Isoperimetric inequalities in high-dimensional convex sets
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## Lecture 6: Log-concave probability measures and isotropic constants

Which probability measures in high dimensions enjoy *concentration phenomena*? With respect to which probability measures on $\mathbb{R}^n$ Lipschitz functions are concentrated near their expectation? For which measures on $\mathbb{R}^n$ most of the mass is located near "any equator", and perhaps even "non-linear equators" which are hypersurfaces partitioning space into two parts of equal mass?

Earlier in these lectures we considered the case of the uniform measure on the sphere unit $S^{n-1}$, as well as the closely related uniform measure on the Euclidean unit ball $B^n$. Furthermore, we know that when

$$X = (X_1, \ldots, X_n) \sim \text{Unif}(\sqrt{n} S^{n-1})$$

and $n$ is very large while $k = o(n)$, the random variables

$$X_1, \ldots, X_k \in \mathbb{R}^k$$

are approximately independent standard Gaussian random variables in the total variation distance (see Diaconis and Freedman [5, Section 6] for this statement and its history). Thus the standard Gaussian probability measure on $\mathbb{R}^n$ enjoys strong concentration properties, which it inherits from the high-dimensional sphere (see exercise below for a better proof of Gaussian concentration of Lipschitz functions).

There are concentration inequalities available for product measures (i.e., independent random variables), in particular for the boolean cube $\{-1, 1\}^n$, and for random variables with weak dependence properties.

Here we study a class of probability measures in $\mathbb{R}^n$ whose concentration properties were understood relatively recently, which are high-dimensional measures with convexity properties. In particular, we focus on *log-concave probability measures*.

We begin with the Prékopa-Leindler inequality, which is a functional version of the Brunn-Minkowski inequality.

**Theorem 6.1** (Prékopa-Leindler)**.** *Suppose that $f, g, h : \mathbb{R}^n \to [0, \infty)$ are measurable functions and $0 < \lambda < 1$ are such that for any $x, y \in \mathbb{R}^n$,*

$$h\left((1-\lambda)x + \lambda y\right) \geq f(x)^{1-\lambda} g(y)^{\lambda}. \qquad (1)$$

*Then,*

$$\int_{\mathbb{R}^n} h \geq \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^{\lambda}, \tag{2}$$

*whenever the integrals on the right-hand side converge.*

*Remarks.*

1. In the case where $A, B \subseteq \mathbb{R}^n$ have finite volume, by setting

   $$f = 1_A, \qquad g = 1_B, \qquad h = 1_{(1-\lambda)A+\lambda B}$$

   we recover the Brunn-Minkowski inequality in its multiplicative form. Indeed, $f, g, h$ satisfy the requirements of Theorem 6.1, and hence by its conclusion

   $$\begin{aligned} Vol_n\left((1-\lambda)A + \lambda B\right) &= \int h \\ &\geq \left( \int f \right)^{1-\lambda} \left( \int g \right)^{\lambda} = \mathrm{Vol}_n(A)^{1-\lambda}\mathrm{Vol}_n(B)^{\lambda}. \end{aligned}$$

   There are also several ways to deduce the Prekopa-Leindler inequality from the Brunn-Minkowski inequality. For example, one may consider convex bodies in higher dimensions whose marginal distributions yield the given functions, and apply Brunn-Minkowski (see, e.g., [10]).

2. The Prékopa-Leindler inequality may be viewed as a certain converse to Hölder's inequality. Indeed, the Hölder inequality implies that

   $$\int_{\mathbb{R}^n} f^{1-\lambda} g^{\lambda} \leq \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^{\lambda}$$

   while the Prékopa-Leindler inequality yields

   $$\int_{\mathbb{R}^n} \left[ \sup_{x=(1-\lambda)y+\lambda z} f(y)^{1-\lambda} g(z)^{\lambda} \right] dx \geq \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^{\lambda}.$$

*Proof of Theorem 6.1 for $n = 1$.* Consider first the case where $f$ and $g$ are bounded functions. If $f$ or $g$ vanish almost everywhere, then there is nothing to prove. Hence we may assume that $\|f\|_\infty$ and $\|g\|_\infty$ are positive numbers. In fact, by homogeneity we may assume that

$$\|f\|_\infty = \|g\|_\infty = 1, \tag{3}$$

since otherwise we may replace $f$ by $f/\|f\|_\infty$, replace $g$ by $g/\|g\|_\infty$ and replace $h$ by $h/\left(\|f\|_\infty^{1-\lambda}\|g\|_\infty^{\lambda}\right)$, without affecting the validity of neither the assumptions nor the conclusions of the theorem.

Recall that we abbreviate $\{h > t\} = \{x \in \mathbb{R}\,;\, h(x) > t\}$. Observe that condition (1) imply that for all $t > 0$,

$$\{h > t\} \supseteq (1-\lambda)\{f > t\} + \lambda\{g > t\}. \tag{4}$$

If $0 < t < 1$ then both sets on the right-hand side of (4) are non-empty. Hence, by the one-dimensional Brunn-Minkowski inequality (which is a triviality), for $0 < t < 1$,

$$m\left(\{h > t\}\right) \geq (1-\lambda) m\left(\{f > t\}\right) + \lambda m\left(\{g > t\}\right),$$

where $m$ is the Lebesgue measure on the real line. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}} h &= \int_0^\infty m(\{h > t\})dt \geq \int_0^1 m(\{h > t\})dt \\
&\geq (1-\lambda)\int_0^1 m(\{f > t\})dt \,+\, \lambda\int_0^1 m(\{g > t\})dt \\
&= (1-\lambda)\int_{\mathbb{R}} f + \lambda\int_{\mathbb{R}} g \geq \left(\int_{\mathbb{R}} f\right)^{1-\lambda}\left(\int_{\mathbb{R}} g\right)^{\lambda}.
\end{aligned}$$

This concludes the proof in the case where $f$ and $g$ are bounded. For the general case, for $M > 0$ we replace $f$ by $\min\{f, M\}$, we replace $g$ by $\min\{g, M\}$ and $h$ by $\min\{g, M\}$. Such a truncation still satisfies the requirements of the Prékopa-Leindler inequality (with the same function $h$). Hence, by the case of the inequality that was already proven,

$$\int_{\mathbb{R}} h \geq \left(\int_{\mathbb{R}} \min\{f, M\}\right)^{1-\lambda}\left(\int_{\mathbb{R}} \min\{g, M\}\right)^{\lambda} \xrightarrow{M\to\infty} \left(\int_{\mathbb{R}} f\right)^{1-\lambda}\left(\int_{\mathbb{R}} g\right)^{\lambda},$$

where we used the monotone convergence theorem in the last passage. □

*Proof of Theorem 6.1 for $n \geq 2$.* By induction on $n$. We use $x = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ as coordinates in $\mathbb{R}^n$ and set

$$\begin{aligned}
F(y) &= \int_{-\infty}^{\infty} f(y,t)\,dt = \int_{-\infty}^{\infty} f_y(t)\,dt, \\
G(y) &= \int_{-\infty}^{\infty} g(y,t)\,dt = \int_{-\infty}^{\infty} g_y(t)\,dt, \\
H(y) &= \int_{-\infty}^{\infty} h(y,t)\,dt = \int_{-\infty}^{\infty} h_y(t)\,dt.
\end{aligned}$$

We claim that if $y = (1-\lambda)y_1 + \lambda y_2$, for $y_1, y_2 \in \mathbb{R}^n$, then,

$$H(y) \geq F(y_1)^{1-\lambda} G(y_2)^{\lambda}. \tag{5}$$

Indeed, if $t = (1-\lambda)t_1 + \lambda t_2$ for $t_1, t_2 \in \mathbb{R}$, then

$$h_y(t) \geq f_{y_1}(t_1)^{1-\lambda} f_{y_2}(t_2)^\lambda.$$

Hence (5) follows by the one-dimensional Prékopa-Leindler inequality. Thanks to (5) and the induction hypothesis, we may apply the $(n-1)$-dimensional Prékopa-Leindler inequality for the functions $F, G$ and $H$ and conclude (2). $\square$

**Definition 6.2.** *A function $\rho : \mathbb{R}^n \to [0, \infty)$ is log-concave if for all $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$,*

$$\rho\left((1-\lambda)x + \lambda y\right) \geq \rho(x)^{1-\lambda} \rho(y)^\lambda,$$

*i.e., if the set $\Omega = \{\rho > 0\}$ is convex and $-\log \rho$ is a convex function on $\Omega$.*

*We say that a probability measure (or a random vector) in $\mathbb{R}^n$ is log-concave if it is supported in an affine subspace of $\mathbb{R}^n$ with a log-concave density in this subspace. Usually this affine subspace is $\mathbb{R}^n$ itself.*

For example, any Gaussian measure in $\mathbb{R}^n$ is log-concave, because its density relative to the affine subspace where it is supported is of the form

$$c_A \exp(-\langle A(x-b), (x-b)\rangle)$$

for a symmetric, positive-definite operator $A$, a number $C_A > 0$ and a vector $b \in \mathbb{R}^n$. The quadratic function $x \to \langle A(x-b), (x-b)\rangle$ is clearly convex, and hence the Gaussian measure is log-concave. The uniform probability measure on any bounded convex set, is log-concave as well. On the real line, it is very common to encounter log-concave distributions; pretty much, a typical distribution that decays exponentially or faster at infinity tends to be log-concave; exponential decay at infinity is indeed a necessary condition for log-concavity. Thus the exponential distribution on $[0, \infty)$ is log-concave, as well as beta and gamma distributions with certain parameters and the double-exponential probability density

$$\exp(-2|x|) \qquad\qquad (x \in \mathbb{R}).$$

Operations that preserve log-concavity include:

1. Linear images. If $X$ is a log-concave random vector in $\mathbb{R}^n$, then for any subspace $E \subseteq \mathbb{R}^n$ also

   $$Proj_E(X)$$

   is log-concave, by Prekopa-Leindler. It follows that for any linear (or affine) map $T : \mathbb{R}^n \to \mathbb{R}^m$, the random vector $T(X)$ is log-concave.

2. Pointwise product. If $f_1, \dots, f_N$ are log-concave functions, then so is the product $\prod_{i=1}^N f_i$. It follows that if a polynomial $P$ has only real roots and is positive on an interval $I$, then its restriction to $I$ is log-concave. Indeed,

$$P(x) = c \cdot 1_I(x) \cdot \prod_{i=1}^N (x - \lambda_i)$$

for some interval $I \subseteq \mathbb{R}$, a real number $c \in \mathbb{R}$ and $\lambda_1, \dots, \lambda_N \in \mathbb{R} \setminus I$. Since $|x - \lambda_i|$ is log-concave on $I$, the same applies for $P$.

3. Convolution. If $f, g : \mathbb{R}^n \to [0, \infty)$ are log-concave, then $f(y)g(x-y)$ is log-concave on $\mathbb{R}^n \times \mathbb{R}^n$, and consequently the same applies for its marginal $f * g$.

4. Weak limits. It is an exercise to deduce from the Prékopa-Leindler inequality that if $(\mu_N)_{N \geq 1}$ is a sequence of log-concave concave probability measures converging weakly to a probability measure $\mu$, then $\mu$ is also log-concave. This is not an obvious fact; think of the case where $\mu_N$ tend to a measure supported on a lower dimensional subspace.

**Proposition 6.3** ("How to think on 1D log-concave random variables")**.** *Let $X \in \mathbb{R}$ be an isotropic, log-concave random variable, i.e., $\mathbb{E}X = 0$ and $Var(X) = 1$. Write $\rho$ for the log-concave density of $X$. Then for all $x \in \mathbb{R}$,*

$$c' 1_{\{|x| \leq c''\}} \leq \rho(x) \leq Ce^{-c|x|},$$

*where $c, c', c'', C > 0$ are universal constants.*

*Sketch of proof.* For the upper bound, if $\rho(b) < \rho(a)/2$ for some $a < b$, then $\rho$ decays exponentially and in fact $\rho(x) \leq \rho(b) 2^{-x/(b-a)}$ for all $x > b$. As for the lower bound, it is enough to show that $\rho(x) > c'$ for some $x > c''$ and for some $x < -c''$. It is an exercise to filling in the details. □

**Corollary 6.4** ("reverse Hölder inequalities", Berwald [1, 3])**.** *For any isotropic, log-concave, real-valued random variable $X$ and for any $p > -1$,*

$$c \cdot \min\{p + 1, 1\} \leq \|X\|_p = (\mathbb{E}|X|^p)^{1/p} \leq C(p + 2), \tag{6}$$

*where $c, C > 0$ are universal constants.*

The case $p = 0$ in (6) is interpreted by continuity, i.e.,

$$\|X\|_0 = \exp(\mathbb{E} \log |X|).$$

This is not a norm, yet a nice feature is its multiplicativity: for any random variables $X$ and $Y$, possibly dependent,

$$\|XY\|_0 = \|X\|_0 \|Y\|_0.$$

*Proof of Corollary 6.4.* Begin with the inequality on the right-hand side. By the monotonicity of $p \mapsto \|X\|_p$, it is enough to look at $p > 0$. In this case,

$$\|X\|_p^p = \int_{-\infty}^{\infty} |t|^p \rho(t) dt \leq C \int_{-\infty}^{\infty} |t|^p e^{-c|t|} dt = \frac{2C}{c^{p+1}} \Gamma(p+1) \leq (\tilde{C}p)^p,$$

where we used the fact that for integer $p$, we have $\Gamma(p+1) = p! \leq p^p$. For the lower bound, by monotonicity it suffices to look at $p < 0$. Setting $q = -p \in (0,1)$ we have

$$\mathbb{E} \frac{1}{|X|^q} \leq C \int_{-\infty}^{\infty} \frac{1}{|t|^q} e^{-c|t|} dt \leq \frac{C'}{1-q}$$

and hence

$$\|X\|_p = \left( \mathbb{E} \frac{1}{|X|^q} \right)^{-1/q} \geq \left( C'(1-q) \right)^{1/q} \geq \tilde{C}(1-q).$$

□

For instance, we learn from Corollary 6.4 that when $X$ is a centered, log-concave random vector in $\mathbb{R}^n$, then

$$\mathbb{E}\langle X, \theta \rangle^4 \leq C \left( \mathbb{E}\langle X, \theta \rangle^2 \right)^2, \tag{7}$$

for a universal constant $C > 0$ (in fact, $C = 9$ is optimal here, see Eitan [6]). Indeed, if $\sigma = (\mathbb{E}\langle X, \theta \rangle^2)^{1/2}$ then $\langle X, \theta \rangle / \sigma$ is an isotropic, log-concave random variable, and (7) follows from Corollary 6.4.

**Corollary 6.5** ("Reverse Hölder inequalities for polynomials")**.** *Let $X$ be a real-valued, log-concave random variable, and let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree at most $d$. Then for any $0 < p \leq q$,*

$$\|f(X)\|_q \leq C_{q,d} \cdot \|f(X)\|_p,$$

*for some constant $C_{q,d}$ depending only on $q$ and $d$.*

*Proof.* Following Bobkov [2], we may assume that $f$ is a monic polynomial in one real variable, hence

$$f(X) = \prod_{i=1}^{d} (X - z_i)$$

for some $z_1, \ldots, z_d \in \mathbb{C}$. Consequently, by Hölder's inequality and by Corollary 6.4,

$$\begin{aligned} \|f(X)\|_q &= \left\| \prod_{i=1}^{d} (X - z_i) \right\|_q \leq \prod_{i=1}^{d} \|X - z_i\|_{dq} \\ &\leq \prod_{i=1}^{d} Cd(q+1)\|X - z_i\|_0 = (Cd(q+1))^d \|f(X)\|_0. \end{aligned}$$

□

We remark that Corollary 6.5 remains valid verbatim if one replaces "real-valued log-concave random variable" by "log-concave random vector in a finite-dimensional normed space"; see Bourgain [4].

**Corollary 6.6** (Hensley [8], Fradelizi [9])**.** *Let $K \subseteq \mathbb{R}^n$ be a centered convex body. Assume that the random vector $X$ that is distributed uniformly in $K$, is isotropic (or more generally, that* $\mathrm{Cov}\,(X)$ *is a scalar matrix). Then for any hyperplanes $H_1, H_2 \subseteq \mathbb{R}^n$ passing through the origin,*

$$Vol_{n-1}(K \cap H_1) \leq C \cdot Vol_{n-1}(K \cap H_2)$$

*where $C > 0$ is a universal constant. In fact, $C \leq \sqrt{6}$.*

*Proof.* Let $\theta \in S^{n-1}$ and denote

$$\rho_\theta(t) = \frac{Vol_{n-1}(K \cap (t\theta + \theta^\perp))}{Vol_n(K)}.$$

Then $\rho_\theta$ is the density of the random variable $X \cdot \theta$, which is log-concave and isotropic. According to Proposition 6.3, for any $x \in \mathbb{R}$,

$$c' 1_{\{|x| \leq c''\}} \leq \sigma \rho_\theta(x\sigma) \leq Ce^{-c|x|}$$

In particular,

$$c \leq \rho_\theta(0) \leq C,$$

for some universal constants $c, C > 0$. Thus, for $\theta_1, \theta_2 \in S^{n-1}$,

$$\frac{Vol_{n-1}(K \cap \theta_1^\perp)}{Vol_{n-1}(K \cap \theta_2^\perp)} = \frac{\rho_{\theta_1}(0)}{\rho_{\theta_2}(0)} \leq \frac{C}{c} \leq C'.$$

$\square$

Thus, up a multiplicative universal constant, volumes of hyperplane sections of $K$ are closely related to the covariance matrix of the uniform distribution of $K$. When $K$ is not isotropic, we have the following:

**Corollary/Exercise 6.7.** *Let $K \subseteq \mathbb{R}^n$ be a convex body of volume one and let $X \sim$ Unif$(K)$. Then,*

$$\sup_H Vol_{n-1}(K \cap H) \sim \sqrt{\|\mathrm{Cov}\,(X)^{-1}\|_{op}},$$

*where the supremum runs over all hyperplanes $H \subseteq \mathbb{R}^n$, and where $\|A\|_{op}$ is the operator norm of the matrix $A \in \mathbb{R}^{n\times n}$.*

A useful geometric parameter is concerned with the relation between the covariance matrix and entropy:

**Definition 6.8.** *For a convex body $K \subseteq \mathbb{R}^n$ we define its isotropic constant to be*

$$L_K = \left(\frac{\det \mathrm{Cov}\,(K)}{Vol_n(K)^2}\right)^{\frac{1}{2n}}$$

*where $\mathrm{Cov}\,(K)$ is the covariance matrix of the uniform probability distribution on $K$. More generally, the isotropic constant of an absolutely continuous, log-concave random vector $X$ in $\mathbb{R}^n$ is*

$$L_X = \left(\frac{\det \mathrm{Cov}\,(X)}{e^{2\mathrm{Ent}(X)}}\right)^{\frac{1}{2n}}, \tag{8}$$

*where $\mathrm{Ent}(X) = -\int_{\mathbb{R}^n} \rho \log \rho$ when $\rho$ is the density of $X$.*

The isotropic constant measures the relation between the (differential) entropy of a distribution, and the determinant of its covariance matrix. These two parameters measure the "size" of a distribution in $\mathbb{R}^n$, in a way which is *invariant* under the group of volume-preserving linear transformations of $\mathbb{R}^n$. Clearly, for any invertible affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$L_{T(X)} = L_X.$$

The entropy of a log-concave random vector in $\mathbb{R}^n$ is a rather robust quantity, and up to an additive error of $O(n)$ may be computed in many ways:

**Lemma 6.9.** *Let $X$ be a log-concave random vector in $\mathbb{R}^n$ with density $\rho = e^{-\psi}$. Then,*

$$\psi(\mathbb{E}X) \leq \mathrm{Ent}(X) \leq \inf \psi + n. \tag{9}$$

*In fact, most of the mass of the random vector $X$ is located in regions where $\psi$ is at most* $\inf \psi + 2n$. *Indeed,*

$$\mathbb{E}e^{\frac{\psi(X)}{2}} \leq e^{\frac{\inf \psi}{2} + (\ln 2)n}.$$

*and hence by Markov-Chebyshev,*

$$\mathbb{P}\left(\psi(X) \geq \inf \psi + 2n\right) \leq e^{-cn},$$

*for a universal constant $c > 0$.*

*Proof.* By approximation, we may assume that $\rho$ is smooth and positive in $\mathbb{R}^n$; this will allow us to neglect boundary terms in integration by parts below. The function $\psi$ is convex, and hence it lies above its tangents. That is, for any $x, y \in \mathbb{R}^n$,

$$\psi(y) \geq \psi(x) + \langle \nabla \psi(x), y - x \rangle. \tag{10}$$

By Jensen's inequality,

$$\psi(\mathbb{E}X) \leq \mathbb{E}\psi(X) = \mathrm{Ent}(X),$$

and the left-hand side inequality in (9) is proven. For proving the right-hand side inequality, note that by (10), for any $y \in \mathbb{R}^n$,

$$\begin{aligned} \mathrm{Ent}(X) = \mathbb{E}\psi(X) &\leq \mathbb{E}\left[\psi(y) - \nabla\psi(X) \cdot (y - X)\right] \\ &= \psi(y) + \mathbb{E}\left[\nabla\psi(X) \cdot (X - y)\right]. \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{E}\langle \nabla\psi(X), X \rangle &= \int_{\mathbb{R}^n} \left[\sum_{i=1}^n \partial^i \psi(x) x_i\right] e^{-\psi(x)} dx \\ &= -\sum_{i=1}^n \int_{\mathbb{R}^n} x_i \partial^i \left(e^{-\psi(x)}\right) = \sum_{i=1}^n \int_{\mathbb{R}^n} e^{-\psi} = n, \end{aligned} \tag{11}$$

and similarly $\mathbb{E}\nabla\psi(X) \cdot y = 0$. Inequality (9) thus follows by taking the infimum in (11) over all $y \in \mathbb{R}^n$. Additionally,

$$\begin{aligned} \mathbb{E}e^{\frac{\psi(X)}{2}} &= e^{\frac{\psi(y)}{2}} \int_{\mathbb{R}^n} e^{-\frac{\psi(x)+\psi(y)}{2}} dx \leq e^{\frac{\psi(y)}{2}} \int_{\mathbb{R}^n} e^{-\psi\left(\frac{x+y}{2}\right)} dx = 2^n e^{\frac{\psi(y)}{2}} \int e^{-\psi} \\ &= 2^n e^{\frac{\psi(y)}{2}}. \end{aligned}$$

The lemma follows by taking the infimum over all $y \in \mathbb{R}^n$. ☐

Which probability measure in $\mathbb{R}^n$ has the smallest isotropic constant?

**Claim 6.10.** *For any absolutely-continuous, log-concave random vector $X$ in $\mathbb{R}^n$,*

$$L_X \geq \frac{1}{\sqrt{2\pi e}}$$

*with equality if and only if $X$ is a Gaussian random vector.*

*Proof.* Log-concavity is irrelevant for this claim, which basically asserts the well-known fact that the Gaussian distribution has maximal entropy among all distributions with a given covariance matrix.

Translating, we may assume that $X$ is centered. Let us write $\rho$ for the density of $X$. Then for any positive probability density $\gamma : \mathbb{R}^n \to (0, \infty)$,

$$\mathbb{E} \log \frac{\gamma(X)}{\rho(X)} \leq \mathbb{E} \left[ \frac{\gamma(X)}{\rho(X)} - 1 \right] = \int_{\mathbb{R}^n} \frac{\gamma(x)}{\rho(x)} \rho(x) dx - 1 = 0.$$

In particular, if $\gamma$ is the density of a centered Gaussian random vector $G$ in $\mathbb{R}^n$ with $\mathrm{Cov}\,(X) = \mathrm{Cov}\,(G)$, then $\log \gamma$ is a quadratic function plus a constant and

$$\mathrm{Ent}(X) = -\mathbb{E} \log \rho(X) \leq -\mathbb{E} \log \gamma(X) = -\mathbb{E} \log \gamma(Z) = \mathrm{Ent}(G).$$

□

By the end of these lectures we will (hopefully) complete the proof that

$$L_X < C$$

for a universal constant $C > 0$, due to Klartag and Lehec [11] following Guan [7]. The optimal value of $C$ remains unknown. The symmetric case of the *strong slicing conjecture* suggests that when $K \subseteq \mathbb{R}^n$ is a centrally-symmetric convex body,

$$L_K \leq L_{[0,1]^n} = \frac{1}{\sqrt{12}}.$$

If true, then the Minkowski lattice conjecture follows, see Magazinov [14] and references therein. The Minkowski lattice conjecture suggests that if $L \subseteq \mathbb{R}^n$ is a lattice of determinant one, then each of its translates intersects the set

$$\left\{ x \in \mathbb{R}^n \, ; \, \prod_{i=1}^n |x_i| \leq \frac{1}{2^n} \right\}.$$

This was proven in two dimensions by Minkowski in 1908. Moreover, the isotropic constant of any simplex $\Delta^n \subseteq \mathbb{R}^n$ equals

$$L_{\Delta^n} = \frac{(n!)^{1/n}}{(n+1)^{(n+1)/(2n)}\sqrt{n+2}} \approx \frac{1}{e}.$$

If the isotropic constant is maximized for the simplex among all convex bodies – this is the non-symmetric strong slicing conjecture – then the Mahler conjecture follows in the non-symmetric case. See [13]. This Mahler conjecture, which we discussed earlier, suggests that among all convex bodies $K \subseteq \mathbb{R}^n$, the volume product

$$Vol_n(K) \cdot Vol_n(K^\circ)$$

is minimized when $K$ is a centered simplex. This was proven in two dimensions by Mahler in 1938.

A useful tool for analyzing a log-concave random vector $X$ in $\mathbb{R}^n$ is its *logarithmic Laplace transform*, defined via

$$\Lambda(y) = \Lambda_X(y) = \log \mathbb{E} e^{\langle X, y\rangle} \qquad\qquad (y \in \mathbb{R}^n).$$

**Proposition 6.11** ("The logarithmic Laplace transform of log-concave measures")**.** *Let $X$ be a log-concave random vector in $\mathbb{R}^n$. Then the following hold:*

*(a) The set $\Omega = \{y \in \mathbb{R}^n \, ; \, \Lambda(y) < \infty\}$ is an open, convex set. The function*

$$\Lambda : \Omega \to \mathbb{R}$$

*is smooth and strongly-convex.*

*(b) If $X$ is absolutely-continuous, then the map $\nabla\Lambda$ is a diffeomorphism from $\Omega$ onto the convex hull of the interior of the support of $X$.*

*(c) Suppose that $X$ has density $\rho$, and for $y \in \Omega$ consider the "tilted probability measure"*

$$\rho_y(x) = \rho(x) e^{x \cdot y - \Lambda(y)}.$$

*Let $X_y$ be a random vector with density $\rho_y$ and denote $a_y = \mathbb{E}X_y$. Then, for any $y \in \Omega$,*

$$\nabla\Lambda(y) = \mathbb{E}X_y, \qquad \nabla^2\Lambda(y) = \mathrm{Cov}\,(X_y), \qquad \nabla^3\Lambda(y) = \mathbb{E}(X_y - a_y)^{\otimes 3}.$$

*Sketch of Proof.* Assume that $X$ is absolutely-continuous. If $y \in \Omega$ then $\rho_y$ is an integrable, log-concave density. As such, it decays exponentially at infinity (see exercise below), and hence $\Omega$ contains a neighborhood of $y$. This shows that $\Omega$ is open, and that $\Lambda$ is smooth. Conclusion (c) is a direct computation, which also implies the strong convexity of $\Lambda$ and the fact that

$$\Omega \ni x \mapsto \nabla \psi(x) \in \mathbb{R}^n$$

is one-to-one and a diffeomorphism onto its image, which is an open set. Write $K$ for the interior of the convex hull of the support of $X$. Then (c) implies that

$$\nabla \psi(\Omega) \subseteq K,$$

since the barycenter of any absolutely-continuous distribution in $K$ is contained in $K$. We leave the proof of the fact that $\nabla \psi(\Omega) = K$ to the reader. This fact will not be used below. $\square$

**Exercises**.

1. The Maurey-Pisier proof of Gaussian concentration.

   (a) Let $X$ and $Y$ be two independent, standard Gaussian random vectors in $\mathbb{R}^n$. For $\theta \in [0, \pi/2]$ set

   $$X_\theta = (\sin\theta)X + (\cos\theta)Y.$$

   Prove that $(X_\theta, \partial X_\theta / \partial\theta)$ coincides in distribution with $(X, Y)$.

   (b) Let $F : \mathbb{R}^n \to \mathbb{R}$ be a locally-Lipschitz function and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that

   $$\mathbb{E}\varphi(F(X) - F(Y)) = \mathbb{E}\varphi\left( \int_0^{\pi/2} \left\langle \nabla F(X_\theta), \frac{\partial X_\theta}{\partial \theta} \right\rangle d\theta \right).$$

   (c) Denote $E = \mathbb{E}F(X)$. Conclude that for any $\lambda > 0$,

   $$\mathbb{E}e^{\lambda(F(X)-E)} \leq \mathbb{E}e^{\lambda\pi\langle \nabla F(X), Y\rangle/2} = \mathbb{E}e^{\lambda^2\pi^2|\nabla F(X)|^2/8}.$$

   (d) Conclude that if $F$ is 1-Lipschitz, then for all $t > 0$,

   $$\mathbb{P}\left(|F(X) - E| \geq t\right) \leq 2e^{-2t^2/\pi^2}.$$

2. Let $\mu, \mu_1, \mu_2, \dots$ be log-concave probability measures on $\mathbb{R}^n$. Assume that $\mu_N \longrightarrow \mu$ weakly, i.e., that for any continuous, compactly-supported function $\varphi : \mathbb{R}^n \to \mathbb{R}$,
$$\lim_{N\to\infty} \int_{\mathbb{R}^n} \varphi d\mu_N = \int_{\mathbb{R}^n} \varphi d\mu.$$
Suppose that $\mu_N$ is log-concave for all $N$. Prove that $\mu$ is log-concave.

3. Complete the proof of Proposition 6.3

4. Let $\rho : \mathbb{R}^n \to [0, \infty)$ be a log-concave probability density. Prove that there exist $A, B > 0$ such that for all $x \in \mathbb{R}^n$,
$$\rho(x) \leq Ae^{-B|x|}.$$

5. Complete the proof of Proposition 6.11.

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*Request.* Please e-mail me at `boaz.klartag@weizmann.ac.il` with any comments, corrections, or suggestions regarding these lecture notes. In addition, if you are able to produce *tikz* figures similar to (or better than) those drawn on the blackboard in class, I would be glad to include them in the notes.

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