$

where we used (27) in the last passage.

The results of Step 2, Step 3 and Step 5 imply the desired bound (22). □

It is a calculus exercise to prove that for any $r \in [2,3]$ and $D > 1$ there exists a smooth, increasing function $f : [0,\infty) \to (0,\infty)$ with

$$f(x) = \begin{cases} e^{D(x-r)} & x \leq r - \frac{1}{D} \\ x^2 & x \geq r \end{cases} \tag{31}$$

and $f''(x) \leq D^2 f(x)$ for all $x \geq 0$. We denote this function f by $f_{r,D}$, and observe that it satisfies condition (20) of From Lemma 10.6. From the conclusion of the lemma we conclude that for any $D > 1, 2 \leq r \leq 3$ and a stopping time τ, if

$$0 < t \leq D^{-4} \tag{32}$$

then $D^2/\sqrt{t} \leq 1/t$ and hence for $f = f_{r,D}$

$$\frac{d}{dt}\mathbb{E}\sum_{i=1}^{n} f(\lambda_i(t \wedge \tau)) \leq \frac{C}{t} \cdot \mathbb{E}\sum_{i=1}^{n} f(\lambda_i(t \wedge \tau)). \tag{33}$$

The function $f = f_{r,D}$ is slightly complicated, and we prefer to reformulate the growth condition (33) in terms of the much simpler function

$$g_r(x) = x^2 \cdot 1_{\{x \geq r\}}.$$

From (31),

$$g_r \leq f_{r,D}. \tag{34}$$

In the other direction, we claim that for any $D > 1$ and $x \geq 0$, if

$$2 \leq r + \frac{1}{D} \leq \tilde{r} \leq 3$$

then

$$f_{\tilde{r},D}(x) \leq \frac{9}{4} g_r(x) + \exp(-D(\tilde{r} - r)). \tag{35}$$

Indeed, if $x \leq r$ then by (31), since $r \leq \tilde{r} - 1/D$,

$$f_{\tilde{r}}(x) \leq f_{\tilde{r}}(r) = \exp(-D(\tilde{r} - r)),$$

and (35) holds true in this case. If $x \geq \tilde{r}$ then both $f_{\tilde{r}}(x)$ and $g_r(x)$ equal x^2, and (35) trivially holds. In the remaining case $r < x < \tilde{r}$ we have

$$f_{\tilde{r}}(x) \leq f_{\tilde{r}}(\tilde{r}) = \tilde{r}^2 \leq \left(\frac{\tilde{r}}{r}\right)^2 \cdot x^2 \leq \frac{9}{4} x^2 = \frac{9}{4} g_r(x),$$

completing the proof of (35).

Proof of Proposition 10.3. We may assume that $t \leq 2^{-8}$ as otherwise there is nothing to prove. We will set $t_0 = t$ and partition the interval $[0, t]$ into intervals

$$[t_1, t_0], [t_2, t_1], \ldots, [t_{k+1}, t_k], \ldots$$

For $k \geq 0$ we define

$$t_k = 2^{-8k} t, \qquad D_k = t_k^{-1/4}, \qquad r_k = 3 - \sum_{i=0}^{k-1} t_i^{1/8} \in [2, 3].$$

Since $t_k \leq D_k^{-4}$ we may use the differential inequality (33) for all $s \in [t_{k+1}, t_k]$. By integrating this differential inequality over this interval, we obtain

$$\mathbb{E} \sum_{i=1}^{n} f_k(\lambda_i(t_k \wedge \tau)) \leq \left(\frac{t_k}{t_{k+1}}\right)^C \mathbb{E} \sum_{i=1}^{n} f_k(\lambda_i(t_{k+1} \wedge \tau)), \tag{36}$$

where $f_k = f_{r_k, D_k}$. Set also $g_k = g_{r_k}$ and define

$$F_k = \mathbb{E} \sum_{i=1}^n f_k(\lambda_i(t_k \wedge \tau)) \qquad \text{and} \qquad G_k = \mathbb{E} \sum_{i=1}^n g_k(\lambda_i(t_k \wedge \tau)).$$

Note that $r_{k+1} + 1/D_k \le r_{k+1} + 1/\sqrt{D_k} = r_k$. From (36), as well as the two inequalities (34) and (35), we obtain for $k \ge 0$,

$$\begin{aligned} G_k \le F_k &\le \left(\frac{t_k}{t_{k+1}} \right)^C \mathbb{E} \sum_{i=1}^n f_{r_k, D_k}(\lambda_i(t_{k+1} \wedge \tau)) \\ &\le \left(\frac{t_k}{t_{k+1}} \right)^C \mathbb{E} \sum_{i=1}^n \left[\frac{9}{4} g_{k+1}(\lambda_i(t_{k+1} \wedge \tau)) + e^{-D_k(r_k - r_{k+1})} \right] \\ &= 2^{8C} \left(\frac{9}{4} G_{k+1} + n \exp(-t_k^{-1/8}) \right) \le \bar{C} \left[G_{k+1} + n \exp(-2^k t^{-1/8}) \right]. \end{aligned} \tag{37}$$

From this recursive inequality we obtain that for $k \ge 0$,

$$G_0 \le \bar{C}^k G_k + n \cdot \sum_{i=0}^{k-1} \bar{C}^{i+1} \exp(-2^i t^{-1/8}) \le \bar{C}^k G_k + \tilde{C} n \cdot e^{-t^{-1/8}}, \tag{38}$$

since the sum in (38) is at most

$$\sum_{i=0}^{k-1} \bar{C}^{i+1} \exp(-2^i t^{-1/8}) \le \sum_{i=0}^{\infty} \bar{C}^{i+1} \exp(-2^i - t^{-1/8}) = \bar{C} \cdot e^{-t^{-1/8}}.$$

We next show that $\bar{C}^k G_k \longrightarrow 0$ as $k \to \infty$. To this end we use (19). Since μ is compactly-supported, for some $C_\mu > 0$ depending on μ and for a sufficiently large k,

$$G_k \le C_\mu \cdot \mathbb{P}\left(\lambda_1(t_k \wedge \tau) \ge 2 \right) \le \tilde{C}_\mu e^{-c_\mu / t_k} = \tilde{C}_\mu e^{-c_\mu \cdot 2^{8k}/t}.$$

Hence indeed $\bar{C}^k G_k \longrightarrow 0$ as $k \to \infty$, and from (38),

$$\sum_{i=0}^n \mathbb{P}(\lambda_i(t \wedge \tau) \ge 3) \le G_0 \le \tilde{C} n \cdot e^{-t^{-1/8}}.$$

□

We end this lecture with an interpretation of our results in the context of the Prékopa-Leindler inequality. Recall that we write

$$\gamma_s(x) = (2\pi s)^{-n/2} \exp(-|x|^2/(2s))$$

for the density of a centered Gaussian random vector of covariance $s \cdot \mathrm{Id}$ in $\mathbb{R}^n$. Let p be an isotropic, log-concave density in $\mathbb{R}^n$ and for $t > 0$ set

$$q_t = p * \gamma_{1/t}.$$

By the Prékopa-Leindler inequality, the probability density q_t is log-concave, since it is a convolution of two log-concave probability measures. A straightforward computation based on (1) shows that

$$\nabla^2(-\log q_t)(x) = t^2 \left(\frac{\mathrm{Id}}{t} - \mathrm{Cov}(p_{t,tx}) \right) = t^2 \left(\frac{\mathrm{Id}}{t} - A_t(tx) \right).$$

Thus the log-concavity of q_t amounts to the inequality $A_t \leq \mathrm{Id}/t$, which was one of the starting points of our analysis today. By using the "Option 1" definition of θ_t, we see that for $t > 0$,

$$\int_{\mathbb{R}^n} \left| \frac{\mathrm{Id}}{t} - \frac{\nabla^2(-\log q_t)(x)}{t^2} \right|^2 q_t(x) \mathrm{d}x = \mathbb{E}|A_t|^2 \leq Cn \tag{39}$$

where $|\cdot|$ is the Hilbert-Schmidt norm, and where the last inequality in (39) follows from Proposition 10.3. Thus, on a quantitative level, inequality (39) is a refinement of the Prékopa-Leindler inequality which amounts to the pointwise bound

$$0 \leq \nabla^2(-\log q_t) \leq t \cdot \mathrm{Id}.$$

Exercises.

1. Why can we assume that μ is compactly-supported when proving the thin-shell theorem?
2. prove that for any $D > 1$ and $r \in [2,3]$ there exists a smooth, increasing function $f : [0,\infty) \to [0,\infty)$ satisfying (20).
3. Consider the isotropic, log-concave probability density

$$p(x_1, \ldots, x_n) = 2^n e^{-\sum_{i=1}^n 2|x_i|}.$$

   (a) Prove that in this case, for any $t > 0$ the matrix A_t is diagonal and its diagonal entries are independent and identically-distributed. Write Z_t for the $(1,1)$-entry of A_t, and explain that its law does not depend on n.

   (b) Prove that the support of the random variable Z_t is *not* uniformly bounded for all $t \in (0,1)$.

(c) Prove that if $x > 0$ is such that $\mathbb{P}(Z_t \geq x) \geq 1/n$, then $\mathbb{E}\|A_t\|_{op} \geq x/2$.

(d) Conclude that $\sup_{0<t<1} \mathbb{E}\|A_t\|_{op} \geq \alpha_n$, for some sequence $\alpha_n \longrightarrow \infty$.

4. Assume that μ is isotropic, compactly-supported probability measure in $\mathbb{R}^n$.

   (a) Use Lemma 10.4 and show that there exists $R = R_\mu > 0$ such that for a convex, smooth, increasing function $f : [0, \infty) \to \mathbb{R}$, and almost all $t > 0$,

$$\frac{d}{dt}\mathbb{E}\sum_{i=1}^{n} f(\lambda_i(t)) \leq R \sum_{i,j=1}^{n} \mathbb{E}f''(\lambda_i(t)).$$

   (b) For $\beta > 0$ and $t > 0$ define

$$F_{\beta,t} = \frac{1}{\beta}\log\mathbb{E}\sum_{i=1}^{n} e^{\beta\lambda_i(t)}.$$

   Prove that

$$F_{\beta,t} \leq tR\beta + \frac{\log n}{\beta} + 1.$$

   (c) Write $p = \mathbb{P}(\lambda_1(t) \geq 2)$. Prove that for $\beta \geq 2\log n$,

$$\log p \leq tR\beta^2 - \frac{\beta}{2}.$$

   Set $\beta = 1/(4tR)$ and conclude that for a sufficiently small $t > 0$,

$$\mathbb{P}(\lambda_1(t) \geq 2) \leq \exp(-c_\mu/t)$$

   for some $c_\mu > 0$ depending on μ.

References

[1] Chen, Y., *An almost constant lower bound of the isoperimetric coefficient in the KLS conjecture*. Geom. Funct. Anal. (GAFA), Vol. 31, no. 1, (2021), 34–61.

[2] Guan, Q., *A note on Bourgain's slicing problem*. arXiv:2412.09075

[3] Guan, Q., *On tail probability of the covariance matrix in Eldan's stochastic localization*. arXiv:2508.14943

[4] Klartag, B., Lehec, J., *Thin-shell bounds via parallel coupling*. arXiv:2507.15495

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