Isoperimetric inequalities in high-dimensional convex sets
Boaz Klartag, ETH Zurich 2025

Lecture 4: Dvoretzky's theorem

This is our second lecture on applications of spherical concentration of measure, which may be used in order to find "regular substructures" in a high-dimensional system; perhaps in the spirit of Ramsey's theorem from graph theory which states that any coloring of the complete graph on N vertices by two colors, contains a monochromatic subgraph of size $c \log N$. Recall that a convex body in $\mathbb{R}^n$ is a compact, convex set with a non-empty interior. It will be convenient to assume that the convex body $K \subseteq \mathbb{R}^n$ is origin-symmetric or centrally-symmetric, i.e.,

$$K = -K.$$

We will discuss the proof by Milman [9], with improvements by Gordon [7] and Schechtman [12], of the following Dvoretzky's theorem [4, 5]:

Theorem 4.1 (Dvoretzky)**.** *Let $K \subseteq \mathbb{R}^n$ be an origin-symmetric convex body and let $0 < \varepsilon < 1/2$ and $1 \leq k \leq n$ be such that*

$$k \leq c\varepsilon^2 \log n. \tag{1}$$

Then there exists a k-dimensional subspace $E \subseteq \mathbb{R}^n$ such that the section $K \cap E$ is ε-spherical, i.e.,

$$(1-\varepsilon)rB_E \subseteq K \cap E \subseteq (1+\varepsilon)rB_E$$

for some $r > 0$. Here, $B_E = B^n \cap E = \{x \in E \,;\, |x| \leq 1\}$ and $c > 0$ is a universal constant.

Thus, any centrally-symmetric convex body in high-dimensions admits sections that are nearly Euclidean balls.

The dependence on the dimension n in the bound (1) is tight for the unit cube $Q^n = [-1/2, 1/2]^n$, see the exercise below. The dependence on ε is probably far from tight; it is conjectured that the bound in (1) may be improved to

$$k \leq c\frac{\log n}{\log 1/\varepsilon}.$$

In the case where $k = 2$, there is a topological argument that attributed to Gromov [10], which yields this better bound.

There is additionally an algebraic version of Dvoretzky's theorem, conjectured in Milman [10] and proven by Dolnikov and Karasaev [3] by using daunting topological tools such as Carlsson's proof of the Segal conjecture. The algebraic version of Dvoretzky's theorem states that for any positive integers k, d there is a number $n_0(k, d)$ such that if $n \geq n_0(k, d)$ then the following holds: For any d-homogeneous polynomial $P : \mathbb{R}^n \to \mathbb{R}$ there exists a k-dimensional subspace $E \subseteq \mathbb{R}^n$ with

$$P(x) = \alpha|x|^d \qquad \text{for all } x \in E,$$

for some $\alpha \in \mathbb{R}$. The case where d is odd is much easier, see Birch [2], as well as the complex case; in these two cases we have $\alpha = 0$. It is an open problem to provide a reasonable (or any) quantitative estimate for the dimension $n_0(k, d)$ in the general, real case.

All known proofs of Theorem 4.1 are probabilistic. Recall that a subset $\mathcal{F}$ of a metric space (X, d) is a δ-net if the δ-neighborhood of $\mathcal{F}$ equals the entire space X, i.e., for any $x \in X$ there is $y \in \mathcal{F}$ such that $d(x, y) < \delta$. We will use a simple volumetric bound for estimating the size of a δ-net of the sphere.

Lemma 4.2. *For any $n \geq 2, 0 < \delta < 1$, there exists a δ-net of S^{n-1} endowed with the Euclidean metric, whose cardinality is at most*

$$\left(\frac{3}{\delta}\right)^n. \tag{2}$$

Proof. Consider a maximal δ-separated subset $\mathcal{F} \subseteq S^{n-1}$, i.e.,

$$\forall x \neq y \in \mathcal{F},\ |x - y| \geq \delta \tag{3}$$

and $\mathcal{F}$ is maximal (under inclusion) subject to the requirement (3). The set $\mathcal{F}$ is a δ-net, by maximality, since for any $x \in S^{n-1} \setminus \mathcal{F}$, the set $\mathcal{F} \cup \{x\}$ is *not* δ-separated, hence there exists $y \in \mathcal{F}$ with $|x - y| < \delta$. Observe that the family of balls

$$x + \frac{\delta}{2}B^n \qquad (x \in \mathcal{F})$$

are disjoint, while all contained in $(1 + \delta/2)B^n$. Therefore, with $N = \#(\mathcal{F})$,

$$N \cdot \kappa_n(\delta/2)^n \leq \sum_{x \in \mathcal{F}} \mathrm{Vol}_n\left(x + \frac{\delta}{2}B^n\right) \leq \mathrm{Vol}_n\left((1 + \delta/2)B^n\right) = \kappa_n(1 + \delta/2)^n.$$

Therefore,

$$N \leq \left(\frac{1 + \delta/2}{\delta/2}\right)^n = \left(\frac{2 + \delta}{\delta}\right)^n \leq \left(\frac{3}{\delta}\right)^n.$$

□

It is easy to describe explicit δ-nets on the sphere S^{n-1} whose size is at most the expression in (2), perhaps with a different universal constant in place of "3". An example for a spherical δ-net of cardinality at most $(C/\delta)^n$ is the set

$$\left\{ \frac{x}{|x|} \, ; \, 1 \leq |x| \leq 2, \ (\sqrt{n}/\delta)x \in \mathbb{Z}^n \right\}. \tag{4}$$

Indeed, the set $\frac{\delta}{\sqrt{n}} \cdot \mathbb{Z}^n$ is a δ-net of $\mathbb{R}^n$, and the radial projection cannot increase distances; see the exercise below. The following theorem lies at the heart of Milman's proof of Dvoretzky's theorem [11], incorporating improvements by Gordon [7] and Schechtman [12]:

Theorem 4.3 ("Lipschitz functions are nearly constant on a random subspace")**.** *Let $f : S^{n-1} \to \mathbb{R}$ be an L-Lipschitz function, let $\varepsilon > 0$, and let $M = \int_{S^{n-1}} f d\sigma_{n-1}$. Denote*

$$k_* = k_*(\varepsilon) = n \left(\frac{\varepsilon}{L} \right)^2$$

and let $1 \leq k \leq n$ satisfy

$$k \leq \tilde{c} k_*. \tag{5}$$

Let $E \in G_{n,k}$ be a uniformly-distributed, random subspace. Then with a probability of at least $1 - C' \exp(-c' k_*)$,

$$\sup_{x \in S^{n-1} \cap E} |f(x) - M| \leq \varepsilon. \tag{6}$$

Here, $C, c, \tilde{c} > 0$ are universal constants.

Theorem 4.3 provides a structured object – a subspace – on which the oscillation of the function f is bounded by 2ε. We begin the proof of Theorem 4.3 with the following lemma:

Lemma 4.4. *Let $f : S^{n-1} \to \mathbb{R}$ be L-Lipschitz, and let U be a random rotation, distributed uniformly in $O(n)$. Then for any fixed $x \neq y \in S^{n-1}$,*

$$\mathbb{P} \left(\left| \frac{f(Ux) - f(Uy)}{|x - y|} \right| \geq tL \right) \leq C \exp(-ct^2 n), \tag{7}$$

where $c, C > 0$ are universal constants.

Proof. We may normalize f and assume that $L = 1$. Set $z = (x+y)/2$ and $w = (x-y)/2$. Note that

$$z \perp w \qquad \text{and} \qquad |z \pm w| = 1. \tag{8}$$

For $u \in \mathbb{R}^n$ with $|u| = |z| = |x+y|/2$ and for $v \in S^{n-1} \cap u^\perp$ we define

$$g_u(v) = g(u;v) = f\left(u + \frac{|x-y|}{2}v\right) - f\left(u - \frac{|x-y|}{2}v\right)$$

Then g_u is an odd, $|x-y|$-Lipschitz function defined on the $(n-2)$-dimensional sphere $S^{n-1} \cap u^\perp$. Being an odd function, the average of g_u on the sphere $S^{n-1} \cap u^\perp$ vanishes. Hence, by spherical concentration of Lipschitz functions,

$$\sigma_{n-2}\left(\left\{v \in S^{n-1} \cap u^\perp\,;\, |g_u(v)| \geq t|x-y|\right\}\right) \leq C\exp(-ct^2 n). \tag{9}$$

The crucial observation is that thanks to (8), if we condition on the value of

$$Uz = u,$$

then the vector

$$U(w/|w|) = v$$

is distributed uniformly in the sphere $S^{n-1} \cap u^\perp$. Hence, (9) implies that

$$\mathbb{P}\left(\left|g\left(Uz; \frac{Uw}{|w|}\right)\right| \geq t|x-y|\right) \leq C\exp(-ct^2 n).$$

However, since $x = z + w$ and $y = z - w$,

$$g\left(Uz; \frac{Uw}{|w|}\right) = f\left(Uz + Uw\right) - f\left(Uz - Uw\right) = f(Ux) - f(Uy),$$

completing the proof. □

Theorem 4.3 will be proved using ideas of “multi-scale approximation” from the theory of suprema of sub-Gaussian processes, which are related to Dudley’s theorem and generic chaining (see e.g. Talagrand [13]).

Proof of Theorem 4.3. Let us assume that k_* exceeds some given universal constant and that $\varepsilon < 2L$, since otherwise the conclusion of the theorem holds trivially. Fix a subspace $E_0 \in G_{n,k}$ and consider the following *sequence of nets*: For $\ell \geq 0$, we use Lemma 4.2 and find a set

$$\mathcal{F}_\ell \subseteq E_0 \cap S^{n-1}$$

which is a $3 \cdot 2^{-\ell}$-net of $S^{n-1} \cap E_0$ of cardinality at most $(9 \cdot 2^\ell)^k$. We may assume that

$$\mathcal{F}_0 = \{z_0\}$$

is a singleton. Let U be a random rotation, distributed uniformly on $O(n)$. Then $E = U(E_0)$ is distributed uniformly in $G_{n,k}$. Moreover, the point Uz_0 is distributed uniformly on the sphere S^{n-1}. We need to show that

$$\sup_{E \cap S^{n-1}} |f - M| \leq C\varepsilon \tag{10}$$

with high probability. (We leave it as an exercise to adjust the constants in the formulation of the theorem in order to eliminate C from (10). Fix $x \in E_0 \cap S^{n-1}$. For $\ell \geq 0$ there exists $x_\ell \in \mathcal{F}_\ell$ satisfying

$$|x - x_\ell| \leq 3 \cdot 2^{-\ell}. \tag{11}$$

Note that we may write

$$f(Ux) = f(Uz_0) + \sum_{\ell=0}^{\infty} \left[f(Ux_{\ell+1}) - f(Ux_\ell) \right].$$

Observe also that $|x_\ell - x_{\ell+1}| \leq 5 \cdot 2^{-\ell}$, by (11). Therefore,

$$\begin{aligned} &\sup_{x \in E \cap S^{n-1}} |f(x) - f(Uz_0)| \\ &\qquad = \sup_{x \in E_0 \cap S^{n-1}} |f(Ux) - f(Uz_0)| \qquad (12) \\ &\qquad \leq \sum_{\ell=0}^{\infty} \max \left\{ |f(Uy_1) - f(Uy_2)| \ ; \ y_1 \in \mathcal{F}_\ell,\ y_2 \in \mathcal{F}_{\ell+1}, |y_1 - y_2| \leq 5 \cdot 2^{-\ell} \right\}. \end{aligned}$$

We need to bound the various terms in the sum in (12).

Step 1. Let us estimate $f(Uz_0)$. By spherical concentration of Lipschitz functions, for all $t > 0$,

$$\mathbb{P}\left(|f(Uz_0) - M| \geq t\right) \leq C \exp(-c(t/L)^2 n).$$

Hence, with probability of at least $1 - C\exp(-ck_*)$, we have

$$|f(Uz_0) - M| \leq \varepsilon. \tag{13}$$

Step 2. Consider the ℓ^{th}-summand in (12). Let us estimate the probability that this summand exceeds

$$5\varepsilon \cdot X \cdot 2^{-\ell/2} = (5 \cdot 2^{-\ell}) \cdot \varepsilon X e^{\ell/2},$$

for some parameter $X > 0$ to be chosen momentarily. By Lemma 4.4 and the union bound, the probability at question is at most

$$\begin{aligned} \#(\mathcal{F}_\ell) \cdot \#(\mathcal{F}_{\ell+1}) \cdot C \exp\left(-c(\varepsilon X e^{\ell/2}/L)^2 n \right) \\ \le C^k \cdot \exp\left(\bar{C}\ell k - cX^2 \cdot 2^\ell \cdot (\varepsilon/L)^2 n \right) \\ \le \exp\left((C\ell - c \cdot X^2 \cdot 2^\ell) k_* \right). \end{aligned}$$

We now choose X to be a large enough universal constant, such that for all $\ell \ge 0$,

$$C\ell - c \cdot X^2 \cdot 2^\ell \le -(\ell+1).$$

Thus, we showed that with probability of at least $1 - C\exp(-(\ell+1)k_*)$, the ℓ^{th}-summand in (12) does not exceed $C\varepsilon \cdot 2^{-\ell/2}$.

Step 3. By Step 1 and Step 2, with probability of at least

$$1 - C\exp(-ck_*) - C\sum_{\ell=0}^{\infty} \exp(-(\ell+1)k_*) \ge 1 - C' \exp(-c' k_*),$$

the following hold:

- The sum in (12) is at most

$$C \sum_{\ell=0}^{\infty} \varepsilon \cdot 2^{-\ell/2} \le \bar{C}\varepsilon.$$

- We have $|f(Uz_0) - M| \le \varepsilon$.

We thus conclude from (12) that with probability of at least $1 - C\exp(-ck_*)$,

$$\sup_{x \in E \cap S^{n-1}} |f(x) - M| \le C'\varepsilon, \tag{14}$$

completing the proof. □

The multi-scale approximation in the proof of Theorem 4.3 is perhaps complicated, but it seems necessary. Arguments that only use a single net seem to miss the desired result by a logarithmic factor. The proof of Dvoretzky's Theorem 4.1 combines Theorem 4.3 with some properties of the John ellipsoid.

For a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ we consider its Minkowski functional

$$\|x\|_K = \inf\{\lambda > 0 \, ; \, x \in \lambda K\},$$

which is the norm on $\mathbb{R}^n$ whose unit ball is K. We also set

$$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_{n-1}(x)$$

and

$$b(K) = \max_{x \in S^{n-1}} \|x\|_K.$$

Clearly,

$$M(K) \leq b(K),$$

and it is a guided exercise to show that $b(K) \leq C\sqrt{n}M(K)$. Observe that $1/b(K)$ is the radius of the largest centered Euclidean ball that is contained in K. Indeed,

$$\frac{1}{b(K)} = \min_{x \in S^{n-1}} \frac{1}{\|x\|_K} = \min_{0 \neq x \in \mathbb{R}^n} \frac{|x|}{\|x\|_K} = \min_{x \in \partial K} |x|.$$

The *Dvoretzky dimension* of K is defined as

$$d(K) := n \left(\frac{M(K)}{b(K)} \right)^2. \tag{15}$$

Roughly speaking, we will see that if k is at most the Dvoretzky dimension, then a random k-dimensional section of K is approximately a Euclidean ball of radius $1/M(K)$.

Examples.

1. We clearly have $d(B_2^n) = n$.

2. In the exercise below you are asked to show that

   $$d(B_1^n) \sim n.$$

   Recall that $B_p^n = \{x \in \mathbb{R}^n \, ; \, \sum_i |x_i|^p \leq 1\}$.

3. As for the cube $B_\infty^n = \{x \in \mathbb{R}^n \, ; \, \max_i |x_i| \leq 1\}$, we have

   $$d(B_\infty^n) \sim \log n.$$

   Additionally, while the regular simplex centered at the origin is not centrally-symmetric, any reasonable definition of its Dvoretzky dimension would give $\log n$

4. If $p \geq 2$, then $d(B_p^n) \sim c_p n^{2/p}$, while if $1 \leq p \leq 2$ then $d(B_p^n) \sim n$.

Corollary 4.5 ("Dvoretzky dimension"). *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body and let $0 < \varepsilon < 1/2$. Assume that $1 \leq k \leq n$ satisfies*

$$k \leq c\varepsilon^2 d(K). \tag{16}$$

Then for a random k-dimensional subspace $E \in G_{n,k}$, with probability of at least $1 - C' \exp(-c'\varepsilon^2 d(K))$,

$$(1-\varepsilon) r B_E \subseteq K \cap E \subseteq (1+\varepsilon) r B_E,$$

where $r = \frac{1}{M(K)}$ and $c, c', C' > 0$ are universal constants.

Proof. Denote $f(x) = \|x\|_K$. We claim that f is an L-Lipschitz function for $L = b(K)$. Indeed,

$$|f(x) - f(y)| = \big|\, \|x\|_K - \|y\|_K \,\big| \leq \|x - y\|_K \leq b(K)|x - y|.$$

Let us apply Theorem 4.3. Abbreviate $M = M(K)$ and observe that from (15) and (16),

$$k \leq c\varepsilon^2 d(K) = cn\frac{(\varepsilon M)^2}{L^2} = k_*(\varepsilon M),$$

for k_* from Theorem 4.3. By the conclusion of Theorem 4.3 (with εM in place of ε), for a random, uniformly-distributed subspace $E \in G_{n,k}$, with probability of at least $1 - C\exp(-ck_*(\varepsilon M))$ we have

$$M - \varepsilon M \leq \|x\|_K \leq M + \varepsilon M \qquad \forall x \in S^{n-1} \cap E.$$

Equivalently,

$$\frac{1}{1+\varepsilon}\frac{1}{M} B_E \subseteq K \cap E \subseteq \frac{1}{1-\varepsilon}\frac{1}{M} B_E.$$

Since $1 - \varepsilon \leq 1/(1+\varepsilon)$ and $1/(1-\varepsilon) \leq 1 + 2\varepsilon$ for $0 < \varepsilon < 1/2$, the proof is complete. □

In order to prove Theorem 4.1, we still need to analyze the Dvoretzky dimension $d(K)$ of an arbitrary centrally-symmetric convex body $K \subseteq \mathbb{R}^n$.

Proposition 4.6. *For any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ there exists an invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ (also known as a "position") such that*

$$d(TK) \geq c \log n.$$

Proof of Dvoretzky's Theorem 4.1 assuming Proposition 4.6. From Corollary 4.5 we know that $T(K)$ admits a section of dimension

$$k \geq c\varepsilon^2 d(TK) \geq c\varepsilon^2 \log n$$

which is ε-spherical. By applying the linear map T^{-1}, we conclude that there exists a k-dimensional subspace $E \subseteq \mathbb{R}^n$ and an origin-symmetric ellipsoid $\mathcal{E} \subseteq E$ such that

$$(1-\varepsilon)\mathcal{E} \subseteq K \subseteq (1+\varepsilon)\mathcal{E}.$$

The conclusion now follows from the little fact that any ellipsoid has a section of (at least) half the dimension which is an exact Euclidean ball. This fact is proven in the next lemma. □

Lemma 4.7. *For any origin-symmetric ellipsoid $\mathcal{E} \subseteq \mathbb{R}^n$ there is a subspace $F \subseteq \mathbb{R}^n$ whose dimension is at least $\lceil n/2 \rceil$ such that the section $F \cap \mathcal{E}$ is a Euclidean ball.*

Proof. We will consider only the case where n is even, leaving the odd case to the reader. We may assume that the ellipsoid $\mathcal{E}$ takes the form

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \, ; \, \sum_{i=1}^{n} \frac{x_i^2}{\alpha_i^2} \leq 1 \right\}$$

for $\alpha_1 \leq \ldots\ldots \alpha_n$. Write R for the median of the α_i's, and denote

$$E_i = sp\{e_i, e_{n+1-i}\}$$

where sp means linear span and $e_1, \ldots, e_n$ are the standard unit vectors in $\mathbb{R}^n$.

The ellipse $E_i \cap \mathcal{E}$ has semi-axes α_i, α_{n+1-i} and the number R belongs to the interval $[\alpha_i, \alpha_{n+1-i}]$. It follows that there exists $\theta_i \in E_i \cap S^{n-1}$ such that the intersection of the line through the origin in direction θ_i with the ellipse $E_i \cap \mathcal{E}$ is a line segment of length $2R$.

Denote $F = sp\{\theta_1, \ldots, \theta_{n/2}\}$. We see that $\mathcal{E} \cap F$ is an origin-symmetric ellipsoid whose axes are in the directions of $\theta_1, \ldots, \theta_{n/2}$. Since the line through the origin in direction θ_i intersects the ellipsoid $\mathcal{E}$ in a line segment of length $2R$, the ellipsoid $\mathcal{E} \cap F$ is in fact a Euclidean ball of radius R. □

We still need to prove Proposition 4.6. Which linear transformation should we choose?

Definition 4.8. *We say that a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ is in John's position, if B^n is contained in K, and B^n is an ellipsoid of maximal volume that is contained in K, that is: for any origin-symmetric ellipsoid $\mathcal{E} \subseteq K$ we have*

$$\mathrm{Vol}_n(\mathcal{E}) \leq \mathrm{Vol}_n(B^n).$$

The ellipsoid of maximal volume that is contained in K is in fact uniquely determined (exercise!), hence "an ellipsoid" may be replaced by "the ellipsoid" in Definition 4.8.

Lemma 4.9. *Any centrally-symmetric convex body has a linear image in John's position.*

Proof. A compactness argument shows that the maximal volume ellipsoid contained in K exists. Let T be any linear transformation that transforms this ellipsoid to B^n. Then TK is in John's position. □

The problem of determining the John ellipsoid of K may be reformulated as a convex optimization problem. Its extremality conditions are known since the days of John [8]. We refer the reader to Ball [1] for a splendid introduction to the John ellipsoid, in which the following is proven:

Theorem 4.10. *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body in John's position. Then there exists an isotropic random vector X in $\mathbb{R}^n$ such that $X/\sqrt{n}$ is supported on the set of contact points $\partial K \cap S^{n-1}$.*

Observe that if $x_0 \in \partial K \cap S^{n-1}$, i.e., if x_0 is a *contact point*, then the outer unit normal to ∂K at x_0 must be x_0 itself. Consequently, by convexity and symmetry,

$$K \subseteq \{x \in \mathbb{R}^n \, ; \, |\langle x, x_0 \rangle| \leq 1\}.$$

It follows (see exercise 4b below) that for any $x \in \mathbb{R}^n$,

$$\|x\|_K \geq |\langle x, x_0 \rangle|. \tag{17}$$

Lemma 4.11 (Dvoretzky-Rogers [6])**.** *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body in John's position. Then there exists orthonormal vectors $y_1, \ldots, y_{\lfloor n/2 \rfloor} \in \mathbb{R}^n$ such that for all i,*

$$\|y_i\|_K > \frac{1}{2}.$$

In other words, the orthogonal vectors $2y_1, \ldots, 2y_{\lfloor n/2 \rfloor}$ of norm 2 lie outside the convex body K.

Proof. We will actually find an orthonormal basis $y_1, \dots, y_n \in \mathbb{R}^n$ such that for any i,

$$\|y_i\|_K \geq \sqrt{\frac{n-i+1}{n}}. \tag{18}$$

We will construct these vectors recursively, as well as a sequence of subspaces $F_1, \dots, F_n$ with $\dim(F_i) = i$. Set $F_0 = \{0\}$. In order to construct the vector y_i, we let X be the isotropic random vector such that $X/\sqrt{n}$ is supported on $S^{n-1} \cap \partial K$. Then,

$$\begin{aligned}\mathbb{E}|Proj_{F_{i-1}^\perp} X|^2 &= \mathbb{E}\langle Proj_{F_{i-1}^\perp} X, X\rangle \\ &= \mathrm{Tr}\left[Proj_{F_{i-1}^\perp} \cdot \mathbb{E} X \otimes X\right] = \mathrm{Tr}[Proj_{F_{i-1}^\perp} \cdot \mathrm{Id}] = n - i + 1.\end{aligned}$$

Since the expectation is at least $n - i$ while $X/\sqrt{n}$ is supported in $S^{n-1} \cap \partial K$, necessarily there exists a point $x \in S^{n-1} \cap K$ with

$$|Proj_{F_{i-1}^\perp} x|^2 \geq \frac{n-i+1}{n}.$$

We define $y_i = Proj_{F_{i-1}^\perp} x / |Proj_{F_{i-1}^\perp} x|$ and $F_i = sp(F_{i-1} \cup \{y_i\})$. Since x is a contact point, by (17),

$$\|y_i\|_K \geq |\langle y_i, x\rangle| = |Proj_{F_{i-1}^\perp} x| \geq \sqrt{\frac{n-i+1}{n}}.$$

Clearly $\dim(F_i) = i$ while y_i is a unit vector orthogonal to $y_1, \dots, y_{i-1}$ and (18) holds true. This completes the proof. □

The existence of $\lfloor n/2 \rfloor$ orthonormal vectors that lie outside $K/2$ implies a lower bound on $M(K)$, as we shall see next:

Proof of Proposition 4.6. For simplicity, suppose that n is even. We will assume that K is in John's position, and show that

$$d(K) \geq c \log n.$$

Since $B^n \subseteq K$ we have $b(K) \leq 1$. Our goal is thus to show that

$$M(K) = \int_{S^{n-1}} \|\theta\|_K d\sigma_{n-1}(\theta) \geq c\frac{\sqrt{\log n}}{\sqrt{n}}. \tag{19}$$

Let $e_1, \dots, e_{n/2} \in \mathbb{R}^n$ be the orthonormal vectors from the Dvoretzky-Rogers Lemma 4.11, so that

$$\|e_i\|_K > \frac{1}{2} \qquad i = 1, \dots, n/2. \tag{20}$$

We complete these orthonormal vectors to an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Let $\delta_1, \ldots, \delta_n$ be i.i.d random variables with $\mathbb{P}(\delta_i = 1) = \mathbb{P}(\delta_i = -1) = 1/2$. Observe that for any $\theta_1, \ldots, \theta_n \in \mathbb{R}$, by the convexity of the norm, for any j,

$$\mathbb{E}\left\|\sum_{i=1}^n \delta_i \theta_i e_i\right\|_K \geq \mathbb{E}_{\delta_j}\left\|\mathbb{E}_{\delta_1,\delta_{h-1},\delta_{j+1},\ldots,\delta_n} \sum_{i=1}^n \delta_i \theta_i e_i\right\|_K = \mathbb{E}\|\delta_j \theta_j e_j\|_K = |\theta_j| \|e_j\|_K.$$

Consequently, by (20) and the symmetries of the sphere,

$$\begin{aligned}\int_{S^{n-1}} \|\theta\|_K d\sigma_{n-1}(\theta) &= \int_{S^{n-1}} \mathbb{E}\left\|\sum_{i=1}^n \delta_i\theta_i e_i\right\|_K d\sigma_{n-1}(\theta) \\ &\geq \int_{S^{n-1}} \max_{i=1,\ldots,n} \left[|\theta_i| \|e_i\|_K\right] d\sigma_{n-1}(\theta) \geq \frac{1}{2}\int_{S^{n-1}} \max_{i=1,\ldots,n/2} |\theta_i| d\sigma_{n-1}(\theta).\end{aligned}$$

Thus, in order to prove (19), it suffices to prove that

$$\int_{S^{n-1}} \max_{i=1,\ldots,n/2} |\theta_i| d\sigma_{n-1}(\theta) \geq c\frac{\sqrt{\log n}}{\sqrt{n}}. \tag{21}$$

Let Z be a standard Gaussian random vector in $\mathbb{R}^n$. Recall from the first lecture that the integral on the left-hand side of (21) may be compared with a Gaussian integral and it equals

$$\frac{1}{\mathbb{E}|Z|} \cdot \mathbb{E}\left[\max_{i=1,\ldots,n/2} |Z_i|\right] \sim \frac{1}{\sqrt{n}} \cdot \mathbb{E}\left[\max_{i=1,\ldots,n/2} |Z_i|\right].$$

A standard estimate for the Gaussian error function is that for $t > 0$,

$$\mathbb{P}(Z_1 \geq t) \geq \frac{e^{-t^2/2}}{\sqrt{2\pi}} \cdot \frac{1}{t+1}. \tag{22}$$

Thus, for an appropriate universal constant $c > 0$, we have

$$\mathbb{P}(|Z_1| > c\sqrt{\log n}) \geq 1/n.$$

Consequently,

$$\mathbb{P}\left(\max_{i=1,\ldots,n/2} |Z_i| \geq c\sqrt{\log n}\right) \geq 1 - (1 - 1/n)^n \geq \frac{1}{2}.$$

Hence,

$$\mathbb{E}\left[\max_{i=1,\ldots,n/2} |Z_i|\right] \geq \frac{1}{2} \cdot c\sqrt{\log n},$$

completing the proof of (21). □

Exercises.

1. Tightness of the dependence on the dimension in Dvoretzky's theorem:

   (a) Let $\theta \in S^{n-1}$, $0 < r < 1$ and consider the cap
   $$C(\theta, r) = \left\{ x \in S^{n-1} \, ; \, \langle x, \theta \rangle \geq r \right\}.$$
   Prove that $\sigma_{n-1}(C(\theta, r)) \leq Ce^{-cr^2 n}$ for universal constants $c, C > 0$.

   (b) Suppose that $P \subseteq \mathbb{R}^n$ is a polytope with N facets such that
   $$\frac{1}{d} B^n \subseteq P \subseteq B^n. \tag{23}$$
   Show that there exists caps $C(\theta_i, r_i)$ for $i = 1, \ldots, N$ with $r_i \geq 1/d$ whose union covers the sphere.

   (c) Prove that any polytope satisfying (23) has at least $\exp(cn/d^2)$ facets (and by duality, also at least that many vertices).

   (d) Conclude that the Dvoretzky dimension of a polytope with N facets is at most $C' \log N$ for a universal constant $C' > 0$.

2. Formulate and prove a version of Lemma 4.2 for the unit sphere in an n-dimensional normed space.

3. Verify that the set in (4) is indeed a δ-net of size at most $(C/\delta)^n$.

4. In this exercise we prove that
   $$b(K) \leq C\sqrt{n} M(K). \tag{24}$$

   (a) Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body such that $b(K) = 1$. Prove that there exists $x_0 \in \partial K$ with $|x_0| = 1$, and that $x_0 \in S^{n-1}$ is the outer normal to ∂K at the point x_0.

   (b) Prove that for any $x \in K$,
   $$\|x\|_K \geq |\langle x, x_0 \rangle|$$

   (c) Deduce that
   $$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_{n-1}(x) \geq \int_{S^{n-1}} |\langle x, x_0 \rangle| d\sigma_{n-1}(x) \geq \frac{c}{\sqrt{n}},$$
   and complete the proof of (24).

5. Prove that $d(B_1^n) \sim n$.

6. Prove that $d(B_\infty^n) \sim \log n$.

7. Recall the proof of (22) e.g. from the "Mills ratio" entry in Wikipedia.

References

[1] Ball, K., *An elementary introduction to modern convex geometry*. Math. Sci. Res. Inst. Publ., Vol. 31, Cambridge, (1997), 1-58.

[2] Birch, B. J., *Homogeneous forms of odd degree in a large number of variables*. Mathematika, Vol. 4, (1957), 102–105.

[3] Dol'nikov, V. L., Karasev, R. N., *Dvoretzky type theorems for multivariate polynomials and sections of convex bodies.* Geom. Funct. Anal. (GAFA), Vol. 21, no. 2, (2011), 301–318.

[4] Dvoretzky, A., *A theorem on convex bodies and applications to Banach spaces. Proc. Nat. Acad. Sci. U.S.A.*, Vol. 45, no. 2, (1959), 223–226 (erratum in Vol. 45, no. 10).

[5] Dvoretzky, A., *Some results on convex bodies and Banach spaces.* In Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), pages 123–160. Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961.

[6] Dvoretzky, A., Rogers, C. A., *Absolute and unconditional convergence in normed linear spaces.* Proc. Nat. Acad. Sci. U.S.A., Vol. 36, (1950), 192–197.

[7] Gordon, Y., *Some inequalities for Gaussian processes and applications*. Israel J. Math., Vol. 50, no. 4, (1985), 265–289.

[8] John, F., *Extremum problems with inequalities as subsidiary conditions.* Interscience Publishers, New York, (1948), 187–204.

[9] Milman, V., *Dvoretzky's theorem—thirty years later.* Geom. Funct. Anal. (GAFA), Vol. 2, no. 4, (1992), 455–479.

[10] Milman, V. D., *A few observations on the connections between local theory and some other fields*. Geom. aspect. of Funct. Anal. – Israel Seminar. Lecture Notes in Math., Vol. 1317, Springer, (1988), 283–289.

[11] Milman, V. D., *A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies.* Funkcional. Anal. i Priložen. 5 (1971), no. 4, 28–37.

[12] Schechtman, G., *A remark concerning the dependence on ε in Dvoretzky's theorem*. Geom. aspect. of Funct. Anal. – Israel Seminar. Lecture Notes in Math., Vol. 1376, Springer, (1989), 274–277.

[13] Talagrand, M., *The generic chaining*. Springer, 2005.

Request. Please e-mail me at `boaz.klartag@weizmann.ac.il` with any comments, corrections, or suggestions regarding these lecture notes. In addition, if you are able to produce *tikz* figures similar to (or better than) those drawn on the blackboard in class, I would be glad to include them in the notes.

October 15, 2025