Isoperimetric inequalities in high-dimensional convex sets
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Lecture 5: Duality and the Santaló and Bourgain-Milman inequalities

Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body. The randomized Dvoretzky theorem tells us that random ℓ-dimensional sections of K for $\ell \ll d(K)$ are approximately Euclidean balls of radius $1/M(K)$. We recall that the Dvoretzky dimension of K is

$$d(K) = n \cdot \left(\frac{M(K)}{b(K)} \right)^2 ,$$

where $M(K)$ is the average of the norm $\| \cdot \|_K$ on the unit sphere S^{n-1} and where $b(K)$ is the maximum of the norm $\| \cdot \|_K$ on this unit sphere. Here, $\| \cdot \|_K$ is the norm whose unit ball is K. In particular, if $b(K) \sim M(K)$, then sections of K of dimension proportional to n are approximately Euclidean. One of the most important examples of such a convex body is the cross polytope

$$B_1^n = \left\{ x \in \mathbb{R}^n \, ; \, \sum_{i=1}^n |x_i| \leq 1 \right\},$$

relevant to Kashin's splitting and to Grothendieck's inequality (see below).

The parameters $b(K)$ and $M(K)$ have a geometric meaning: the radius of the Euclidean ball circumscribed in K equals $1/b(K)$, while $1/M(K)$ is the harmonic average on the sphere of the radial function of K, i.e.,

$$\frac{1}{M(K)} = \left(\int_{S^{n-1}} \frac{1}{r_K(\theta)} d\sigma_{n-1}(\theta) \right)^{-1},$$

and $r_K(\theta) = 1/\|\theta\|_K = \sup \{ t > 0 \, ; \, t\theta \in K \}$ measures how far K extends in direction θ. Next we interpret Dvoretzky's theorem and these two geometric parameters via *duality*.

5.1 Convex duality

Convex sets $K \subseteq \mathbb{R}^n$ come to the world in pairs; this is especially true for *centrally-symmetric* convex sets or convex *cones*. We recall that the polar body to a convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior is

$$K^\circ = \{ x \in \mathbb{R}^n \, ; \, \forall y \in K, \ \langle x, y \rangle \leq 1 \} .$$

We have $(K^\circ)^\circ = K$ with $K = K^\circ$ if and only if $K = B^n$ (exercise). When K is a polytope, there is a one-to-one correspondence between the vertices of K and the $(n-1)$-dimensional facets of K°. In particular, the number of vertices of K equals the number of facets of K°. We denote the supporting functional of K by

$$h_K(x) = \sup_{y \in K} \langle x, y \rangle.$$

The supporting functional h_K determines the convex body K. In fact, the supporting functional provides a one-to-one correspondence convex bodies on one hand, and 1-positively-homogenous, convex functions on the other hand.

Observe that for $\theta \in S^{n-1}$, the quantity $h_K(\theta) + h_K(-\theta)$ is the *width* of K in the direction θ; that is, it equals the distance between the two hyperplanes orthogonal to θ such that K is "sandwiched" between them. Moreover, note that

$$h_{K+T} = h_K + h_T. \tag{1}$$

When $K \subseteq \mathbb{R}^n$ is centrally-symmetric,

$$\|x\|_{K^\circ} = \|x\|_K^* = h_K(x).$$

An important geometric parameter of a convex body $K \subseteq \mathbb{R}^n$ is its (half) *mean width*, defined by

$$M^*(K) = M(K^\circ) = \int_{S^{n-1}} h_K(x) d\sigma_{n-1}(x).$$

The mean-width is *Minkowski-additive*, i.e.,

$$\begin{aligned} M^*(K+T) &= \int_{S^{n-1}} h_{K+T}(x) d\sigma_{n-1}(x) = \int_{S^{n-1}} \left[h_K(x) + h_T(x)\right] d\sigma_{n-1}(x) \\ &= M^*(K) + M^*(T). \end{aligned}$$

It is an exercise to verify that the dual operation to section is projection, i.e., if $E \subseteq \mathbb{R}^n$ is a subspace, then,

$$(K \cap E)^\circ = Proj_E K^\circ, \tag{2}$$

where $Proj_E : \mathbb{R}^n \to E$ is the orthogonal projection operator. Moreover, when K is centrally-symmetric,

$$b(K^\circ) = \sup_{x \in K} |x| = \frac{1}{2} \mathrm{diam}(K).$$

Since $M(K^\circ) = M^*(K)$ while $b(K^\circ) = \mathrm{diam}(K)/2$, the randomized version of Dvoretzky's theorem has the following immediate corollary, obtained by dualizing:

Corollary 5.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body with $K = -K$, let $0 < \varepsilon < 1/2$ and assume that $1 \leq \ell \leq n$ satisfies*

$$\ell \leq c\varepsilon^2 \cdot d(K^\circ)$$

for

$$d(K^\circ) = n \cdot \left(\frac{2M^*(K)}{\mathrm{diam}(K)} \right)^2.$$

Let $E \in G_{n,\ell}$ be a random, uniformly-distributed subspace. Then with probability of at least $1 - \tilde{C}\exp(-\tilde{c}\varepsilon^2 d(K^\circ))$,

$$(1-\varepsilon)M^*(K)B_E \subseteq Proj_E K \subseteq (1+\varepsilon)M^*(K)B_E,$$

where $c, \tilde{c}, \tilde{C} > 0$ are universal constants.

Similarly, one may deduce from Dvoretzky's theorem that any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ has a *orthogonal projection*

$$Proj_E(K)$$

of dimension $\dim(E) \geq c\varepsilon^2 \cdot \log n$ which is ε-close to a Euclidean ball. In addition to the geometric projection, we may also consider the measure projection and ask how regular they are. That is, if $X \sim Unif(K)$, then we can ask whether the random vector

$$Proj_E(X) \tag{3}$$

is close to Gaussian. A result in this direction, in spirit of the thin-shell theorem and the strong thin-shell bounds satisfies by convex bodies, was proven by Eldan and Klartag [3]. The theorem states that for any convex body $K \subseteq \mathbb{R}^n$ with barycenter at the origin, there exists a subspace $E \subseteq \mathbb{R}^n$ with $\dim(E) \geq n^\alpha$ such that the random vector in (3) is close to a Gaussian random vector, with total-variation distance at most $Cn^{-\beta}$. Here, $C, \alpha, \beta > 0$ are universal constants.

Thus far we considered four lengthscales associated with K, which are

$$1/b(K), 1/M(K), M^*(K) \qquad \text{and} \qquad \mathrm{diam}(K).$$

We now add a fifth lengthscale, the *volume-radius* of K, defined via

$$v.rad.(K) = \left(\frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(B^n)} \right)^{1/n}$$

which is the volume of the Euclidean ball with the same radius as K. Quite a few geometric questions on high-dimensional convex sets involve at least one of the three middle parameters from (4) in the following lemma:

Lemma 5.2. *For any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ we have the chain of basic inequalities*

$$\frac{1}{b(K)} \le \frac{1}{M(K)} \le v.rad.(K) \le M^*(K) \le \frac{\mathrm{diam}(K)}{2}, \tag{4}$$

and equalities hold when K is a Euclidean ball.

Proof. The inequality $M(K) \le b(K)$ holds true since the spherical average of the norm $\|\cdot\|_K$ is at most its maximum. Similarly

$$M^*(K) = M(K^\circ) \le b(K^\circ) = \mathrm{diam}(K)/2.$$

We move on to the second inequality from the left in (4). Recall that

$$\mathrm{Vol}_n(K) = \mathrm{Vol}_n(B^n) \int_{S^{n-1}} \|x\|_K^{-n} d\sigma_{n-1}(x). \tag{5}$$

Indeed, by integrating in polar coordinates,

$$\begin{aligned}\mathrm{Vol}_n(K) &= \mathrm{Vol}_{n-1}(S^{n-1}) \int_{S^{n-1}} \int_0^\infty 1_K(r\theta) r^{n-1} dr d\sigma_{n-1}(\theta) \\ &= n\mathrm{Vol}_n(B^n) \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} dr \right) d\sigma_{n-1}(\theta),\end{aligned}$$

and (5) follows. Hence, by Jensen's inequality,

$$v.rad.(K) = \left(\int_{S^{n-1}} \|x\|_K^{-n} d\sigma_{n-1}(x) \right)^{1/n} \ge \frac{1}{\int_{S^{n-1}} \|x\|_K d\sigma_{n-1}(x)} = \frac{1}{M(K)}.$$

The third inequality from the left on (4), which is the inequality

$$v.rad.(K) \le M^*(K), \tag{6}$$

is called *Urysohn's inequality*. It is valid for any Borel subset $K \subseteq \mathbb{R}^n$ of finite volume. In order to prove it, we recall the multiplicative Brunn-Minkowski inequality,

$$\mathrm{Vol}_n\left(\lambda K_1 + (1-\lambda) K_2\right) \ge \mathrm{Vol}_n(K_1)^\lambda \mathrm{Vol}_n(K_2)^{1-\lambda}.$$

By a simple induction argument, if $N \ge 1$ and the numbers $\lambda_1, \ldots, \lambda_N \ge 0$ add to one, then for any Borel sets $K_1, \ldots, K_N \subseteq \mathbb{R}^n$ of finite volume,

$$\mathrm{Vol}_n\left(\sum_{i=1}^N \lambda_i K_i \right) \ge \prod_{i=1}^N \mathrm{Vol}_n(K_i)^{\lambda_i}. \tag{7}$$

Consider the particular case where $U_1, \dots, U_N \in O(n)$ and $K_i = U_i(K)$. By (7),

$$\mathrm{Vol}_n\left(\frac{1}{N}\sum_{i=1}^{N} U_i(K)\right) \geq \mathrm{Vol}_n(K). \tag{8}$$

We interpret the convex body on the left-hand side as

$$\int_{O(n)} U(K)d\nu(U) = \frac{1}{N}\sum_{i=1}^{N} U_i(K),$$

where $\nu = (1/N)\sum_i \delta_{U_i}$ is a discrete measure on $O(n)$, and where in view of (1), we define the convex body

$$\int_{O(n)} U(K)d\nu(U)$$

via its supporting functional, i.e., it is the unique convex body whose supporting functional is

$$\int_{O(n)} h_{U(K)}(x)d\nu(x) \qquad (x \in \mathbb{R}^n).$$

We may now let the discrete measures $\nu_N = \nu_{U_1,\dots,U_N}$ tend to the uniform probability measure on $O(n)$, denoted by μ_n. We deduce from (8) that

$$\begin{aligned}\mathrm{Vol}_n\left(\int_{O(n)} U(K)d\mu_n(U)\right) &= \lim_{N\to\infty} \mathrm{Vol}_n\left(\int_{O(n)} U(K)d\nu_N(U)\right)\\ &\geq \mathrm{Vol}_n(K). \end{aligned} \tag{9}$$

The convex body $\int_{O(n)} U(K)d\mu_n(U)$ is rotationally-invariant, and hence it is a Euclidean ball centered at the origin. In order to determine its radius, we compute its mean-width:

$$\begin{aligned}M^*\left(\int_{O(n)} U(K)d\mu_n(U)\right) &= \int_{O(n)} M^*(UK)d\mu_n(U)\\ &= \int_{O(n)} M^*(K)d\mu_n(U) = M^*(K).\end{aligned}$$

Thus the radius of this Euclidean ball equals $M^*(K)$, and by (9),

$$\mathrm{Vol}_n(M^*(K)B^n) \geq \mathrm{Vol}_n(K),$$

which proves (6). □

5.2 Volume-diameter balance

The following theorem complements Dvoretzky's theorem, and is useful for understanding the diameter of random sections whose dimension exceeds the Dvoretzky dimension of K.

Theorem 5.3 ("volume-diameter balance [7]"). *Let $K \subseteq \mathbb{R}^n$ be a convex body containing the origin, and let $1 \le \ell \le n$. Let $E \in G_{n,\ell}$ be a random subspace, distributed uniformly. Set*

$$\lambda = \frac{\ell}{n}.$$

Then with probability of at least $1 - \tilde{C}\exp(-\tilde{c}n)$,

$$\mathrm{diam}(K \cap E)^{1-\lambda} v.rad.(K \cap E)^{\lambda} \le C v.rad.(K), \tag{10}$$

where, $c, \tilde{c}, \tilde{C} > 0$ are universal constants. In fact, the random variable

$$X = \mathrm{diam}(K \cap E)^{1-\lambda} v.rad.(K \cap E)^{\lambda}$$

satisfies

$$(\mathbb{E}|X|^n)^{1/n} \sim v.rad.(K). \tag{11}$$

Thus, if a random section of K typically has a non-negligible volume, then its diameter is not too large.

Proof of Theorem 5.3. Integrating in polar coordinates,

$$\mathrm{Vol}_n(K) = \frac{n\kappa_n}{\ell\kappa_\ell}\mathbb{E}\int_{K\cap E} |x|^{n-\ell} dx, \tag{12}$$

where $\kappa_n = \mathrm{Vol}_n(B^n)$. Indeed, as before,

$$\begin{aligned}
\mathrm{Vol}_n(K) &= n\kappa_n \int_{S^{n-1}} \int_0^\infty 1_K(r\theta) r^{n-1} dr d\sigma_{n-1}(\theta) \\
&= n\kappa_n \mathbb{E}\int_{S^{n-1}\cap E} \int_0^\infty 1_K(r\theta)|r\theta|^{n-\ell} \cdot r^{\ell-1} dr d\sigma_E(\theta) \\
&= \frac{n\kappa_n}{\ell\kappa_\ell}\mathbb{E}\int_{K\cap E} |x|^{n-\ell} dx,
\end{aligned}$$

where σ_E is the uniform probability measure on the $(\ell-1)$-dimensional sphere $S^{n-1} \cap E$. Next, we claim that almost surely

$$\int_{K\cap E} |x|^{n-\ell} dx \ge c^n \cdot \mathrm{Vol}_\ell(K \cap E) \cdot \mathrm{diam}(K \cap E)^{n-\ell}. \tag{13}$$

Observe that the reverse inequality to (13) trivially holds true if we remove the c^n factor. In order to prove (13), we denote $T = K \cap E$. Pick $x_0 \in T$ with

$$|x_0| \geq \frac{\mathrm{diam}(T)}{4}.$$

Set

$$T_0 = \frac{8}{9}x_0 + \frac{1}{9}T \subseteq T.$$

Then for any $x \in T_0$,

$$|x| \geq \frac{8}{9}|x_0| - \frac{1}{9}\mathrm{diam}(T) \geq c \cdot \mathrm{diam}(T).$$

Hence,

$$\begin{aligned}\int_T |x|^{n-\ell}dx \geq \int_{T_0} |x|^{n-\ell}dx &\geq c^{n-\ell} \cdot \mathrm{Vol}_\ell(T_0) \cdot \mathrm{diam}(T)^{n-\ell} \\ &= c^{n-\ell}\frac{\mathrm{Vol}_\ell(T)}{9^\ell}\mathrm{diam}(T)^{n-\ell},\end{aligned}$$

proving (13). By substituting (13) into (12), we see that

$$\mathrm{Vol}_n(K) \geq c^n \cdot \frac{n\kappa_n}{\ell\kappa_\ell} \cdot \mathbb{E}\mathrm{Vol}_\ell(K \cap E)\mathrm{diam}(K \cap E)^{n-\ell}, \tag{14}$$

where the reverse inequality to (14) holds true without the c^n factor. Hence

$$v.rad.(K)^n \geq c^n\mathbb{E}\left(v.rad.(K \cap E)^\lambda \mathrm{diam}(K \cap E)^{1-\lambda}\right)^n.$$

We now deduce (10) by the Markov-Chebyshev inequality. The "In fact" part follows from the fact that the reverse inequality to (14) holds true without the c^n factor. □

In the important case of $K = B_1^n$, Theorem 5.3 can be used to prove the following (see guided exercise below).

Corollary 5.4 (Kashin's splitting [14])**.** *There exists an orthogonal decomposition* $\mathbb{R}^n = E_1 \oplus E_2$ *with* $\dim(E_i) \geq \lfloor n/2 \rfloor$ *for* $i = 1, 2$ *such that*

$$\forall x \in E_1 \cup E_2, \qquad c\sqrt{n}|x| \leq \|x\|_1 \leq \sqrt{n}|x|,$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$ *for* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and where* $c > 0$ *is a universal constant.*

By considering a Kashin splitting into three pieces and dualizing, we obtain the following:

Corollary 5.5 (Grothendieck's inequality [6]). *For any numbers $M_{ij} \in \mathbb{R}$ $(i, j = 1, \ldots, n)$,*

$$\max_{\substack{u_i, v_j \in \mathbb{R}^n \\ |u_i|, |v_j| \leq 1}} \left| \sum_{i,j=1}^n M_{ij} \langle u_i, v_j \rangle \right| \leq C \max_{s_i, t_k \in \{-1,1\}} \left| \sum_{i,j=1}^n M_{ij} s_i t_j \right|,$$

where $C > 0$ is a universal constant.

5.3 The Santaló and the Bourgain-Milman inequalities

The bodies K and K° are kind-of "inverses" to each other. For instance, for any invertible, linear map $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$(T(K))^\circ = (T^{-1})^*(K^\circ).$$

Theorem 5.6 (Santaló and Bourgain-Milman). *For any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$,*

$$c \leq v.rad.(K) v.rad.(K^\circ) \leq 1, \tag{15}$$

where $c \geq 0$ is a universal constant. In fact, $c = 1/2$ works according to Kuperberg [8].

The left-hand side inequality in (15) holds true without the central-symmetry assumption, assuming only that 0 lies in the interior of K. The right-hand side inequality in (15) holds true whenever $K \subseteq \mathbb{R}^n$ is a centered convex body, i.e., its barycenter lies at the origin.

In the case of a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$, the Mahler conjecture [9, 10] suggests that

$$\mathrm{Vol}_n(K) \cdot \mathrm{Vol}_n(K^\circ) \geq \frac{4^n}{n!}. \tag{16}$$

Equality in (16) is attained when $K = [-1, 1]^n$. Inequality (16) was proven thus far for $n = 2, 3$, see Iriyeh and Shibata[5]. For a general convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior, the Mahler conjecture [9, 10] suggests that

$$\mathrm{Vol}_n(K) \cdot \mathrm{Vol}_n(K^\circ) \geq \frac{(n+1)^{n+1}}{(n!)^2}. \tag{17}$$

There is equality in (16) when K is a centered simplex. Inequality (17) was proven in [9, 10] for $n = 2$.

Sketch of proof of the Santaló inequality. Since the optimizer is a Euclidean ball, a symmetrization proof comes to mind. Recall the Steiner symmetrization from Lecture 2. When $K \subseteq \mathbb{R}^n$ is a convex body and $H \subseteq \mathbb{R}^n$ is a hyperplane through the origin with $H = \theta^\perp$ for $\theta \in S^{n-1}$, we write

$$S_H(K)$$

for the closure of

$$\left\{ y + t\theta \, ; \, y \in H, t \in \mathbb{R}, \ |t| < \frac{\text{Length}((y + \mathbb{R}\theta) \cap K)}{2} \right\}.$$

Recall from Lecture 2 that there is a sequence of consecutive Steiner symmetrizations of K that converges to a Euclidean ball. By Fubini's theorem,

$$\text{Vol}_n(S_H(K)) = \text{Vol}_n(K).$$

We furthermore claim that when K is centrally-symmetric,

$$\text{Vol}_n(S_H(K^\circ)) \leq \text{Vol}_n(S_H(K)^\circ). \tag{18}$$

Inequality (18) implies that $\text{Vol}_n(K) \cdot \text{Vol}_n(K^\circ)$ cannot decrease under Steiner symmetrization. Since there a sequence of Steiner symmetrizations of K converging to a ball, the Santaló inequality follows. It still remains to prove (18). Denote $K(t) = \{y \in H \, ; \, y + t\theta \in K\}$, and claim that for $t \in \mathbb{R}$,

$$\frac{K^\circ(t) + K^\circ(-t)}{2} \subseteq (S_H K)^\circ(t),$$

as may be readily checked from the definitions. The non-centrally-symmetric case is proven in Meyer and Pajor [11]. □

5.4 Sketch of proof of the Bourgain-Milman inequality

There are several proof of the Bourgain-Milman inequality, all quite startling, using tools such as harmonic analysis and K-convexity, or complex analysis, or properties of the log-Laplace transform Here we discuss Berndtsson's simplification [1] of Kuperberg's proof [8]. Other proofs may be found in Bourgain and Milman [2], Nazarov [12] and Giannopoulos, Paouris and Vritsiou [4].

We begin with duality of convex functions, which is yet another manifestation of convex duality. For a convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ with $\lim_{x\to\infty} \psi(x)/|x| = \infty$, we consider its *Legendre transform*

$$\psi^*(x) = \sup_{y \in \mathbb{R}^n} \left[x \cdot y - \psi(y) \right] \qquad (x \in \mathbb{R}^n). \tag{19}$$

The supremum is attained by continuity, since ψ grows super-linearly at infinity, and $\psi^* : \mathbb{R}^n \to \mathbb{R}$ is a convex function with $\lim_{x\to\infty} \psi(x)/|x| = \infty$ satisfying

$$(\psi^*)^* = \psi,$$

see exercise below. The role of the Euclidean ball, as the unique fixed point of the polarity transform, is played by the function

$$\psi_0(x) = \frac{|x|^2}{2}.$$

Indeed, we have that $\psi = \psi^*$ if and only if $\psi = \psi_0$. In fact, for any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$,

$$\psi(x) = \frac{\|x\|_K^2}{2} \qquad \Longrightarrow \qquad \psi^*(x) = \frac{\|x\|_{K^\circ}^2}{2}. \tag{20}$$

In the case where ψ is smooth, the supremum of the concave function in (19) is attained at a point $y \in \mathbb{R}^n$ with $\nabla\psi(y) = x$. It follows that for any $x \in \mathbb{R}^n$,

$$\psi(x) + \psi^*(\nabla\psi(x)) = x \cdot \nabla\psi(x). \tag{21}$$

Another important property of the Legendre transform (exercise) is that when ψ is C^1-smooth and strictly-convex, the continuous map

$$\mathbb{R}^n \ni x \mapsto \nabla\psi(x) \in \mathbb{R}^n$$

is invertible, and its inverse is the map

$$x \mapsto \nabla\psi^*(x).$$

Thus the Legendre transform provides a convenient way to invert the gradient-map of a convex function.

Theorem 5.7 (Berndtsson)**.** *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an even, convex function such that $\lim_{x\to\infty} \psi(x)/|x| = +\infty$. Then,*

$$\int_{\mathbb{R}^n} e^{-\psi} \int_{\mathbb{R}^n} e^{-\psi^*} \geq \pi^n. \tag{22}$$

In order to deduce Kuperberg's bound for the Bourgain-Milman inequality from Theorem 5.7, we apply the theorem with $\psi(x) = \|x\|_K^2/2$ and note that

$$\int_{\mathbb{R}^n} e^{-\|x\|_K^2/2} dx = \int_0^\infty \mathrm{Vol}_n\left(\left\{x \in \mathbb{R}^n\,;\, e^{-\|x\|_K^2/2} \geq t\right\}\right) dt = C_n \mathrm{Vol}_n(K),$$

for $C_n = (2\pi)^{n/2}/\kappa_n = 2^{n/2}/\Gamma(n/2+1)$. The left-hand side inequality in (15) now follows from (20) and Theorem 5.7.

We proceed with a sketch of proof of Theorem 5.7. Let us assume that ψ is smooth and strongly convex; this means that the Hessian matrix $\nabla^2\psi(x)$ is positive definite everywhere, rather than merely positive semidefinite as for a general convex function. In fact, we may even assume that there exists some absurdly large $R > 0$ such that

$$\psi(x) = \frac{|x|^2}{2} \qquad \text{for all } |x| \geq R.$$

All of this can be acheived by an approximation argument which only has an arbitrarily small effect on the integrals in Theorem 5.7. Our assumptions imply that the gradient map

$$x \to \nabla\psi(x)$$

is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, which equals the identity map outside a ball of radius R centered at the origin.

We will work in $\mathbb{C}^{2n} \cong \mathbb{C}^n \times \mathbb{C}^n$, and use

$$z = x + iy, \qquad w = \xi + i\eta \qquad (x, y, \xi, \eta \in \mathbb{R}^n) \tag{23}$$

as complex coordinates in $\mathbb{C}^{2n}$. Consider the $2n$-dimensional submanifold of $\mathbb{C}^{4n}$,

$$\begin{aligned}\Lambda = \Lambda_\psi &= \text{Graph}(\nabla\psi) \times \text{Graph}(\nabla\psi) \\ &= \left\{(x, y, \xi, \eta)\,;\, x, y, \xi, \eta \in \mathbb{R}^n,\ \xi = \nabla\psi(x),\ \eta = \nabla\psi(y)\right\}.\end{aligned}$$

For instance, with $\psi_0(x) = |x|^2/2$, the submanifold Λ_{ψ_0} is the diagonal subspace $x = \xi, \eta = y$. Define

$$t = \frac{x+y}{2} \qquad \text{and} \qquad s = \frac{\xi - \eta}{2}. \tag{24}$$

Lemma 5.8. *For $(x, y, \xi, \eta) \in \Lambda$, defining t and s via (24) and z and w via (23), we have*

$$\psi(t) + \psi^*(s) \leq \frac{x \cdot \xi + y \cdot \eta}{2} = \text{Re}\left(\frac{z \cdot \bar{w}}{2}\right),$$

where $z \cdot w = \sum_j z_j w_j$ for $z, w \in \mathbb{C}^n$ while $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_n)$.

Proof. Indeed, by the convexity and evenness of ψ and ψ^*,

$$\psi(t) = \psi\left(\frac{x+y}{2}\right) \leq \frac{\psi(x) + \psi(y)}{2}$$

while

$$\psi^*(s) = \psi^*\left(\frac{\xi - \eta}{2}\right) \le \frac{\psi^*(\xi) + \psi^*(-\eta)}{2} = \frac{\psi^*(\xi) + \psi^*(\eta)}{2}.$$

This is the only place in the whole proof where the assumption that ψ is even is used. Since $\xi = \nabla\psi(x)$ and $\eta = \nabla\psi(y)$, by property (21) of the Legendre transform,

$$\psi(t) + \psi^*(s) \le \frac{\psi(x) + \psi^*(\xi)}{2} + \frac{\psi(y) + \psi^*(\eta)}{2} = \frac{x \cdot \xi + y \cdot \eta}{2}.$$

□

We view the coordinates x, y, ξ, η as $\mathbb{R}^n$-valued functions on $\mathbb{C}^{2n}$. Thus t and s are also $\mathbb{R}^n$-valued functions on $\mathbb{C}^{2n}$, while z and w are $\mathbb{C}^n$-valued. We leave it as an exercise to show that our assumptions imply that the map

$$(t, s) : \Lambda \to \mathbb{R}^n \times \mathbb{R}^n \tag{25}$$

is a diffeomorphism. By Lemma 5.8 we may integrate with respect to the standard orientation in $\mathbb{R}^n$ and obtain

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\psi} \int_{\mathbb{R}^n} e^{-\psi^*} &= \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-\psi(t)-\psi^*(s)} dt_1 \wedge \ldots \wedge dt_n \wedge ds_1 \wedge \ldots \wedge ds_n \\ &= \int_\Lambda e^{-\psi(t)-\psi^*(s)} dt \wedge ds \ge \int_\Lambda e^{-\mathrm{Re}(z\cdot\bar{w})/2} dt \wedge ds, \end{aligned} \tag{26}$$

where we abridge $dt = dt_1 \wedge \ldots \wedge dt_n$ and $ds = ds_1 \wedge \ldots \wedge ds_n$, and also $dz = dz_1 \wedge \ldots \wedge dz_n$ and $d\bar{w} = d\bar{w}_1 \wedge \ldots \wedge d\bar{w}_n$ and similarly for dx and dy.

Lemma 5.9. *In $\mathbb{C}^{2n}$ we have the following relation:*

$$dt \wedge ds = (4i)^{-n} dz \wedge d\bar{w}.$$

Proof. The submanifold Graph$(\nabla\psi)$ is Lagrangian relative to the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$. In other words, since $\xi = \nabla\psi(x)$ we have $\xi_j = \partial_j\psi(x)$ and

$$\sum_{j=1}^n dx_j \wedge d\xi_j = \sum_{j,k=1}^n dx_j \wedge (\partial_{jk}\psi(x) dx_k) = 0,$$

as $\partial_{jk}\psi = \partial_{kj}\psi$. Hence, by (24),

$$\sum_{j=1}^n dt_j \wedge ds_j = \frac{1}{4}\sum_{j=1}^n (dx_j + dy_j) \wedge (d\xi_j - d\eta_j) = \frac{1}{4}\sum_{j=1}^n \left[-dx_j \wedge d\eta_j + dy_j \wedge d\xi_j\right],$$

since the sum over $dx_j \wedge d\xi_j$ vanishes, as well as the sum over $dy_j \wedge d\eta_j$. By the same reason,

$$\sum_{j=1}^{n} dz_j \wedge d\bar{w}_j = \sum_{j=1}^{n} (dx_j + i dy_j) \wedge (d\xi_j - i d\eta_j) = i \sum_{j=1}^{n} \left[-dx_j \wedge d\eta_j + dy_j \wedge d\xi_j \right].$$

Consequently,

$$\sum_{j=1}^{n} dt_j \wedge ds_j = \frac{1}{4i} \sum_{j=1}^{n} dz_j \wedge d\bar{w}_j.$$

By considering the n^{th} exterior power of both 2-forms, the lemma follows. □

Since the map in (25) is a diffeomorphism, the $(2n)$-form $dt \wedge ds$ does not vanish on the submanifold Λ, and in particular it does not change sign. From (26) and Lemma 5.9, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\psi} \int_{\mathbb{R}^n} e^{-\psi^*} &\geq \int_{\Lambda} \left| e^{-z \cdot \bar{w}/2} \right| dt \wedge ds = \int_{\Lambda} \left| e^{-z \cdot \bar{w}/2} \right| |dt \wedge ds| \\ &\geq \left| \int_{\Lambda} e^{-z \cdot \bar{w}/2} dt \wedge ds \right| = 4^{-n} \left| \int_{\Lambda} e^{-z \cdot \bar{w}/2} dz \wedge d\bar{w} \right|. \qquad (27)\end{aligned}$$

The beauty is that the $(2n)$-form

$$e^{-z \cdot \bar{w}/2} dz \wedge d\bar{w} \qquad (28)$$

is holomorphic in z and anti-holomorphic in w, and as such it is a closed form. Indeed, it is an exercise to write $d = \partial_z + \partial_{\bar{z}} + \partial_w + \partial_{\bar{w}}$ in $\mathbb{C}^{2n}$ and verify that the form in (28) is closed, by using only the fact that $e^{-z \cdot \bar{w}/2}$ is a holomorphic function of z and an anti-holomorphic function of w. In particular, by Stokes theorem, the integral

$$\int_{\Lambda} e^{-z \cdot \bar{w}/2} dz \wedge d\bar{w}$$

does not change in value when we deform the sub-manifold $\Lambda = \Lambda_\psi$ in a compact region of $\mathbb{C}^n$. However, outside of a large ball, the function ψ coincides with ψ_0, and hence Λ_ψ coincides with the flat subspace Λ_{ψ_0} which is given by the equations $x = \xi, y = \eta$. Hence,

$$\begin{aligned}\int_{\Lambda_\psi} e^{-z \cdot \bar{w}/2} dz \wedge d\bar{w} &= \int_{\Lambda_{\psi_0}} e^{-z \cdot \bar{w}/2} dz \wedge d\bar{w} \qquad (29) \\ &= \pm (2i)^n \int_{\Lambda_{\psi_0}} e^{-(|x|^2 + |y|^2)/2} dx \wedge dy = \pm (2i)^n \cdot (2\pi)^n,\end{aligned}$$

where in the last passage we replace the integral over Λ_{ψ_0} by the integral over $\mathbb{R}^n \times \mathbb{R}^n$ since the map $(x, y) : \Lambda_{\psi_0} \to \mathbb{R}^n \times \mathbb{R}^n$ is a diffeomorphism. Theorem 5.7 follows from (27) and (29).

Exercises.

1. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $F(\lambda x) = \lambda F(x)$ for all $x \in \mathbb{R}^n$ and $\lambda \geq 0$. Prove that there exists a unique convex body $K \subseteq \mathbb{R}^n$ such that
$$h_K = F.$$
2. Let $K \subseteq \mathbb{R}^n$ be a convex body containing the origin in its interior.
   (a) Prove that $(K^\circ)^\circ = K$ with $K = K^\circ$ if and only if $K = B^n$.
   (b) Prove that $(K \cap E)^\circ = Proj_E K^\circ$ for any subspace $E \subseteq \mathbb{R}^n$.
3. Kashin's splitting. Let $K = B_1^n$.
   (a) Show that $b(K) = \sqrt{n}$ and $\text{Vol}_n(K) = 2^n/n!$, and conclude that
   $$v.rad.(K) \leq C/b(K).$$
   (b) Apply Theorem 5.3 and deduce Corollary 5.4.
   (c) Split into three pieces and dualize: Prove that we may decompose $\mathbb{R}^{3n} = E_1 \oplus E_2 \oplus E_3$, an orthogonal decomposition with $\dim(E_i) = n$ for all i, such that for any $i, j \in \{1, 2, 3\}$, setting $E = E_i \oplus E_j$,
   $$c\sqrt{n}B_E \subseteq Proj_E\left([-1, 1]^n\right) \subseteq \sqrt{n}B_E. \qquad (30)$$
4. Grothendieck's inequality.
   (a) Deduce from (30) that for any $u_1, \ldots, u_n, v_1, \ldots, v_n \in B^n \cap E_1$ there exist $f_1, \ldots, f_n \in E_1 \oplus E_2$ and $g_1, \ldots, g_n \in E_1 \oplus E_3$ such that for all i,
   $$\|f_i\|_\infty \leq \frac{C}{\sqrt{n}}, \qquad \|g_i\|_\infty \leq \frac{C}{\sqrt{n}} \qquad (i = 1, \ldots, n)$$
   and $Proj_{E_1} f_i = u_i, Proj_{E_1} g_i = v_i$ for all i. Conclude that for all i, j,
   $$\langle f_i, g_j \rangle = \langle u_i, v_j \rangle.$$

(b) Prove that for any numbers $M_{ij} \in \mathbb{R}$ $(i, j = 1, \ldots, n)$,

$$\max_{\substack{u_i, v_j \in \mathbb{R}^n \\ |u_i|, |v_j| \leq 1}} \left| \sum_{i,j=1}^{n} M_{ij} \langle u_i, v_j \rangle \right| \leq \frac{C}{n} \max_{f_i, g_j \in [-1,1]^{3n}} \left| \sum_{i,j=1}^{n} M_{ij} \langle f_i, g_j \rangle \right|.$$

(c) Prove Corollary 5.5.

5. Prove the basic properties of the Legendre transform mentioned above. Show that if $F : \mathbb{R}^n \to \mathbb{R}$ is smooth, strongly-convex and $\lim_{x \to \infty} F(x)/|x| = +\infty$, then $\nabla F : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

6. Prove that the map in (25) is a diffeomorphism (hint: fix $t \in \mathbb{R}^n$, consider the convex function $\varphi(x) = \psi(x) + \psi(2t - x)$, and observe that if $(x + y)/2 = t$ then $s = 2\nabla\varphi(x)$).

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