Isoperimetric inequalities in high-dimensional convex sets
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Lecture 1: The cube and the sphere in high dimensions

In these lectures we study geometry in an n-dimensional Euclidean space when the dimension n is very large, tending to infinity. We will encounter high-dimensional phenomena that do not arise in dimension 3 or 7, say, such as concentration of measure or the emergence of approximately symmetric substructures.

The simplest examples of geometric shapes in $\mathbb{R}^n$ are perhaps the unit cube

$$Q^n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n,$$

and the Euclidean unit sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \, ; \, |x| = 1\},$$

where $|x| = \sqrt{\langle x, x \rangle}$ is the Euclidean norm of the vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and we denote the scalar product of $x, y \in \mathbb{R}^n$ by $\langle x, y \rangle = x \cdot y = \sum_i x_i y_i$. Additional examples of geometric shapes in $\mathbb{R}^n$ include the Euclidean unit ball

$$B^n = \{x \in \mathbb{R}^n \, ; \, |x| \leq 1\},$$

whose features are rather similar to those of the unit sphere, the *cross-polytope* which is the convex hull of the $2n$ vectors

$$\pm e_1, \ldots, \pm e_n \in \mathbb{R}^n,$$

and simplices, where an n-dimensional simplex in $\mathbb{R}^n$ is the convex hull of $n+1$ vectors that affinely span $\mathbb{R}^n$. Here, $e_i \in \mathbb{R}^n$ is the standard i^{th} unit vector. Note that a regular $(n-1)$-dimensional simplex is conveniently represented in $\mathbb{R}^n$ as the convex hull of $e_1, \ldots, e_n \in \mathbb{R}^n$.

1.1 The unit cube

Consider the unit cube $Q^n = [-1/2, 1/2]^n \subseteq \mathbb{R}^n$. There are two relevant length-scales for this cube: its sidelength, which is 1, and its diameter, which is

$$\sqrt{n} = \left| \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) - \left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right) \right|.$$

Here, the diameter of $K \subseteq \mathbb{R}^n$ is

$$\mathrm{diam}(K) = \sup_{x,y \in K} |x - y|.$$

The $\sqrt{n}$ lengthscale is slightly more prevalent in the analysis of the high-dimensional cube; if we are forced to compare the unit cube to a Euclidean ball of a certain radius, then we should choose a ball of radius on the order of $\sqrt{n}$ (or in some cases $\sqrt{n/\log n}$). For example, what is the typical distance between two random points in the unit cube? That is, let

$$X = (X_1, \ldots, X_n) \sim \mathrm{Unif}(Q^n)$$

and

$$Y = (Y_1, \ldots, Y_n) \sim \mathrm{Unif}(Q^n)$$

be two independent random vectors, each distributed uniformly in the unit cube Q^n. We are interested in typical values of the random variable $|X - Y|$. Its L^2-norm is easy to compute. Indeed, since $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are independent random variables, all distributed uniformly in the interval $[-1/2, 1/2]$, we have

$$\sqrt{\mathbb{E}|X-Y|^2} = \sqrt{\mathbb{E}\sum_{i=1}^n (X_i - Y_i)^2} = \sqrt{n \cdot 2\mathrm{Var}(X_1)} = \sqrt{n/6}.$$

The random variable $|X - Y|$ is actually *concentrated* around the value $\sqrt{n/6}$, and in fact, for any $t > 0$, the probability that it deviates from this value by more than t may be bounded as follows:

$$\mathbb{P}\left(\left| |X-Y| - \sqrt{\frac{n}{6}} \right| \geq t\right) \leq C\exp(-ct^2), \tag{1}$$

for some universal constants $c, C > 0$. Inequality (1) shows that most of the mass of the random vector $X - Y$ is contained in a thin spherical shell of radius $\sqrt{n/6}$ and width $O(1)$. Here $B = O(A)$ means that $|B| \leq CA$, where $C > 0$ is some universal constant. Two sources for such concentration inequalities to be discussed in these lectures are *independence* and *convexity*. Let us describe a proof of (1) which relies on statistical independence: Observe that for any $u \in \mathbb{R}$ and $t > 0$,

$$\left| u - \sqrt{\frac{n}{6}} \right| \geq t \qquad \Longrightarrow \qquad \left| u^2 - \frac{n}{6} \right| \geq t\sqrt{\frac{n}{6}}$$

and hence (1) would follow once we prove that

$$\mathbb{P}\left(\left| \frac{|X-Y|^2 - \frac{n}{6}}{\sqrt{n}} \right| \geq t\right) \leq C\exp(-ct^2), \tag{2}$$

for some universal constants $c, C > 0$. Since the random variable $|X - Y|^2 = \sum_{i=1}^n (X_i - Y_i)^2$ is a sum of independent, identically-distributed (i.i.d), bounded random variables, the random variable $|X - Y|^2$ is approximately *a Gaussian random variable* of mean $n/6$ and standard deviation $C\sqrt{n}$. The deviation inequality (2) fits with this Gaussian approximation; more precisely, it states that the random variable $|X-Y|^2$ has a uniformly sub-gaussian tail, relative to its mean and variance. This follows from the Bernstein (or Hoeffding) concentration inequality for sums of bounded, independent random variables, which is the subject of a guided exercise below.

Our next question about the cube concerns the volumes of its hyperplane sections. For any $\theta \in S^{n-1}$, writing $\theta^\perp = \{x \in \mathbb{R}^n \,;\, \langle x, \theta \rangle = 0\}$ for its orthogonal complement, we have

$$1 \leq \mathrm{Vol}_{n-1}\left(\theta^\perp \cap Q^n\right) \leq \sqrt{2} \tag{3}$$

where the inequality on the left-hand side is due to Hensley [5] and equality is attained when $\theta = e_i$; we will (probably) prove Vaaler's stronger version [9] when we study the Prékopa-Leindler inequality. The inequality on the right-hand side of (3) is due to Ball [1] (see also the simpler proof in [7]) and equality is attained when $\theta = (1, 1, 0, \ldots, 0)/\sqrt{2}$.

We thus know that volumes of central hyperplane sections of the unit cube can fluctuate between the values 1 and $\sqrt{2}$. What is the "typical value" within this interval $[1, \sqrt{2}]$?

Claim 1.1. *For a typical $\theta \in S^{n-1}$, and in particular for $\theta = (1, \ldots, 1)/\sqrt{n}$, we have*

$$\mathrm{Vol}_{n-1}\left(\theta^\perp \cap Q^n\right) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{12} \cdot \left(1 + O\left(\frac{1}{n}\right)\right). \tag{4}$$

Here, "typical" refers to the uniform probability measure on S^{n-1}, to be described shortly. (The $1/\sqrt{2\pi}$ factor in (4) comes from the Gaussian distribution, while the $\sqrt{12}$ factor is one over a certain standard deviation.)

Claim 1.1 is related to the classical *Central Limit Theorem* (CLT). Indeed, if

$$X = (X_1, \ldots, X_n) \sim \mathrm{Unif}(Q^n),$$

i.e., the random variables $X_1, \ldots, X_n \sim \mathrm{Unif}([-1/2, 1/2])$ are independent, then

$$\sum_{i=1}^n \theta_i X_i = \langle X, \theta \rangle$$

is approximately Gaussian for $\theta = (1, \ldots, 1)/\sqrt{n}$, as well as for other choices of a vector of coefficients $\theta \in S^{n-1}$. More precisely, we have the following classical result:

Theorem 1.2. *(CLT, version 1) For any $\theta \in S^{n-1}$ and $t \in \mathbb{R}$,*

$$\left| \mathbb{P}\left(\sqrt{12}\langle X, \theta\rangle \leq t\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2} ds \right| \leq C \sum_{i=1}^{n} \theta_i^4, \tag{5}$$

where $C > 0$ is a universal constant. (Note that $\sqrt{12}\langle X, \theta\rangle$ is a random variable of mean zero and variance one.)

The usual proof of Theorem 1.2 involves the Fourier inversion formula, see e.g. Feller [3, Chapter XVI] or the guided exercise below. If $\theta = (1, 0, \ldots, 0)$ then $\sum_i \theta_i^4 = 1$ and inequality (5) is vacuous. However, for a typical $\theta \in S^{n-1}$, including the case $\theta = (1, \ldots, 1)/\sqrt{n}$, we have

$$\sum_i \theta_i^4 = O\left(\frac{1}{n}\right), \tag{6}$$

which is the correct rate of approximation in the CLT for the high-dimensional cube.[1] Let us provide a geometric interpretation of the CLT for the cube. Write $f_\theta : \mathbb{R} \to [0, \infty)$ for the density of the random vector $\langle X, \theta\rangle$. A moment of reflection reveals that

$$f_\theta(t) = \mathrm{Vol}_{n-1}\left(H_{\theta,t} \cap Q^n\right)$$

where

$$H_{\theta,t} = \{x \in \mathbb{R}^n \,;\, \langle x, \theta\rangle = t\} \tag{7}$$

is a hyperplane orthogonal to $\theta \in S^{n-1}$ of distance $|t|$ from the origin. By Fubini's theorem, for $s < t$,

$$\mathrm{Vol}_n\left(\{x \in Q^n \,;\, s \leq \langle x, \theta\rangle \leq t\}\right) = \mathbb{P}(s \leq \langle X, \theta\rangle \leq t) = \int_s^t f_\theta(r) dr.$$

Thus Theorem 1.2 provides Gaussian asymptotic estimates for the volume of the intersection of the unit cube with various planks; a plank is the region in space bounded by two parallel hyperplanes. Observe that

$$\frac{1}{\sqrt{12}} f_\theta\left(\frac{t}{\sqrt{12}}\right) \qquad (t \in \mathbb{R})$$

is the density of the random variable $\sqrt{12}\langle X, \theta\rangle$. Theorem 1.2 admits the following little variant:

[1] It is a better (faster) rate than the $O(1/\sqrt{n})$ rate that we have for the CLT for the *discrete* cube $\{-1, 1\}^n$, and which also appears in the Berry-Esseen bound, see Feller [3, Chapter XVI].

Theorem 1.3. *(CLT, version 2) Under the assumptions of Theorem 1.2,*

$$\left| \frac{1}{\sqrt{12}} f_\theta\left(\frac{t}{\sqrt{12}}\right) - \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \right| \leq C \sum_{i=1}^n \theta_i^4,$$

where $C > 0$ is a universal constant.

Theorem 1.3 with $t = 0$ justifies Claim 1.1 above, and may be used in order to show that the volume of typical central hyperplane sections of the cube concentrate around the value $\sqrt{6/\pi}$. To summarize, when considering volumes of hyperplane sections, we observe a simpler behavior for the high-dimensional cube than for the cube in dimension 5, say. Let us also mention that the question of volumes of *hyperplane projections* of the cube is easier to analyze; for any $\theta \in S^{n-1}$ we have the McMullen formula (see [6]),

$$\mathrm{Vol}_{n-1}\left(Proj_{\theta^\perp}(Q^n)\right) = \sum_{i=1}^n |\theta_i|, \tag{8}$$

where $Proj_{\theta^\perp} : \mathbb{R}^n \to \theta^\perp$ is the orthogonal projection operator, i.e., $Proj_{\theta^\perp} x = x - \langle x, \theta \rangle \theta$.

1.2 The Euclidean unit ball and sphere

The unit cube in $\mathbb{R}^n$ has volume one. By contrast, the volume of the Euclidean unit ball $B^n = \{x \in \mathbb{R}^n \,;\, |x| \leq 1\}$ is rather tiny:

$$\kappa_n := \mathrm{Vol}_n(B^n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} = \left(\frac{\sqrt{2\pi e} + o(1)}{\sqrt{n}} \right)^n. \tag{9}$$

We thus need to scale the Euclidean unit ball by a factor of the order of $\sqrt{n}$ in order to obtain a body of volume one. More precisely, the radius of the Euclidean ball of volume one is

$$r_n = \kappa_n^{-1/n} \approx \frac{\sqrt{n}}{\sqrt{2\pi e}},$$

since $\mathrm{Vol}_n(r_n B^n) = r_n^n \cdot \kappa_n = 1$. We scale the Euclidean unit ball of a factor of $\sqrt{n}$, and consider a random vector

$$X \sim \mathrm{Unif}(\sqrt{n} B^n),$$

where $\lambda K = \{\lambda x \,;\, x \in K\}$ for $\lambda \in \mathbb{R}$ and $K \subseteq \mathbb{R}^n$.

Is it true that the random vector $\langle X, \theta \rangle$ is approximately Gaussian for $\theta \in S^{n-1}$, like in the case of the high-dimensional cube?

The answer is YES. In fact, by symmetry, the distribution of $\langle X, \theta \rangle$ does not depend on $\theta \in S^{n-1}$, and we may write $f_\theta(t) = f(t)$ for the density of $\langle X, \theta \rangle$. Thus,

$$f_\theta(t) = \frac{\mathrm{Vol}_{n-1}(H_{\theta,t} \cap \sqrt{n}B^n)}{\mathrm{Vol}_n(\sqrt{n}B^n)} \qquad (t \in \mathbb{R}),$$

with $H_{\theta,t}$ as in (7). When $|t| \leq \sqrt{n}$, the slice

$$H_{\theta,t} \cap \sqrt{n}B^n$$

is an $(n-1)$-dimensional ball of radius $\sqrt{n - t^2}$, by the Pythagoras theorem. Consequently,

$$f(t) = \frac{\kappa_{n-1}(\sqrt{(n-t^2)_+})^{n-1}}{n^{n/2}\kappa_n} = c_n \left(1 - \frac{t^2}{n}\right)_+^{\frac{n-1}{2}} \tag{10}$$

with $c_n = \kappa_{n-1}/(\sqrt{n}\kappa_n) = 1/\sqrt{2\pi} + O(1/n)$ by the Stirling formula. The proof of the CLT for the uniform distribution on the cube requires indirect tools such as the Fourier transform. In contrast, the case of the Euclidean ball is conceptually simpler, even though the random variables $X_1, \ldots, X_n$ are no longer independent:[2]

Proposition 1.4. *For any $t \in \mathbb{R}$,*

$$\left| f(t) - \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \right| \leq \frac{C}{n}, \tag{11}$$

where $C > 0$ is a universal constant.

Proof. If $|t| \geq n^{1/4}$ then $e^{-t^2/2} \leq e^{-\sqrt{n}/2} \leq C/n$ while

$$\left(1 - \frac{t^2}{n}\right)_+^{\frac{n-1}{2}} \leq e^{-t^2(n-1)/(2n)} \leq \frac{C}{n},$$

and the bound (11) holds true. If $|t| \leq n^{1/4}$ then we may use the Taylor approximation $\log(1-x) = -x + O(x^2)$ for $|x| \leq 1/2$, which yields

$$\frac{n-1}{2} \log\left(1 - \frac{t^2}{n}\right) = -\frac{n-1}{2}\frac{t^2}{n} + O\left(\frac{t^4}{n}\right) = -\frac{t^2}{2} + O\left(\frac{t^4+1}{n}\right).$$

Therefore, for $|t| \leq n^{1/4}$,

$$\left(1 - \frac{t^2}{n}\right)_+^{\frac{n-1}{2}} = \exp\left[-t^2/2 + O\left(\frac{t^4+1}{n}\right)\right] = e^{-\frac{t^2}{2}} + O\left(\frac{1}{n}\right). \tag{12}$$

□

[2] When was it discovered that the marginals of the high-dimensional sphere are approximately Gaussian? Diaconis and Freedman [2, Section 6] searched in vain for this observation in Poincaré's writings, but found it in Borel's book from 1914 in connection with the kinetic theory of gas.

Where is the "bulk" of the mass of the high-dimensional Euclidean ball located? One answer is "near the boundary". Recall that a star body in $\mathbb{R}^n$ is a subset $K \subseteq \mathbb{R}^n$ such that $tK \subseteq K$ for $0 \le t \le 1$. One property of the high-dimensional Euclidean ball, or any star body in $\mathbb{R}^n$, is that most of its mass lies near the boundary. Indeed, when $X \sim \text{Unif}(B^n)$, for any $0 \le t \le 1$,

$$\mathbb{P}(|X| \le t) = \frac{\text{Vol}_n(tB^n)}{\text{Vol}_n(B^n)} = t^n. \tag{13}$$

It follows that for $n \ge 2$,

$$\mathbb{P}\left(1 - \frac{1}{n} \le |X| \le 1\right) = 1 - \left(1 - \frac{1}{n}\right)^n \ge \frac{1}{2}. \tag{14}$$

We see from (14) that most of the mass of the unit ball is located at distance only $O(1/n)$ from its boundary. The distribution of volume on the high-dimensional Euclidean ball is rather close to that on the high-dimensional sphere, and results on S^{n-1} can sometimes be translated to corresponding results on B^n and vice versa.

Another answer for the above question is "near the equator". We slightly prefer to work now with the unit sphere S^{n-1}, since it is a *homogenous space*, admitting a transitive group of symmetries. In other words, all points of the sphere S^{n-1} lie on the same footing, while the ball B^n contains "special points" like the origin.

What is the volume of the Euclidean unit sphere? By integrating in polar coordinates,

$$\text{Vol}_n(B^n) = \text{Vol}_{n-1}(S^{n-1}) \cdot \int_0^1 r^{n-1} dr = \frac{1}{n} \cdot \text{Vol}_{n-1}(S^{n-1})$$

or more succinctly:

Claim 1.5.

$$\text{Vol}_{n-1}(S^{n-1}) = n\kappa_n.$$

We write σ_{n-1} for the uniform probability measure on S^{n-1}, which can either be viewed as the normalized surface area measure on S^{n-1}, or as the unique rotationally-invariant (Haar) probability measure on S^{n-1}.

It is convenient to replace spherical integrals with a Gaussian computation via integration in polar coordinates. Indeed, let

$$Z = (Z_1, \ldots, Z_n)$$

be a standard Gaussian vector in $\mathbb{R}^n$ (i.e., its components are independent, standard Gaussian random variables). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positively-p-homogeneous function (i.e., $f(\lambda x) = \lambda^p f(x)$ for $\lambda > 0$ and $x \in \mathbb{R}^n$). Then,

$$\begin{aligned}\mathbb{E} f(Z) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-|x|^2/2} dx \\ &= (2\pi)^{-n/2} \cdot n\kappa_n \cdot \int_0^\infty \int_{S^{n-1}} f(r\theta) e^{-|r\theta|^2/2} r^{n-1} dr d\sigma_{n-1}(\theta) \\ &= C_{n,p} \int_{S^{n-1}} f(\theta) d\sigma_{n-1}(\theta),\end{aligned}$$

where $C_{n,p} = 2^{p/2-1} \cdot n \cdot \Gamma(\frac{n+p}{2})/\Gamma(\frac{n+2}{2})$. For instance, in order to show that a typical vector $\theta \in S^{n-1}$ satisfies (6), we may argue as follows:

$$\int_{S^{n-1}} \left(\sum_{i=1}^n \theta_i^4 \right) d\sigma_{n-1}(\theta) = \frac{1}{C_{n,4}} \mathbb{E} \sum_{i=1}^n Z_i^4 = \frac{1}{n(n+2)} \cdot 3n = \frac{3}{n+2},$$

as $\mathbb{E} Z_1^4 = 3$. See the exercises below for more information on the distribution of $\sum_i \theta_i^4$ where $\theta = (\theta_1, \ldots, \theta_n)$ is a uniformly-distributed random vector in the sphere S^{n-1}. Another relation between the uniform measures on the ball and the sphere is the following fact, going back to Archimedes in the case $n = 3$.

Proposition 1.6. *For $n \geq 3$, if*

$$X = (X_1, \ldots, X_n) \sim \textit{Unif}(S^{n-1})$$

then

$$(X_1, \ldots, X_{n-2}) \sim \textit{Unif}(B^{n-2}).$$

An analytic way to prove this is to note that by calculus, $(X_1, \ldots, X_{n-1})$ has density on B^{n-1} which equals $\tilde{c}_n / \sqrt{1 - |x|^2}$. We now integrate the density of $(X_1, \ldots, X_{n-1})$ along a suitable segment and obtain that the density of $(X_1, \ldots, X_{n-2})$ at the point $y \in B^{n-2}$ is

$$\tilde{c}_n \int_{-\sqrt{1-|y|^2}}^{\sqrt{1-|y|^2}} \frac{dt}{\sqrt{1 - |y|^2 - t^2}} = \tilde{c}_n \int_{-1}^1 \frac{ds}{\sqrt{1 - s^2}}$$

which is independent of $y \in B^{n-2}$. Here we changed variables $s = t/\sqrt{1 - |y|^2}$.

Corollary 1.7. *If $X = (X_1, \ldots, X_n) \sim \textit{Unif}(S^{n-1})$, then for $t \geq 0$,*

$$\mathbb{P}\left(\sqrt{n}|X_1| \geq t\right) \leq Ce^{-t^2/2} \tag{15}$$

where $C > 0$ is a universal constant.

Proof I. Since $(X_1, \ldots, X_{n-2}) \sim \text{Unif}(B^{n-2})$, the density of $\sqrt{n}X_1$ equals

$$c_n \left(1 - \frac{x^2}{n}\right)_+^{\frac{n-3}{2}} \qquad\qquad (x \in \mathbb{R})$$

with $c_n = 1/\sqrt{2\pi} + O(1/n)$. Hence, for $0 \leq t \leq \sqrt{n}$,

$$\mathbb{P}\left(\sqrt{n}|X_1| \geq t\right) = 2\int_t^{\sqrt{n}} c_n \left(1 - \frac{x^2}{n}\right)_+^{\frac{n-3}{2}} dx \leq C\int_t^{\infty} e^{-x^2(n-3)/(2n)} dx.$$

For $t \notin [1, \sqrt{n}]$ conclusion (15) is trivial, while for $1 \leq t \leq \sqrt{n}$ we may use that $\int_t^\infty xe^{-x^2/2}dx = e^{-t^2/2}$ and elementary manipulations to conclude (15). $\square$

Proof II (which I essentially learned from Afonso Bandeira). We may assume that $t \leq \sqrt{n}$, since otherwise the probability in question vanishes. Since $\sum_{i=1}^n X_i^2 = 1$ we have

$$\mathbb{P}\left(|X_1| \geq t/\sqrt{n}\right) \leq \mathbb{P}\left(X_1^2 + X_2^2 \geq \frac{t^2}{n}\right) = \mathbb{P}\left(\sum_{i=3}^n X_i^2 \leq 1 - \frac{t^2}{n}\right).$$

The random vector $(X_3, \ldots, X_n)$ is distributed uniformly in B^{n-2}, according to Proposition 1.6. Therefore, by (13),

$$\mathbb{P}\left(\sum_{i=3}^n X_i^2 \leq 1 - \frac{t^2}{n}\right) = \left(1 - \frac{t^2}{n}\right)^{\frac{n-2}{2}} \leq e^{-\frac{t^2}{2}\cdot\frac{n-2}{n}} \leq Ce^{-\frac{t^2}{2}}$$

with $C = e$. $\square$

In particular, we learn from Corollary 1.7 that when $X \sim \text{Unif}(S^{n-1})$,

$$\mathbb{P}\left(|X_1| \geq 1/10\right) \leq Ce^{-cn},$$

which is exponentially small in the dimension n. Thus,

Proposition 1.8. *Most of the mass of the high-dimensional sphere S^{n-1} is located rather close to the equator*

$$\{x \in S^{n-1}\,;\, x_1 = 0\},$$

i.e., at distance roughly $O(1/\sqrt{n})$ from this equator. By the symmetries of the sphere, the same applies for any equator

$$\{x \in S^{n-1}\,;\, \langle x, \theta\rangle = 0\},$$

with $\theta \in S^{n-1}$.

This startling high-dimensional effect is a manifestation of the *concentration of measure* phenomenon on the high-dimensional sphere.

1.3 The isoperimetric inequality on the sphere

One way to harness this concentration phenomenon is via the isoperimetric inequality on the sphere. For $A \subseteq S^{n-1}$ and $\varepsilon > 0$ consider the ε-neighborhood of the set A, defined as

$$A_\varepsilon = \left\{ x \in S^{n-1} \, ; \, d(x, A) < \varepsilon \right\}$$

where $d(x, A) = \inf_{y \in A} d(x, y)$ and $d(x, y) = |x - y|$ is the Euclidean distance between $x, y \in S^{n-1}$. Another option is to work with the geodesic distance on the sphere, namely $\rho(x, y) = \arccos\langle x, y \rangle \in [0, \pi]$. The Euclidean distance (also called here the "tunnel distance") is always shorter than the geodesic distance, though not by much: it is shorter by a multiplicative factor that does not exceed $\pi/2$. These two distances are essentially equivalent for our needs; note that $\cos \rho(x, y) = 1 - d^2(x, y)/2$.

For example, the ε-neighborhood of the hemisphere

$$H = \left\{ x \in S^{n-1} \, ; \, x_1 \leq 0 \right\},$$

is

$$H_\varepsilon = \left\{ x \in S^{n-1} \, ; \, x_1 \leq \varepsilon \cdot \sqrt{1 - \varepsilon^2/4} \right\}.$$

Clearly $\sigma_{n-1}(H) = 1/2$, while by the concentration of measure bound (15),

$$\sigma_{n-1}(H_\varepsilon) \ = \ \mathbb{P}(X_1 \leq \varepsilon \cdot \sqrt{1 - \varepsilon^2/4}) \ \geq \ \mathbb{P}\left(X_1 \leq \varepsilon/2\right) \ \geq \ 1 - Ce^{-c\varepsilon^2 n}. \quad (16)$$

Thus, the measure of the ε-neighborhood of the hemisphere is very close to one if, say, $\varepsilon = 1/10$ and n is large. The isoperimetric inequality of P. Lévy [4, 8] states that among all sets of σ_{n-1}-measure equal to $1/2$, the hemisphere *minimizes* the measure of the ε-neighborhood. Since the ε-neighborhood of the hemisphere already has relatively large measure, this fact has far-reaching consequences.

Theorem 1.9 (spherical isoperimetric inequality)**.** *For any measurable subset $A \subseteq S^{n-1}$ and any $\varepsilon > 0$,*

$$\sigma_{n-1}(A) \geq \frac{1}{2} \qquad \Longrightarrow \qquad \sigma_{n-1}(A_\varepsilon) \geq \sigma_{n-1}(H_\varepsilon) \qquad (17)$$

where $H \subseteq S^{n-1}$ is a hemisphere. Moreover, for any $0 < t < 1$,

$$\sigma_{n-1}(A) \geq t \qquad \Longrightarrow \qquad \sigma_{n-1}(A_\varepsilon) \geq \sigma_{n-1}(H_\varepsilon^{(t)})$$

where $H^{(t)} \subseteq S^{n-1}$ is a spherical cap with $\sigma_{n-1}(H^{(t)}) = t$. A spherical cap is the intersection of S^{n-1} with a half-space in $\mathbb{R}^n$.

We will discuss the proof of Theorem 1.9 in the next lecture. Thanks to Theorem 1.9 and the bound (16), we may leverage the concentration of measure phenomenon as follows:

Corollary 1.10. *For any $A \subseteq S^{n-1}$ and $\varepsilon > 0$,*

$$\sigma_{n-1}(A) \geq \frac{1}{2} \qquad \Longrightarrow \qquad \sigma_{n-1}(A_\varepsilon) \geq 1 - C\exp(-c\varepsilon^2 n), \tag{18}$$

where $C, c > 0$ are universal constants.

The constant $1/2$ in (18) may be replaced by $1/10$ or any other universal constant, at the expense of adjusting the values of the universal constants C and c. Corollary 1.10 tells us that for any measurable set $A \subseteq S^{n-1}$ with $1/10 \leq \sigma_{n-1}(A) \leq 9/10$, most of the mass of the sphere is located near the boundary of A, i.e., at distance on the order of $O(1/\sqrt{n})$ from the "non-linear equator" ∂A. This provides a rather striking answer to our question: where is the "bulk" of the mass of the high-dimensional sphere located?

Exercises.

1. *Bernstein inequalities* (closely related to Bennet, Hoeffding and Chernoff inequalities; see [10, Chapter 2]): Let $M > 0$ and let $X_1, \dots, X_n$ be independent random variables. Assume that $\mathbb{E}X_i = 0$ and $\mathbb{P}(|X_i| \leq M) = 1$ for all i. We will prove that for all $t > 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geq t\sqrt{n}\right) \leq Ce^{-c(t/M)^2},$$

where $c, C > 0$ are universal constants. By scaling, we may reduce matters to the case $M = 1$.

(a) We will apply Markov's inequality for exponential moments. Begin by proving that for any $s > 0$,

$$\mathbb{E}e^{sX_1} = \sum_{k=0}^\infty \frac{\mathbb{E}(sX_1)^k}{k!} \leq e^s - s \leq e^{s^2},$$

where the last inequality is obvious for $s > 1$ and follows from $e^s \leq 1 + s + s^2 \leq s + e^{s^2}$ for $0 < s < 1$.

(b) Given $t > 0$, find an appropriate $s > 0$ so that

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq t\right) = \mathbb{P}\left(e^{\sum_{i=1}^{n} sX_i} \geq e^{st}\right) \leq e^{-st} \prod_{i=1}^{n} \mathbb{E} e^{sX_i} \leq e^{-t^2/(4n)}.$$

2. Recall the proof of (9) that you might have learned in your undergraduate studies:

$$\begin{aligned}(2\pi)^{n/2} = \int_{\mathbb{R}^n} e^{-|x|^2/2} dx &= \mathrm{Vol}_{n-1}(S^{n-1}) \cdot \int_0^\infty e^{-r^2/2} r^{n-1} dr \\ &= n\kappa_n \cdot 2^{(n-2)/2} \Gamma(n/2).\end{aligned}$$

3. Show that c_n from (10) satisfies $c_n = 1/\sqrt{2\pi} + O(1/n)$ without using the Stirling formula, but rather by using the Taylor approximation (12) as well as the formula

$$c_n = \int_{-\sqrt{n}}^{\sqrt{n}} \left(1 - \frac{t^2}{n}\right)^{\frac{n-1}{2}} dt.$$

4. In this exercise we outline the proof of Theorem 1.3 in the case

$$\theta = \frac{(1, \ldots, 1)}{\sqrt{n}} \in S^{n-1}.$$

(a) Abbreviate $f(t) = f_\theta\left(t/\sqrt{12}\right)/\sqrt{12}$, write $\mathrm{sinc}(x) = \sin(x)/x$ and assume that $n \geq 2$. Use the Fourier inversion formula in order to show that for $t \in \mathbb{R}$,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{sinc}^n\left(\sqrt{\frac{3}{n}} x\right) e^{itx} dx.$$

Conclude that

$$\left| f(t) - \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \right| \leq \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \left| \mathrm{sinc}^n\left(\sqrt{\frac{3}{n}} x\right) - e^{-x^2/2} \right| dx. \quad (19)$$

(b) We bound the integral in (19) by C/n by dividing it into three intervals. Consider first the interval $|x| \leq n^{1/4}$, and use Taylor's theorem in order to show that in this interval, the integrand in (19) is at most $C\frac{t^4}{n} e^{-t^2/2}$.

(c) Bound the integral in (19) also for $n^{1/4} \leq |x| \leq \sqrt{n}$ and for $|x| \geq \sqrt{n}$ and conclude the proof.

5. Let $Y \sim \mathrm{Unif}(S^{n-1})$, and let $Z \sim N(0,1)$ be a standard Gaussian. Prove that for any $t \in \mathbb{R}$,
$$\left|\mathbb{P}\left(\sqrt{n}Y_1 \le t\right) - \mathbb{P}\left(Z \le t\right)\right| \le \frac{C}{n}.$$
6. (*) Let $\Theta = (\Theta_1, \dots, \Theta_n) \in S^{n-1}$ be a uniformly distributed random vector. Show that
$$\mathbb{P}\left(\sum_{i=1}^{n} \Theta_i^4 \ge \frac{C}{n}\right) \le \exp(-c\sqrt{n})$$
for some universal constants $C, c > 0$.

   (Hint: maybe try to show that $\mathbb{E}\left(\sum_{i=1}^{n} \Gamma_i^4\right)^p \le (Cn)^p$ for $p \le c\sqrt{n}$ and $\Gamma_1, \dots, \Gamma_n$ being i.i.d standard Gaussians, using that $\mathbb{E}\Gamma_i^{4k} \le (Ck)^{2k}$).

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Request. Please e-mail me at `boaz.klartag@weizmann.ac.il` with any comments, corrections, or suggestions regarding these lecture notes. In addition, if you are able to produce *tikz* figures similar to (or better than) those drawn on the blackboard in class, I would be glad to include them in the notes.

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