Isoperimetric inequalities in high-dimensional convex sets
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Lecture 2: Geometric inequalities and symmetrization processes

In this lecture we study classical geometric inequalities, such as the isoperimetric inequality on on the sphere that was formulated in the last lecture together with its following corollary:

Corollary 2.1. *For any measurable $A \subseteq S^{n-1}$ and $\varepsilon > 0$,*

$$\sigma_{n-1}(A) \geq \frac{1}{2} \qquad \Longrightarrow \qquad \sigma_{n-1}(A_\varepsilon) \geq 1 - C\exp(-c\varepsilon^2 n), \tag{1}$$

where $C, c > 0$ are universal constants, where A_ε is the ε-neighborhood of A, and where σ_{n-1} is the uniform probability measure on S^{n-1}.

Corollary 2.1 admits direct proofs, one of which we can explain right away. It is based on the classical Brunn-Minkowski inequality from 1887:

Theorem 2.2. *(Brunn-Minkowski) Let $S, T \subseteq \mathbb{R}^n$ be non-empty Borel sets. Then,*

$$\mathrm{Vol}_n(S+T)^{1/n} \geq \mathrm{Vol}_n(S)^{1/n} + \mathrm{Vol}_n(T)^{1/n}, \tag{2}$$

where $S + T = \{s + t\,;\, s \in S, t \in T\}$ is the Minkowski sum.[1]

Note that for any *convex* set $K \subseteq \mathbb{R}^n$ and $r_1, r_2 > 0$,

$$r_1 K + r_2 K = (r_1 + r_2)K. \tag{3}$$

In fact, (3) is the very definition of a convex set. Therefore, when K is convex,

$$\mathrm{Vol}_n(r_i K)^{1/n} = r_i \cdot \mathrm{Vol}_n(K)^{1/n} \qquad\qquad (i = 1, 2)$$

while

$$\mathrm{Vol}_n(r_1 K + r_2 K)^{1/n} = (r_1 + r_2) \cdot \mathrm{Vol}_n(K)^{1/n}.$$

Thus equality holds in the Brunn-Minkowski inequality when S and T are $r_1 K$ and $r_2 K$, respectively. In fact, when S and T are assumed compact, equality in (2) holds true if and only if S and T are convex and homothetic, see Henstock and Macbeath [6]. Thus the Brunn-Minkowski inequality is closely related to convex sets, even though convexity does not appear in its formulation. A dimension-free corollary of the Brunn-Minkowski inequality is the following:

[1] The Minkowski sum of two Borel sets in $\mathbb{R}^n$ is Lebesgue measurable. For more information, see e.g. the first pages in Carleson [2]. Observe that by inner regularity of the Lebesgue measure, the Brunn-Minkowski inequality for Borel sets follows from the corresponding inequality for compact sets.

Corollary 2.3. *For any Borel sets $S, T \subseteq \mathbb{R}^n$ and $0 < \lambda < 1$,*

$$\mathrm{Vol}\left((1-\lambda)S + \lambda T\right) \geq \mathrm{Vol}(S)^{1-\lambda}\mathrm{Vol}(T)^{\lambda}. \tag{4}$$

This *multiplicative* Brunn-Minkowski inequality holds true also when S or T are empty, as opposed to Theorem 2.2. In order to prove (4), say in the case $\lambda = 1/2$, we apply Theorem 2.2 and the arithmetic/geometric means inequality as follows:

$$\mathrm{Vol}_n\left(\frac{S+T}{2}\right)^{1/n} \geq \frac{\mathrm{Vol}_n(S)^{1/n} + \mathrm{Vol}_n(T)^{1/n}}{2} \geq \left(\sqrt{\mathrm{Vol}_n(S)\mathrm{Vol}_n(T)}\right)^{1/n}.$$

The case of a general $\lambda \in (0,1)$ is similar.

Proof of Corollary 2.1 using the Brunn-Minkowski inequality. We follow Gromov and Milman [4]. We may assume that $n \geq 3$ and

$$\varepsilon \geq 2/\sqrt{n} \tag{5}$$

since otherwise the conlusion is vacuous. Let $A \subseteq S^{n-1}$ satisfy $\sigma_{n-1}(A) \geq 1/2$, and let $B \subseteq S^{n-1}$ be the complement of A_ε. Thus, for $x \in A$ and $y \in B$,

$$|x-y| \geq \varepsilon. \tag{6}$$

In order to use the Brunn-Minkowski inequality on volumes in $\mathbb{R}^n$ we need to pass from $(n-1)$-dimensional subsets of the sphere to n-dimensional sets in the unit ball. Fortunately, the uniform probability measure on the sphere is very close to that of the ball. That is, consider the following slight radial extension of the sets A and B into the unit ball:

$$S = \bigcup_{1-\frac{1}{n} \leq r \leq 1} rA, \qquad T = \bigcup_{1-\frac{1}{n} \leq r \leq 1} rB.$$

Then,

$$\frac{\mathrm{Vol}_n(S)}{\mathrm{Vol}_n(B^n)} = \left(1 - (1-1/n)^n\right)\sigma_{n-1}(A) \geq \frac{\sigma_{n-1}(A)}{2} \geq \frac{1}{4} \tag{7}$$

and similarly

$$\frac{\mathrm{Vol}_n(T)}{\mathrm{Vol}_n(B^n)} \geq \frac{\sigma_{n-1}(B)}{2}. \tag{8}$$

By (5) and (6), for any $x \in S$ and $y \in T$,

$$|x-y| \geq \varepsilon - \frac{2}{n} \geq c\varepsilon$$

for, say, $c = 1/4$. Since x and y are far apart, the *uniform convexity* of the sphere implies that their midpoint is deep inside the ball. That is, for any $x, y \in B^n$ with $|x-y| \geq c\varepsilon$,

$$\left|\frac{x+y}{2}\right|^2 = \frac{|x|^2+|y|^2}{2} - \frac{|x-y|^2}{4} \leq 1 - \tilde{c}\varepsilon^2$$

for some universal constant $c > 0$. Hence,

$$\frac{S+T}{2} \subseteq \sqrt{1-\tilde{c}\varepsilon^2} \cdot B^n \subseteq \left(1-\bar{c}\varepsilon^2\right) \cdot B^n.$$

Consequently, from the multiplicative Brunn-Minkowski inequality,

$$(1-\bar{c}\varepsilon^2)^n \geq \frac{\mathrm{Vol}_n\left(\frac{S+T}{2}\right)}{\mathrm{Vol}_n(B^n)} \geq \frac{\sqrt{\mathrm{Vol}_n(S)\mathrm{Vol}_n(T)}}{\mathrm{Vol}_n(B^n)} \geq \sqrt{\frac{1}{4} \cdot \frac{\sigma_{n-1}(B)}{2}},$$

where we used (7) and (8) in the last passage. Hence,

$$1 - \sigma_{n-1}(A_\varepsilon) = \sigma_{n-1}(B) \leq C(1-\bar{c}\varepsilon^2)^n \leq Ce^{-\bar{c}\varepsilon^2 n},$$

and (1) is proven. □

This proof of Corollary 2.1 relies heavily on the *uniform convexity* of the Euclidean ball/sphere, the fact that the midpoint between two points in the ball that are far apart, must lie deep inside the ball. It admits generalization to other uniformly convex sets, see the exercises below.

Hadwiger-Ohman proof of Theorem 2.2. Consider first the case where $S, T \subseteq \mathbb{R}^n$ are two parallel boxes, of edge length $a_1, \ldots, a_n > 0$ and $b_1, \ldots, b_n > 0$ respectively, we have

$$\mathrm{Vol}_n(S+T) = \prod_{i=1}^n (a_i + b_i).$$

Here the boxes may be open or closed; for concreteness let us work here with boxes of the form $\prod_{i=1}^n [c_i, d_i)$ where $c_i < d_i$ for all i. The Brunn-Minkowski inequality for two parallel boxes thus amounts to the inequality

$$\left(\prod_{i=1}^n (a_i+b_i)\right)^{1/n} \geq \left(\prod_{i=1}^n a_i\right)^{1/n} + \left(\prod_{i=1}^n b_i\right)^{1/n}.$$

This inequality follows from the arithmetic/geometric means inequality, since

$$\left(\prod_{i=1}^n \frac{a_i}{a_i+b_i}\right)^{1/n} + \left(\prod_{i=1}^n \frac{b_i}{a_i+b_i}\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n \left[\frac{a_i}{a_i+b_i} + \frac{b_i}{a_i+b_i}\right] = 1. \quad (9)$$

We move on to the case of general S and T. By approximation, we may assume that both S and T can be written as the union of finitely many disjoint boxes, all parallel to the axes. Consider representations of S and of T as disjoint unions of finitely many boxes, and write N for the number of boxes appearing in the representation of S plus the number of boxes appearing in the representation of T. We prove (2) by induction on N.

Since S and T are non-empty, the base of the induction is the case $N = 2$. In this case, S and T must be two parallel boxes, and the Brunn-Minkowski inequality follows from the arithmetic/geometric means inequality (9).

Suppose that $N \geq 3$. Then the representation of the S or of the set T consists of at least two disjoint boxes; without loss of generality assume that it is the set S. Let Q and $\tilde{Q}$ be two disjoint boxes from the representation of S. A crucial observation is that since the boxes Q and $\tilde{Q}$ are disjoint, there exists a hyperplane

$$H \subseteq \mathbb{R}^n$$

parallel to the axes that separates Q from $\tilde{Q}$. Writing H of the form $\{x \in \mathbb{R}^n\,;\, x_i = t\}$ for some $i = 1, \ldots, n$ and $t \in \mathbb{R}$, we look at the half-spaces,

$$H_1 = \{x \in \mathbb{R}^n\,;\, x_i < t\}, \qquad H_2 = \{x \in \mathbb{R}^n\,;\, x_i \geq t\}. \tag{10}$$

These are two disjoint half-spaces whose union equals $\mathbb{R}^n$. Each of these two halfs-paces is disjoint either from the box Q or from the box $\tilde{Q}$. For $i = 1, 2$ denote

$$S_i = S \cap H_i.$$

Clearly, S_i may be represented as a disjoint union of finitely many boxes; in fact, each of the boxes of S contributes at most one box to the representation S_i, with either Q or $\tilde{Q}$ not contributing at all. Thus the total number of disjoint boxes in the representation of S_i is *strictly smaller* than in the representation of S. Set

$$\lambda = \frac{\mathrm{Vol}_n(S_1)}{\mathrm{Vol}_n(S)} \in (0,1), \qquad 1 - \lambda = \frac{\mathrm{Vol}_n(S_2)}{\mathrm{Vol}_n(S)}.$$

For $s \in \mathbb{R}$ we consider the hyperplane $\tilde{H} = \tilde{H}(s) = \{x \in \mathbb{R}^n\,;\, x_i = s\}$ that is parallel to H, and we define $\tilde{H}_1 = \tilde{H}_1(s)$ and $\tilde{H}_2 = \tilde{H}_2(s)$ analogously to (10), i.e., with t replaced by s. Consider the fraction

$$\frac{\mathrm{Vol}_n(T \cap \tilde{H}_1(s))}{\mathrm{Vol}_n(T)}. \tag{11}$$

When we let s vary continuously, the fraction in (11) varies continuously from 0 to 1. By the mean value theorem, there exists a hyperplane $\tilde{H}$ parallel to H such that denoting

$$T_i = T \cap \tilde{H}_i \qquad\qquad (i = 1, 2)$$

we have

$$\frac{\mathrm{Vol}_n(T_1)}{\mathrm{Vol}_n(T)} = \lambda, \qquad 1 - \lambda = \frac{\mathrm{Vol}_n(T_2)}{\mathrm{Vol}_n(T)}.$$

For $i = 1, 2$, the set T_i may be represented as a disjoint union of finitely many boxes, where each of the boxes in the representation of T contributes at most one box to the representation T_i. Thus the number of boxes in the representation of T_i is *not larger* than in that of T.

Hence the total number of boxes in the representations of S_i and T_i combined is at most $N - 1$. By the induction hypothesis,

$$\mathrm{Vol}_n(S_i + T_i)^{1/n} \geq \mathrm{Vol}_n(S_i)^{1/n} + \mathrm{Vol}_n(T_i)^{1/n}. \tag{12}$$

Observe that the Minkowski sum $S_i + T_i$ is contained in the set $H_i + \tilde{H}_i$, which is a halfspace. Moreover, the halfspace $H_2 + \tilde{H}_2$ is the complement in $\mathbb{R}^n$ to the halfspace $H_1 + \tilde{H}_1$. Consequently $S_1 + T_1$ and $S_2 + T_2$ are two disjoint subsets of $S + T$. Thus

$$\begin{aligned}\mathrm{Vol}_n(S+T) \geq \sum_{i=1}^{2} \mathrm{Vol}_n(S_i + T_i) &\geq \sum_{i=1}^{2} (\mathrm{Vol}_n(S_i)^{1/n} + \mathrm{Vol}_n(T_i)^{1/n})^n \\ &= [\lambda + (1-\lambda)] \left(\mathrm{Vol}_n(S)^{1/n} + \mathrm{Vol}_n(T_i)^{1/n}\right)^n,\end{aligned}$$

completing the proof of (2). □

The Brunn-Minkowski inequality implies the isoperimetric inequality in $\mathbb{R}^n$, as we shall now explain. Let $A \subseteq \mathbb{R}^n$ be an open set with a smooth boundary. For $0 < \varepsilon < 1$, the Minkowski sum

$$A + \varepsilon B^n$$

equals the ε-neighborhood of A, which is of course the set

$$A_\varepsilon = \{x \in \mathbb{R}^n \,;\, d(x, A) < \varepsilon\}$$

where $d(x, A) = \inf_{y \in A} |x - y|$. Assuming that A is bounded and connected, it is proven in multivariate calculus class that

$$\mathrm{Vol}_{n-1}(\partial A) = \lim_{\varepsilon \to 0^+} \frac{\mathrm{Vol}_n(A_\varepsilon) - \mathrm{Vol}_n(A)}{\varepsilon}. \tag{13}$$

Corollary 2.4. *For any connected, bounded, open set $A \subseteq \mathbb{R}^n$ with a smooth boundary,*

$$\frac{\mathrm{Vol}_{n-1}(\partial A)}{\mathrm{Vol}_n(A)^{\frac{n-1}{n}}} \geq \frac{\mathrm{Vol}_{n-1}(\partial B)}{\mathrm{Vol}_n(B)^{\frac{n-1}{n}}}, \tag{14}$$

where $B \subseteq \mathbb{R}^n$ is any Euclidean ball. Moreover, if $B \subseteq \mathbb{R}^n$ is a ball with $\mathrm{Vol}_n(B) = \mathrm{Vol}_n(A)$ then for any $\varepsilon > 0$,

$$\mathrm{Vol}_n(A_\varepsilon) \geq \mathrm{Vol}_n(B_\varepsilon). \tag{15}$$

Proof of Corollary 2.4. We prove (15) by the Brunn-Minkowski inequality as follows:

$$\mathrm{Vol}_n(A_\varepsilon) = \mathrm{Vol}_n(A + \varepsilon B^n) \geq \left[\mathrm{Vol}_n(A)^{1/n} + \mathrm{Vol}_n(\varepsilon B^n)^{1/n}\right]^n \tag{16}$$
$$= \left[\mathrm{Vol}_n(B)^{1/n} + \varepsilon \mathrm{Vol}_n(B^n)\right]^n = \mathrm{Vol}_n(B + \varepsilon B^n) = \mathrm{Vol}_n(B_\varepsilon),$$

where we used the fact that B is homothetic to B^n and convex, and this yields equality in Brunn-Minkowski.

In order to deduce (14), we use that $\mathrm{Vol}_{n-1}(S^{n-1}) = n\mathrm{Vol}_n(B^n)$ and hence

$$\frac{\mathrm{Vol}_{n-1}(\partial B)}{\mathrm{Vol}_n(B)^{\frac{n-1}{n}}} = \frac{\mathrm{Vol}_{n-1}(\partial B^n)}{\mathrm{Vol}_n(B^n)^{\frac{n-1}{n}}} = n\mathrm{Vol}_n(B^n)^{1/n}.$$

Consequently, by (16), for any $\varepsilon > 0$,

$$\mathrm{Vol}_n(A_\varepsilon) \geq \left[\mathrm{Vol}_n(A)^{1/n} + \varepsilon \mathrm{Vol}_n(B^n)\right]^n$$
$$= \mathrm{Vol}_n(A) + n\varepsilon \mathrm{Vol}_n(A)^{\frac{n-1}{n}} \mathrm{Vol}_n(B^n)^{\frac{1}{n}} + o(\varepsilon)$$

as $\varepsilon \to 0$. Therefore, from formula (13) for the surface area,

$$\mathrm{Vol}_{n-1}(\partial A) \geq n\mathrm{Vol}_n(A)^{\frac{n-1}{n}} \mathrm{Vol}_n(B^n)^{\frac{1}{n}} = \mathrm{Vol}_n(A)^{\frac{n-1}{n}} \cdot \frac{\mathrm{Vol}_{n-1}(\partial B)}{\mathrm{Vol}_n(B)^{\frac{n-1}{n}}}.$$

□

Corollary 2.1 seems to suffice for all of the applications to high-dimensional geometry that we have in mind, though for completeness let us now prove the spherical isoperimetric inequality with sharp constants.[2] We proceed by discussing a *symmetrization* or *rearrangement* proof of the following:

[2] There are variants of the Brunn-Minkowski inequality for the sphere S^{n-1} rather than for $\mathbb{R}^n$. For instance, the multiplicative version holds true if we replace $(A+B)/2$ by the set of all midpoints of minimizing geodesics connecting a point in A with a point in B, see Cordero-Erausquin, McCann and Schmuckenschlaeger [3] and Sturm [8]. Nevertheless, we are not aware of a geometric inequality as elementary and easily proven as Theorem 2.2 in spherical (or hyperbolic) geometry which immediately implies the isoperimetric inequality.

Theorem 2.5 (spherical isoperimetric inequality)**.** *For any measurable subset $A \subseteq S^{n-1}$ with $\sigma_{n-1}(A) \geq 1/2$ and any $\varepsilon > 0$,*

$$\sigma_{n-1}(A_\varepsilon) \geq \sigma_{n-1}(H_\varepsilon) \tag{17}$$

where $H \subseteq S^{n-1}$ is a hemisphere and σ_{n-1} is the uniform probability measure on the sphere S^{n-1}. Equality in (17) holds if and only if the closure of A is a hemisphere.

Our proof generalizes almost verbatim to the case $\sigma_{n-1}(A) \geq t$ for $t \in (0,1)$ discussed last week. In proving Theorem 2.5 we may assume that A is closed, since $(\overline{A})_\varepsilon = A_\varepsilon$ where $\overline{A}$ is the closure of A. Let us fix $\varepsilon > 0$ and abbreviate $\sigma = \sigma_{n-1}$.

Claim 2.6. *The infimum*

$$\inf\left\{\sigma(A_\varepsilon)\,;\,\sigma(A) \geq \frac{1}{2},\ A \subseteq S^{n-1} \textit{ is closed}\right\} \tag{18}$$

is attained. We refer to any set attaining the minimum as an isoperimetric minimizer.

Proof. The proof relies on the well-known fact the collection of non-empty closed subsets of S^{n-1} is *compact* in the Hausdorff metric, where we recall that the Hausdorff distance between two closed subsets $A, B \subseteq S^{n-1}$ is

$$d(A,B) = \inf\left\{\varepsilon > 0\,;\, A \subseteq B_\varepsilon, B \subseteq A_\varepsilon\right\}. \tag{19}$$

See the exercise below. We claim that the map

$$A \mapsto \sigma(A) \tag{20}$$

is upper semi-continuous in the Hausdorff metric. Indeed if $A^{(N)} \longrightarrow A$, then for any $\delta > 0$ we have $A^{(N)} \subseteq A_\delta$ for a sufficiently large N, and hence

$$\limsup_{N\to\infty} \sigma(A^{(N)}) \leq \sigma(A_\delta) \xrightarrow{\delta \to 0^+} \sigma(A),$$

where the last convergence follows from the fact that $A_\delta \searrow A$, i.e., the set A_δ is non-decreasing with $\delta > 0$ with respect to inclusion while $A = \cap_{\delta>0} A_\delta$. Next, it is an exercise to show that the map $A \mapsto \sigma(A_\varepsilon)$ is continuous. Therefore, in (18) we minimize a continuous function over a compact set, and the infimum is attained. □

It follows from the proof of Claim 2.6 that the collection of isoperimetric minimizers is compact. Write

$$H = \{(x_1, \ldots, x_n) \in S^{n-1}\,;\, x_1 \leq 0\}$$

for the "southern" hemisphere. The map $A \mapsto \sigma(A \cap H)$ is upper-semi continuous, exactly like the map in (20), essentially because $\sigma(\partial H) = 0$ (exercise!). We have thus obtained the following:

Corollary 2.7. *The supremum*

$$\sup\{\sigma(A \cap H)\,;\, A \textit{ is an isoperimetric minimizer}\}$$

is attained. We refer to an optimizer of this supremum as a "most southern isoperimetric minimizer".

The crux of the proof of Theorem 2.5 is the following:

Proposition 2.8. *Any most southern minimizer is the southern hemisphere H itself.*

It clearly suffices to show that any most southern minimizer contains the southern hemisphere H. In order to prove Proposition 2.8 we will introduce a process that transforms a given subset $A \subseteq S^{n-1}$ into another subset, denoted by

$$S(A) = S_\theta(A) \subseteq S^{n-1},$$

which also depends on a vector $\theta \in S^{n-1}$. This process will possess several desirable properties:

(a) Preservation of measure: $\sigma(S(A)) = \sigma(A)$.

(b) The ε-neighborhood does not grow in measure:

$$\sigma(S(A)_\varepsilon) \leq \sigma(A_\varepsilon)$$

(c) In some sense, the set $S(A)$ is "at least as southern" as the set A.

The rearrangement process that we selected for our proof of Proposition 2.8 is the following:

2.1 Two-point rearrangement (or symmetrization)

We follow the presentation of Benyamini [1]. For $x, \theta \in S^{n-1}$, let us write

$$\pi_\theta(x) = x - 2\langle x, \theta\rangle\theta \in S^{n-1}$$

for the reflection of the point x across the hyperplane $\theta^\perp$. Clearly the involution $\pi_\theta : S^{n-1} \to S^{n-1}$ is measure-preserving, i.e., $\sigma(A) = \sigma(\pi_\theta(A))$.

The idea of the two-point rearrangement of a set $A \subseteq S^{n-1}$ in the direction $\theta \in S^{n-1}$ is as follows: for each point $x \in A$ lying in the hemisphere centered at θ, we attempt to reflect it across the hyperplane $\theta^\perp$. If the place is already occupied by another point of A, we keep the original point x, but otherwise we replace x with its reflection. In other words, we have the following:

Definition 2.9. *For $A \subseteq S^{n-1}$ and $\theta \in S^{n-1}$ set*

$$T(x) = T_A(x) = T_{A,\theta}(x) = \begin{cases} \pi_\theta(x) & x \cdot \theta > 0 \text{ and } \pi_\theta(x) \notin A \\ x & \text{otherwise} \end{cases}$$

and the two-point rearrangement of A is

$$S_\theta(A) = \{T(x)\,;\, x \in A\}.$$

Note that if we define

$$A^+ = \{x \in A\,;\, x \cdot \theta \geq 0\}, \qquad A^- = \{x \in A\,;\, x \cdot \theta < 0\}$$

then we can decompose A into three disjoint parts as follows:

$$A = A^- \cupdot A^+ = A^- \cupdot \left(A^+ \cap \pi_\theta(A^-)\right) \cupdot \left(A^+ \setminus \pi_\theta(A^-)\right). \tag{21}$$

By the definition of $S_\theta(A)$,

$$S_\theta(A) = A^- \cupdot \left(A^+ \cap \pi_\theta(A^-)\right) \cupdot \pi_\theta \left(A^+ \setminus \pi_\theta(A^-)\right), \tag{22}$$

i.e., we keep the first two components of the decomposition (21) intact, while reflecting the third component. The two-point rearrangement of A has the same spherical measure as A, as follows from (21), (22) and the fact the reflection is a measure-preserving transformation. Thus property (a) above holds true, and in fact, the map

$$T_{A,\theta} : A \to S_\theta(A)$$

is measure-preserving. Two-point rearrangement is also monotone under inclusion, i.e., $A \subseteq B$ implies that $S_\theta(A) \subseteq S_\theta(B)$.

For example, for $x_0 \in S^{n-1}$ and $r > 0$ let us write here

$$B(x_0, r) = \{x \in S^{n-1}\,;\, |x - x_0| \leq r\}$$

for the spherical cap centered at the point $x_0 \in S^{n-1}$, which is in fact a ball relative to the metric $d(x, y) = |x - y|$ on S^{n-1}. Observe that

$$\pi_\theta(B(x, r)) = B(\pi_\theta(x), r).$$

By drawing a little figure depicting the decomposition (21) corresponding to the set $A = B(x_0, r)$ we see that

$$S_\theta(B(x_0,r)) = \begin{cases} B(x_0,r) & x_0\cdot\theta \le 0 \\ B(\pi_\theta(x_0),r) & x_0\cdot\theta \ge 0 \end{cases} \tag{23}$$

Property (b) above of two-point rearrangement follows from the following:

Lemma 2.10. *For any $A \subseteq S^{n-1}$,*

$$(S_\theta(A))_\varepsilon \subseteq S_\theta(A_\varepsilon) \tag{24}$$

and consequently, for any $\varepsilon > 0$ the ε-neighborhood of A does not grow in measure under rearrangement, i.e.,

$$\sigma\left((S_\theta(A))_\varepsilon\right) \le \sigma(A_\varepsilon).$$

Proof. Pick $x \in S_\theta(A)$. In order to prove (24) it suffices to show that

$$B(x,\varepsilon) \subseteq S_\theta(A_\varepsilon). \tag{25}$$

This is done by case analysis.

Case 1. Assume $x \in A$ and $\pi_\theta(x) \in A$. In this case,

$$B(x,\varepsilon) \subseteq A_\varepsilon \qquad \text{and} \qquad \pi_\theta(B(x,\varepsilon)) \subseteq A_\varepsilon.$$

This implies that $T_{A_\varepsilon}(x) = x$ for any $y \in B(x,\varepsilon)$, and hence (25) holds true.

Case 2. Assume $x \in A, \pi_\theta(x) \notin A$. Observe that since $x \in S_\theta(A)$, necessarily $x\cdot\theta \le 0$ in this case. Therefore,

$$S_\theta(A_\varepsilon) \supseteq S_\theta(B(x,\varepsilon)) = B(x,\varepsilon),$$

proving (25), where we used (23) in the last passage.

Case 3. Assume $x \notin A, \pi_\theta(x) \in A$. In this case again $x\cdot\theta \le 0$. We have

$$A_\varepsilon \supseteq B(\pi_\theta(x),\varepsilon)$$

so that by monotonicity and (23),

$$S_\theta(A_\varepsilon) \supseteq S_\theta(B(\pi_\theta(x),\varepsilon)) = B(x,\varepsilon).$$

The 4^{th} case where $x, \pi_\theta(x) \notin A$ cannot occur as $x \in S_\theta(A)$. This completes the proof. ☐

It seems that the geometric property of the sphere that is relevant for our analysis is the fact that if $x, y, \theta \in S^{n-1}$ are such that both x and y lie in the hemisphere centered at θ, then

$$|x - y| \leq |\pi_\theta x - y|.$$

This property holds true also in Euclidean and hyperbolic geometries, and hence two-point rearrangement leads to proofs of the isoperimetric inequalities in such geometries as well. Lemma 2.10 admits the following:

Corollary 2.11. *The collection of isoperimetric minimizers is closed under two-point rearrangement.*

Our next goal is to show that any southern minimizer contains the southern hemisphere H.

Lemma 2.12. *Let $A \subseteq S^{n-1}$ be a measurable set and let $\theta \in S^{n-1}$ satisfy $\theta_1 \geq 0$. Let*

$$E \subseteq \{x \in A\, ;\, x_1 > 0, x \cdot \theta > 0, \pi_\theta(x) \in H \setminus A\}.$$

be any measurable set. Then,

$$\sigma(S_\theta(A) \cap H) \geq \sigma(A \cap H) + \sigma(E).$$

Lemma 2.12 tells us that if A is a most southern isoperimetric minimizer then necessarily $\sigma(E) = 0$, since otherwise $S_\theta(A)$ is an isoperimetric minimizer that is more southern than A.

Proof of Lemma 2.12. Our goal is to show that $T_A(A \cap H)$ and $T_A(E)$ are two disjoint subsets of $S_\theta(A) \cap E$. Begin by noting that since $\theta_1 \geq 0$,

$$x \in H, x \cdot \theta > 0 \qquad \Longrightarrow \qquad \pi_\theta(x) \in H. \tag{26}$$

Indeed, under the assumptions appearing in the left-hand side of (26), we have $x_1 \leq 0$ and

$$\pi_\theta(x) \cdot e_1 = x_1 - 2(x \cdot \theta)\theta_1 \leq 0.$$

It follows from (26) and the definition of rearrangement that for any $x \in A \cap H$,

$$T_A(x) = T_{A,\theta}(x) \in H.$$

Consequently,

$$T_A(A \cap H) \subseteq S_\theta(A) \cap H \tag{27}$$

Next, from the definition of E,

$$T_A(E) = \pi_\theta(E) \subseteq H, \tag{28}$$

while the set $\pi_\theta(E)$ is disjoint both from $A \cap H$ and from $\pi_\theta(A \cap H)$. Thus

$$\pi_\theta(E) \cap T_A(A \cap H) = \emptyset. \tag{29}$$

From (27), (28) and (29) we see that $T_A(E)$ and $T_A(A \cap H)$ are two disjoint sets that are contained in the two-point rearrangement $S_\theta(A)$ as well as in the hemisphere H. Consequently,

$$\sigma(S_\theta(A) \cap H) \geq \sigma(T_A(A \cap H)) + \sigma(T_A(E)) = \sigma(A \cap H) + \sigma(E).$$

□

Proof of Proposition 2.8. Let $A \subseteq S^{n-1}$ be a closed set which is an isoperimetric minimizer that does not contain H. We will show that A is *not* a most southern isoperimetric minimizer. The set int$(H)/A$ is non-empty and open, where int(H) is the interior of H. Hence we can find a ball of positive radius

$$B(x_0, \varepsilon) \subseteq H \setminus A.$$

Since $\sigma(A) \geq 1/2$ while A does not contain H, the set A must contain some mass in the northern hemisphere. Hence there exists $B(y_0, \varepsilon) \subseteq S^{n-1} \setminus \overline{H}$ such that the set

$$E := A \cap B(y_0, \varepsilon)$$

has a positive σ-measure. Clearly

$$B(x_0, \varepsilon) \cap B(y_0, \varepsilon) = \emptyset.$$

Define

$$\theta := \frac{y_0 - x_0}{|y_0 - x_0|},$$

so that $\pi_\theta(y_0) = x_0$ and $\pi_\theta(B(y_0, \varepsilon)) = B(x_0, \varepsilon)$. Note that $\theta_1 \geq 0$ as $x_0 \in H$ while $y_0 \notin H$. Observe that for any $x \in E$,

$$x_1 > 0 \qquad \text{and} \qquad x \cdot \theta > 0$$

while

$$\pi_\theta(E) \subseteq B(x_0, \varepsilon) \subseteq H \setminus A$$

Thus E satisfies all of the requirements of Lemma 2.12. By the conclusion of that lemma, $S_\theta(A)$ is an isoperimetric minimizer whose intersection with H has larger measure than that of the isoperimetric minimizer A. This completes the proof. □

Theorem 2.5 is thus proven.

2.2 Steiner symmetrization

There are additional symmetrization procedures beyond the two-point rearrangement described above. A symmetrization process that was invented in 1838 by Jakob Steiner [7] is defined as follows. Let $K \subseteq \mathbb{R}^n$ be a Borel set and let $\theta \in S^{n-1}$. Clearly, each vector $x \in \mathbb{R}^n$ may be uniquely represented as $x = y + t\theta$ for $y \in \theta^\perp$ and $t \in \mathbb{R}$. Let us momentarily use

$$x = (y, t) \in \theta^\perp \times \mathbb{R}$$

as coordinates in $\mathbb{R}^n$. The *Steiner symmetral* of K in direction θ is defined as

$$S_\theta(K) = \left\{ (y,t) \, ; \, K \cap (y + \mathbb{R}\theta) \neq \emptyset, \ |t| \leq \frac{1}{2} \cdot \text{Length}(K \cap (y + \mathbb{R}\theta)) \right\},$$

where Length is the one-dimensional Lebesgue measure in the line $y + \mathbb{R}\theta$. The symmetral $S_\theta(K) \subseteq \mathbb{R}^n$ is indeed symmetric with respect to the hyperplane $\theta^\perp$, i.e.,

$$S_\theta(K) = \pi_\theta \left(S_\theta(K) \right).$$

Moreover, by Fubini's theorem, Steiner symmetrization preserves volume, i.e.,

$$\text{Vol}_n(S_\theta(K)) = \text{Vol}_n(K).$$

It is an exercise to see that Steiner symmetrization interacts nicely with Minkowski sum: for any Borel sets $A, B \subseteq \mathbb{R}^n$,

$$S_\theta(A + B) \supseteq S_\theta(A) + S_\theta(B). \tag{30}$$

The inclusion (30) may be used to give another proof of the Brunn-Minkowski inequality. Let us sketch the argument.

It is intuitively clear that successive Steiner symmetrizations make the shape K more and more symmetric. With a bit of effort (see exercise!), one may prove the existence of a sequence of Steiner symmetrizations of a given compact K that converges to a Euclidean ball of the same volume as that of K. In fact, a random choice of symmetrizations work, see Volčič [9]. Let us begin with two compact sets $A, B \subseteq \mathbb{R}^n$ of finite measure, and apply the same sequence of symmetrizations to both, so that they simultaneously converge to Euclidean balls. Let us write A_i and B_i for the i^{th} symmetrals in the process. By induction, we deduce from (30) that

$$\text{Vol}_n(A + B) \geq \text{Vol}_n(A_i + B_i) \xrightarrow{i \to \infty} \text{Vol}_n(r_1 B^n + r_2 B^n), \tag{31}$$

where $r_1, r_2 > 0$ are such that $\kappa_n r_1^n = \text{Vol}_n(A)$ and $\kappa_n r_2^n = \text{Vol}_n(B)$ with $\kappa_n = \text{Vol}_n(B^n)$. Observing that the right-hand side of (31) equals $(r_1 + r_2)^n \kappa_n$, the Brunn-Minkowski inequality follows. This is essentially the original proof method employed by Brunn.

Exercises.

1. For a convex polygon $P \subseteq \mathbb{R}^2$ and $t > 0$ and for the unit disc $D = \{x \in \mathbb{R}^2 \, ; \, |x| < 1\}$, prove that for any $t > 0$,
$$\text{Area}(P + tD) = \text{Area}(P) + t \cdot \text{Length}(\partial P) + \pi t^2.$$

2. Use the Brunn-Minkowski inequality in order to show that for any convex body $K \subseteq \mathbb{R}^n$ that is centrally-symmetric (i.e. $K = -K$) and any $\theta \in S^{n-1}, t \in \mathbb{R}$,
$$\text{Vol}_{n-1}(K \cap \theta^\perp) \geq \text{Vol}_{n-1}(K \cap (t\theta + \theta^\perp)).$$

3. Recall that the Hausdorff metric defined in (19) is indeed a metric, and that the collection of all non-empty closed subsets of S^{n-1} is a compact in the Hausdorff measure. Find a sequence of finite sets that converges to S^{n-1}.

4. Prove that for any $\varepsilon > 0$, the map $A \mapsto \sigma(A_\varepsilon)$ is continuous in the Hausdorff metric, and that the map $A \mapsto \sigma(A \cap H)$ is upper-semi continuous, where $H \subseteq S^{n-1}$ is a given hemisphere.

5. Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body (i.e., $K = -K$), and consider the norm $\|x\|_K = \inf \{\lambda \geq 0 \, ; \, x \in \lambda K\}$ whose unit ball of K. The *modulus of convexity* of K is defined for $0 < \varepsilon < 1$ via
$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\|_K \, ; \, x, y \in K, \|x - y\|_K \geq \varepsilon \right\}.$$
   (a) Verify that if $K \subseteq \mathbb{R}^n$ is an origin-symmetric ellipsoid, then $\delta(\varepsilon) \geq \varepsilon^2/8$.
   (b) Write μ_K for the Lebesgue measure on K, normalized to a probability measure. Prove that for any measurable set $A \subseteq K$ with $\mu_K(A) \geq 1/2$ and any $\varepsilon > 0$,
$$\mu_K(A_\varepsilon) \geq 1 - 2e^{-2n\delta(\varepsilon)},$$
   where $A_\varepsilon = \{x \in \mathbb{R}^n \, ; \, \inf_{y \in A} \|x - y\|_K < \varepsilon\}$ is the ε-neighborhood of A with respect to the norm $\| \cdot \|_K$.

6. Prove the relation (30) between Steiner symmetrization and Minkowski sum.

7. Convergence of Steiner Symmetrization. Let $K \subset \mathbb{R}^n$ be a compact. Set
$$R(K) = \max_{x \in K} |x|,$$
and assume that $R(K) > \text{v.rad.}(K) := (\text{Vol}_n(K)/\text{Vol}_n(B^n))^{1/n}$. We refer to v.rad.$(K)$ as the "volume-radius" of K, since it is the radius of the Euclidean ball with the same volume as K.

(a) Prove that there exists a finite sequence of Steiner symmetrizations, with respect to hyperplanes through the origin, that arrive at another compact set $T \subset \mathbb{R}^n$ with

$$R(T) < R(K).$$

Hint: The set $K \cap S^{n-1}$ can only lose points when we symmetrize, and we may "empty" a cap after cap.

(b) Write $\mathcal{F}$ for the collection of all compacts obtained from K by applying a finite sequence of Steiner symmetrizations. Argue that $\mathcal{F}$ contains elements that are arbitrarily close to a Euclidean ball, in the Hausdorff metric.

(c) Prove that for any two compacts $K_1, K_2 \subseteq \mathbb{R}^n$, there exists a sequence of hyperplanes such that if we consecutively symmetrize both compacts with respect to these hyperplanes, we obtain a sequence of compacts converging to Euclidean balls in the Hausdorff metric. Fill in the details of the proof of the Brunn-Minkowski inequality using Steiner symmetrizations.

References

[1] Benyamini, Y., *Two-point symmetrization, the isoperimetric inequality on the sphere and some applications.* Texas functional analysis seminar (1983–1984), Longhorn Notes, University of Texas Press (1984), 53–76.

[2] Carleson, L., *Selected problems on exceptional sets.* D. Van Nostrand Co., 1967.

[3] Cordero-Erausquin, D., McCann, Schmuckenschläger, M., *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb.* Invent. Math., Vol. 146, (2001), 219–257.

[4] Gromov, M., Milman, V. D., *Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces.* Compositio Math., Vol. 62, no. 3, (1987), 263–282.

[5] Hadwiger, H., Ohmann, D., *Brunn-Minkowskischer Satz und Isoperimetrie.* Math. Z., Vol. 66, (1956), 1–8.

[6] Henstock, R., Macbeath, A. M., *On the measure of sum-sets. I. The theorems of Brunn, Minkowski, and Lusternik.* Proc. London Math. Soc. (3), Vol. 3 (1953), 182–194.

[7] Steiner, J., *Einfacher Beweis der isoperimetrischen Hauptsätze.* J. Reine Angew. Math. (Crelle's journal), Vol. 18, (1838), 281–296.

[8] Sturm, K.-T., *On the geometry of metric measure spaces. I+II.* Acta Math., Vol. 196, (2006), 65–177.

[9] Volčič, A., *Random Steiner symmetrizations of sets and functions.* Calc. Var. Partial Differential Equations, Vol. 46, no. 3-4, (2013), 555–569.

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Request. Please e-mail me at `boaz.klartag@weizmann.ac.il` with any comments, corrections, or suggestions regarding these lecture notes. In addition, if you are able to produce *tikz* figures similar to (or better than) those drawn on the blackboard in class, I would be glad to include them in the notes.

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