Isoperimetric inequalities in high-dimensional convex sets
Boaz Klartag, ETH Zurich 2025

Lecture 7: The logarithmic Laplace transform and M-ellipsoids

A useful tool for analyzing a log-concave random vector X in $\mathbb{R}^n$ is its *logarithmic Laplace transform*, defined via

$$\Lambda(y) = \Lambda_X(y) = \log \mathbb{E} e^{\langle X, y \rangle} \qquad\qquad (y \in \mathbb{R}^n), \tag{1}$$

where we set $\Lambda(y) = \infty$ when $\mathbb{E} \exp(x \cdot Y) = \infty$.

Proposition 7.1 ("log-Laplace transform of log-concave measures")**.** *Let X be an absolutely-continuous log-concave random vector in $\mathbb{R}^n$. Then the following hold:*

(a) The set $\Omega = \{y \in \mathbb{R}^n \,;\, \Lambda(y) < \infty\}$ is an open, convex set, and the function

$$\Lambda : \Omega \to \mathbb{R}$$

is smooth and strongly-convex.

(b) The map $\nabla \Lambda : \Omega \to \mathbb{R}^n$ is a diffeomorphism from Ω onto the interior of the convex hull of the support of X.

(c) Write ρ for the density of X, and for $y \in \Omega$ consider the "tilted probability measure"

$$\rho_y(x) = e^{x \cdot y - \Lambda(y)} \rho(x). \tag{2}$$

Let X_y be a random vector with density ρ_y, and denote $a_y = \mathbb{E} X_y$. Then, for any $y \in \Omega$,

$$\nabla \Lambda(y) = \mathbb{E} X_y, \qquad \nabla^2 \Lambda(y) = \mathrm{Cov}(X_y), \qquad \nabla^3 \Lambda(y) = \mathbb{E}(X_y - a_y)^{\otimes 3},$$

i.e., the log-Laplace transform is the "cumulant generating function".

Proof. If $y_1, y_2 \in \Omega$ then for any $0 < \lambda < 1$, by Hölder's inequality and (1),

$$e^{\Lambda(\lambda y_1 + (1-\lambda) y_2)} \leq e^{\lambda \Lambda(y_1)} \cdot e^{(1-\lambda)\Lambda(y_2)} < \infty.$$

Hence Ω is a convex set. When $y \in \Omega$ the probability density ρ_y is log-concave. As such, it decays exponentially at infinity; by the exercise from last week, there exists $A, B > 0$ such that

$$\rho_y(x) \leq A e^{-B|x|} \qquad\qquad (x \in \mathbb{R}^n).$$

Therefore, for any $z \in \mathbb{R}^n$ with $|z-y| < B/2$,

$$\int_{\mathbb{R}^n} e^{x\cdot z}\rho(x)dx = e^{\Lambda(y)}\int_{\mathbb{R}^n} e^{x\cdot(z-y)}\rho_y(x)dx \quad \leq Ae^{\Lambda(y)}\int_{\mathbb{R}^n} e^{x\cdot(z-y)-B|x|}dx < \infty.$$

Thus the ball around $y \in \mathbb{R}^n$ of radius $B/2$ is contained in Ω, and consequently Ω is an open set. A similar argument shows that $\Lambda : \Omega \to \mathbb{R}$ is a smooth function, and that we may differentiate

$$e^{\Lambda(y)} = \int_{\mathbb{R}^n} e^{x\cdot y}\rho(x)dx$$

under the integral sign any finite number of times. For instance, for any $y \in \Omega$ we obtain,

$$\nabla_y\Lambda(y) = \frac{\int_{\mathbb{R}^n}\nabla_y e^{x\cdot y}\rho(x)dx}{\int_{\mathbb{R}^n} e^{x\cdot y}\rho(x)dx} = \frac{\int_{\mathbb{R}^n} xe^{x\cdot y}\rho(x)dx}{\int_{\mathbb{R}^n} e^{x\cdot y}\rho(x)dx} = \mathbb{E}X_y.$$

Conclusion (c) follows by similar direct computations. The Hessian matrix $\nabla^2\Lambda(y) \in \mathbb{R}^{n\times n}$ is positive-definite, since it is a covariance matrix of an absolutely-continuous random vector. Therefore Λ is strongly-convex. From the inverse function theorem we thus conclude that

$$\nabla\Lambda : \Omega \to \mathbb{R}^n \tag{3}$$

is an open map, i.e., it maps open sets to open sets. The map in (3) is also one-to-one, since for $x, y \in \Omega$,

$$\langle\nabla\Lambda(x) - \nabla\Lambda(y), x-y\rangle = \int_0^1 \nabla^2\Lambda((1-t)y+tx)(x-y)\cdot(x-y)dt > 0,$$

and consequently $\nabla\Lambda(x) \neq \nabla\Lambda(y)$. Thus the map $\nabla\Lambda$ is a diffeomorphism from Ω onto its image, which is an open set. Write K for the interior of the convex hull of the support of X. For any $y \in \Omega$, the random vector X_y is a supported in K, and hence

$$\nabla\Lambda(y) = \mathbb{E}X_y \in K.$$

Therefore

$$\nabla\Lambda(\Omega) \subseteq K. \tag{4}$$

While we will not use this fact here, it is indeed true that (4) is in fact an equality, see the exercise below. $\square$

We refer to probability densities of the form (2) as "exponential tilts" of the density ρ. Observe again that they are log-concave.

The isotropic constant of an absolutely-continuous, random vector X in $\mathbb{R}^n$ with a log-concave density ρ was defined last week via

$$L_X = L_\rho = \left(\frac{\det \mathrm{Cov}(X)}{e^{2\mathrm{Ent}(X)}} \right)^{\frac{1}{2n}} \sim \left(\det \mathrm{Cov}(X) \cdot \sup \rho^2 \right)^{\frac{1}{2n}},$$

where and $A \sim B$ means that $cA \leq B \leq CA$ for universal constants $C, c > 0$. We proved that $L_X \geq 1/\sqrt{2\pi e}$, and by the end of the semester we will (hopefully) prove that $L_X < C$. In the meantime we prove the easier:

Proposition 7.2 ("exponential tilts with bounded isotropic constant")**.** *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body with $\mathrm{Vol}_n(K) = 1$. Then there exists a log-concave probability density $\rho : K \to [0, \infty)$ such that the following hold:*

(i) $L_\rho < C$.

(ii) For any $x \in K$,

$$\tilde{c} \leq \rho^{1/n}(x) \leq \tilde{C}.$$

Here, $C, \tilde{C}, c > 0$ are universal constants.

The log-concave probability density from Proposition 7.2 is actually an exponential tilt of 1_K. Proposition 7.2 admits a generalization, from the case of centrally-symmetric convex bodies to arbitrary log-concave measures; this generalization will not be discussed here.

Proof of Proposition 7.2. Consider the log-Laplace transform

$$\Lambda(y) = \log \int_K e^{x \cdot y} dx.$$

Observe that $\Lambda(y) < \infty$ for any $y \in \mathbb{R}^n$. By Jensen's inequality,

$$\Lambda(y) \geq \int_K \log e^{x \cdot y} dx = \int_K (x \cdot y) dx = 0. \tag{5}$$

Moreover, for any $y \in nK^\circ$,

$$\Lambda(y) \leq \log \sup_{x \in K} e^{x \cdot y} = \sup_{x \in K} x \cdot y \leq n. \tag{6}$$

The map $\nabla \Lambda : \mathbb{R}^n \to K$ is smooth, open and one-to-one. By changing variables $x = \nabla \Lambda(y)$ we see that

$$\begin{aligned} 1 = \mathrm{Vol}_n(K) &\geq \int_{\nabla \Lambda(\mathbb{R}^n)} dx = \int_{\mathbb{R}^n} \det \nabla^2 \Lambda(y) dy \\ &= \int_{\mathbb{R}^n} \det \mathrm{Cov}(X_y) dy \geq \int_{nK^\circ} \det \mathrm{Cov}(X_y) dy. \end{aligned}$$

In particular, there exists $y \in nK^\circ$ with

$$\det \mathrm{Cov}(X_y) \leq \frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(nK^\circ)} = \frac{1}{\mathrm{Vol}_n(K)\mathrm{Vol}_n(nK^\circ)} \leq C^n, \tag{7}$$

where we used the Bourgain-Milman inequality in the last passage. Write

$$\rho(x) = \rho_y(x) = e^{x \cdot y - \Lambda(y)} 1_K(x)$$

for the corresponding exponential tilt, which is a log-concave probability density. Then by (5), for any $x \in K$,

$$\rho(x) \leq \sup_{z \in K} e^{z \cdot y} \leq e^n \tag{8}$$

while similarly, by (6),

$$\rho(x) \geq \inf_{z \in K} e^{z \cdot y - n} \geq 2e^{-n}.$$

By (7) and (8),

$$L_\rho = L_{X_y} \sim \left(\det \mathrm{Cov}(X_y) \cdot \sup_{x \in \mathbb{R}^n} \rho^2(x) \right)^{1/(2n)} \leq \tilde{C}.$$

$\square$

Proposition 7.2 will be used in order to construct a remarkable ellipsoid which approximates a given a convex body $K \subseteq \mathbb{R}^n$ on a rough scale. For instance, this ellipsoid represents K in volumetric computations such as $\mathrm{Vol}_n(K+T)$ or $\mathrm{Vol}_n(K \cap T)$ with a multiplicative error of at most C^n. Recall that an ellipsoid in $\mathbb{R}^n$ is an affine image of a Euclidean ball, and thus an ellipsoid appears much simpler than a general convex body in high dimensions.

Definition 7.3 ("M-ellipsoid")**.** *Let $K \subseteq \mathbb{R}^n$ be a convex body and let $\alpha > 0$. An ellipsoid $\mathcal{E} \subseteq \mathbb{R}^n$ is called an M-ellipsoid of K with constant α if the following hold:*

(i) $\mathrm{Vol}_n(\mathcal{E}) \leq \alpha^{-n} \mathrm{Vol}_n(K)$.

(ii) $\mathrm{Vol}_n(K \cap \mathcal{E}) \geq \alpha^n \mathrm{Vol}_n(K)$.

It follows from Definition 7.3 that if $K_2 = T(K_1)$ for an invertible, affine map $T : \mathbb{R}^n \to \mathbb{R}^n$, and $\mathcal{E}_1$ is an M-ellipsoid of K_1 with constant α, then $\mathcal{E}_2$ is an M-ellipsoid of K_2 with the same constant α.

Theorem 7.4 (Milman)**.** *For any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$, there exists an centrally-symmetric M-ellipsoid of K with constant c, where $c > 0$ is a universal constant.*

Let us abbreviate $|K| = \mathrm{Vol}_n(K)$. For the proof of Theorem 7.4 we need the following:

Lemma 7.5. *Let $K, T \subseteq \mathbb{R}^n$ be centrally-symmetric convex bodies. Then,*

$$\sup_{x \in \mathbb{R}^n} |K \cap (T + x)| = |K \cap T|.$$

Proof. By the convexity of K and T, for any $x_1, x_2 \in \mathbb{R}^n$ and $0 < \lambda < 1$,

$$(1-\lambda)\,[K \cap (T + x_1)] \,+\, \lambda\,[K \cap (T + x_2)] \subseteq K \cap (T + (1-\lambda)x_1 + \lambda x_2)\,.$$

The main observation is that thanks to this inclusion, the Brunn-Minkowski inequality implies that the function

$$f(x) = |K \cap (T + x)|^{1/n}$$

is concave on its support. Note that this function is even, by the central symmetry of K and T. Hence its maximum is attained at the origin. $\square$

Proof of Theorem 7.4. We may apply a linear transformation and assume that

$$|K| = 1.$$

From Proposition 7.2, there exists a log-concave probability density $f : K \to [0, \infty)$ with

$$\frac{1}{\sqrt{2\pi e}} \leq L_f < C \tag{9}$$

such that for any $x \in K$,

$$c^n \leq f(x) \leq C^n. \tag{10}$$

Write X for the random vector with density ρ, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible, affine map such that the random vector

$$Y = T(X)$$

is isotropic. (For instance, we may pick the affine map $T(z) = \mathrm{Cov}(X)^{-1/2}(z - \mathbb{E}X)$ for $z \in \mathbb{R}^n$). We will show that the ball

$$B = 2\sqrt{n}B^n$$

is an M-ellipsoid for the convex body $T(K)$ with constant c. First, since Y is isotropic we have $\mathbb{E}|Y|^2 = n$. Hence, by Markov's inequality,

$$\mathbb{P}(Y \in B) = 1 - \mathbb{P}(|Y|^2 > 4n) \geq 1 - \frac{\mathbb{E}|Y|^2}{4n} = \frac{3}{4}. \tag{11}$$

Write g for the density of Y. By the change-of-variables formula, the relation of g to the density f of the random vector X is given via

$$|\det T| \cdot g(Tx) = f(x) \qquad\qquad (x \in K), \tag{12}$$

where $\det T$ stands for the determinant of the linear part of the affine map T. Let us show that the density g enjoys a bound similar to (10). Since the isotropic constant is affinely invariant,

$$L_f = L_g \sim \det \mathrm{Cov}(Y)^{1/(2n)} \sup_{T(K)} g^{1/n} = |\det T|^{1/n} \sup_K f^{1/n} \sim |\det T|^{1/n}, \tag{13}$$

where we used (10) in the last passage. From (9) and (13) we see that

$$|\det T|^{1/n} \sim 1. \tag{14}$$

Thus, from (10) and (12), for any $x \in T(K)$,

$$\bar{c}^n \leq g(x) \leq \bar{C}^n. \tag{15}$$

Since g is supported on K, by (15),

$$|T(K) \cap B| \geq c^n \int_B g(x) dx = c^n \mathbb{P}(Y \in B) \geq \tilde{c}^n, \tag{16}$$

where we used (11) in the last passage. Since $|K| = 1$, the volume of $T(K)$ is at most C^n by (14). Thus the volume of $T(K)$ is at most $\tilde{C}^n$ times the volume of

$$B = 2\sqrt{n} B^n.$$

We thus conclude from (16) that B is an M-ellipsoid of $T(K)$ with constant C. By linear equivariance, the ellipsoid

$$\mathcal{E} = T^{-1}(B)$$

is an M-ellipsoid of K with the same constant C. From Lemma 7.5 we deduce that the centrally-symmetric translate of the ellipsoid $\mathcal{E}$ is an M-ellipsoid of K with the same constant C. $\square$

Let us explore the properties of the M-ellipsoid, using relatively "soft" arguments. We would need the following:

Lemma 7.6 (Rogers-Shephard)**.** *Let $K, T \subseteq \mathbb{R}^n$ be centrally-symmetric, convex bodies. Then,*

$$|K| \cdot |T| \leq |K \cap T| \cdot |K + T| \leq 4^n |K| \cdot |T|. \tag{17}$$

Proof. The function $f(x) = |K \cap (T + x)|^{1/n}$ is even and concave on its support, as we remember from the proof of Lemma 7.5. Observe that the support of the function f is the Minkowski sum $K + T$. Thus,

$$\int_{K+T} f^n = \int_{\mathbb{R}^n} |K \cap (T + x)| dx = \int_{\mathbb{R}^n} (1_K * 1_T)(x) dx = |K| \cdot |T|. \tag{18}$$

Since the maximum of the concave, even function f is attained at the origin,

$$\int_{K+T} f^n \leq |K + T| f(0)^n = |K + T| \cdot |K \cap T|,$$

proving the left-hand side inequality in (17). For the right-hand side inequality, note that for any $x \in K + T$, by the concavity and non-negativity of f,

$$f(x/2) \geq \frac{f(0) + f(x)}{2} \geq \frac{f(0)}{2} = \frac{|K \cap T|^{1/n}}{2}.$$

Hence, by (18),

$$|K| \cdot |T| \geq \int_{(K+T)/2} f^n \geq \left| \frac{K + T}{2} \right| \cdot \frac{|K \cap T|}{2^n} = 4^{-n} |K \cap T| \cdot |K + T|.$$

□

In this lecture we adapt the following jargon: when we say that the ellipsoid $\mathcal{E}$ is an M-ellipsoid of the convex body $K \subseteq \mathbb{R}^n$, we mean "with a universal constant c".

Corollary 7.7 ("Summing a convex body with its M-ellipsoid")**.** *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body, and let $\mathcal{E} \subseteq \mathbb{R}^n$ be its centrally-symmetric M-ellipsoid. Then,*

$$|K + \mathcal{E}|^{1/n} \leq C |K|^{1/n},$$

where $C > 0$ is a universal constant.

Proof. By Definition 7.3 we know that

$$|K \cap \mathcal{E}|^{1/n} \sim |K|^{1/n} \sim |\mathcal{E}|^{1/n}.$$

The conclusion thus follows from Lemma 7.6. □

For two convex bodies $K, T \subseteq \mathbb{R}^n$, the covering number of K by T is defined as

$$N(K,T) = \min\left\{ N > 0\,;\, \exists x_1, \ldots, X_N \in \mathbb{R}^n,\ K \subseteq \bigcup_{i=1}^{N} (x_i + T) \right\},$$

i.e., it is the minimal number of translates of T that are required in order to cover K. Recall that in Lecture 4 we proved that for any centrally-symmetric convex bodies $K, T \subseteq \mathbb{R}^n$,

$$N(K,T) \leq \frac{|K + T/2|}{|T/2|}.$$

Corollary 7.8 ("Covering a convex body by its M-ellipsoid, and vice versa")**.** *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body, and let $\mathcal{E} \subseteq \mathbb{R}^n$ be its centrally-symmetric M-ellipsoid. Then,*

$$N(K, \mathcal{E}) \leq C^n,$$

and

$$N(\mathcal{E}, K) \leq C^n$$

where $C > 0$ is a universal constant.

Proof. By Corollary 7.7,

$$N(K,\mathcal{E}) \leq \frac{|K + \mathcal{E}/2|}{|\mathcal{E}/2|} \leq 2^n \frac{|K + \mathcal{E}|}{|\mathcal{E}|} \leq \tilde{C}^n,$$

and

$$N(\mathcal{E},K) \leq \frac{|\mathcal{E} + K/2|}{|K/2|} \leq 2^n \frac{|K + \mathcal{E}|}{|K|} \leq \tilde{C}^n.$$

□

Proposition 7.9 ("Representing a convex body by its M-ellipsoid in volumetric computations")**.** *Let $K, T \subseteq \mathbb{R}^n$ be centrally-symmetric, convex bodies. Let $\mathcal{E}_K, \mathcal{E}_T$ be their centrally-symmetric M-ellipsoids, respectively. Then,*

$$|K + T|^{1/n} \sim |\mathcal{E}_K + \mathcal{E}_T|^{1/n} \tag{19}$$

and

$$|K \cap T|^{1/n} \sim |\mathcal{E}_K \cap \mathcal{E}_T|^{1/n}. \tag{20}$$

Proof. Use Corollary 7.8 in order to find $x_1, \ldots, x_{N_1} \in \mathbb{R}^n$, for $N_1 \leq C^n$, with

$$K \subseteq \bigcup_{i=1}^{N_1} (x_i + \mathcal{E}_K).$$

Similarly, there exist $y_1, \ldots, y_{N_2} \in \mathbb{R}^n$, for $N_2 \leq C^n$, such that

$$T \subseteq \bigcup_{j=1}^{N_2} (y_j + \mathcal{E}_T).$$

Therefore,

$$K + T \subseteq \bigcup_{i=1}^{N_1} \bigcup_{j=1}^{N_2} (x_i + y_j + \mathcal{E}_K + \mathcal{E}_T)$$

and hence

$$|K + T| \leq N_1 N_2 |\mathcal{E}_K + \mathcal{E}_T| \leq C^{2n} |\mathcal{E}_K + \mathcal{E}_T|,$$

proving that

$$|K + T|^{1/n} \leq C |\mathcal{E}_K + \mathcal{E}_T|^{1/n}.$$

This proves one half of the equivalence in (19), and the other half is proven similarly, by covering $\mathcal{E}_K$ by translates of K and $\mathcal{E}_T$ by translates of T. Next, by (19) and the Rogers-Shephard lemma,

$$|K \cap T|^{1/n} \sim \frac{|K|^{1/n} |T|^{1/n}}{|K + T|^{1/n}} \sim \frac{|\mathcal{E}_K|^{1/n} |\mathcal{E}_T|^{1/n}}{|\mathcal{E}_K + \mathcal{E}_T|^{1/n}} \sim |\mathcal{E}_K \cap \mathcal{E}_T|^{1/n},$$

proving (20). $\square$

Recall that the volume-radius of $K \subseteq \mathbb{R}^n$ is the radius of the Euclidean ball with the same volume as K, namely,

$$v.rad.(K) = \left(\frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(B^n)} \right)^{1/n}.$$

Proposition 7.10 ("Duality and M-ellipsoid"). *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body, and let $\mathcal{E} \subseteq \mathbb{R}^n$ be its centrally-symmetric M-ellipsoid. Then $\mathcal{E}^\circ$ is an M-ellipsoid of K° with a universal constant $c' > 0$.*

Proof. Write conv(K, T) for the convex hull of K and T. It is an exercise to show that

$$(K \cap T)^\circ = \text{conv}(K^\circ, T^\circ)$$

and

$$\frac{K+T}{2} \subseteq \text{conv}(K, T) \subseteq K + T.$$

Therefore, by the Santaló and Bourgain-Milman inequalities combined with Corollary 7.7,

$$\begin{aligned} v.rad.(\mathcal{E}^\circ \cap K^\circ) &\sim \frac{1}{v.rad.(conv(K, \mathcal{E}))} \sim \frac{1}{v.rad.(K+\mathcal{E})} \\ &\sim \frac{1}{v.rad.(K)} \sim v.rad.(K^\circ), \end{aligned}$$

while

$$v.rad(\mathcal{E}^\circ) = \frac{1}{v.rad.(\mathcal{E})} \sim \frac{1}{v.rad.(K)} \sim v.rad.(K^\circ).$$

Thus $\mathcal{E}^\circ$ is an M-ellipsoid of K°, with a universal constant $c > 0$. □

Definition 7.11 ("M-position"). *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body. We say that K is in M-position with constant $\alpha > 0$ if B^n is an M-ellipsoid of K with constant $\alpha > 0$.*

As usual, when we say that K is in M-position we mean "with a universal constant $c > 0$". It follows from Theorem 7.4 that any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ has a linear image which is in M-position.

Corollary 7.12 ("reverse Brunn-Minkowski"). *Let $K, T \subseteq \mathbb{R}^n$ be centrally-symmetric convex bodies in M-position. Then,*

$$|K + T|^{1/n} \sim |K|^{1/n} + |T|^{1/n}.$$

Proof. Since B^n is an M-ellipsoid of K and of T, the conclusion follows from Proposition 7.9. □

By Proposition 7.10, if K is in M-position, then so is K°. In the following proposition we assume for simplicity that n is divisible by 10.

Theorem 7.13 ("Randomized Quotient of Subspace")**.** *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body in M-position. Let $k_1 = 8n/10$ and $k_2 = 9n/10$. Let $E \subseteq \mathbb{R}^n$ be a uniformly-distributed, random k_2-dimensional subspace. Let $F \subseteq E$ be a random subspace, distributed uniformly in the Grassmannian of all k_1-dimensional subspaces of F. Then, with probability of at least $1 - Ce^{-cn}$,*

$$cB_F \subseteq F \cap Proj_E(K) \subseteq CB_F \tag{21}$$

where $B_F = B^n \cap F$ and

$$cB_F \subseteq Proj_F(K \cap E) \subseteq CB_F. \tag{22}$$

Theorem 7.13 will be proven shortly. While Dvoretzky's theorem implies the existence of nearly spherical sections of dimension $c \log n$, if we combine the operations of section and projection we can go all the way up to dimension cn.

Corollary 7.14 ("Quotient of Subspace")**.** *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body. Then there exists subspaces $F \subseteq E \subseteq \mathbb{R}^n$ with $\dim(F) \geq n/3$ such that*

$$cB_F \subseteq F \cap Proj_E(K) \subseteq CB_F,$$

where $C, c > 0$ are universal constants.

Proof of Corollary 7.14 assuming Theorem 7.13. Recall from Lecture 4 that any ellipsoid has a section of half the dimension which is an exact Euclidean ball. Hence it suffices to find subspaces $F \subseteq E \subseteq \mathbb{R}^n$ with $\dim(F) \geq 2n/3$, and a centrally-symmetric ellipsoid $\mathcal{E} \subseteq F$, such that

$$c\mathcal{E} \subseteq F \cap Proj_E(K) \subseteq C\mathcal{E}.$$

By applying a linear transformation, we may assume that K is in M-position. The conclusion thus follows from Theorem 7.13. $\square$

The proof of Theorem 7.13 requires the following two lemmas:

Lemma 7.15. *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body in M-position. Let $0 < \lambda < 1$ and set $k = \lceil \lambda n \rceil$. Let $E \subseteq \mathbb{R}^n$ be any k-dimensional subspace. Then,*

$$c_\lambda \leq v.rad.(K \cap E) \leq v.rad.(Proj_E K) \leq C_\lambda. \tag{23}$$

Moreover, both $K \cap E$ and $Proj_E K$ are in M-position with constant $\tilde{C}_\lambda$. Here, $c_\lambda, C_\lambda, \tilde{C}_\lambda > 0$ are constants depending solely on λ.

Proof. Write $B_E = B^n \cap E$. Recall that $N(K, B^n) \leq e^{Cn}$. Thus for appropriate points $x_1, \ldots, x_N \in \mathbb{R}^n$, with $N \leq e^{Cn}$,

$$K \subseteq \bigcup_{i=1}^{N} (x_i + B^n).$$

Consequently,

$$Proj_E K \subseteq \bigcup_{i=1}^{N} (Proj_E x_i + B_E) \tag{24}$$

and hence

$$v.rad.(Proj_E K) \leq N^{\frac{1}{k}} \leq e^{Cn/(\lambda n)} = C_\lambda.$$

This proves the inequality on the right-hand side of (23). Since K° is in M-position, we conclude from the right-hand side of (23) and the Bourgain-Milman inequality that

$$C_\lambda \geq v.rad.(Proj_E K^\circ) \geq \frac{c}{v.rad.((Proj_E K^\circ)^\circ)} = \frac{c}{v.rad.(K \cap E)},$$

proving the left-hand side of (23). It follows from (24) that

$$Proj_E K + B_E \subseteq \bigcup_{i=1}^{N} (Proj_E x_i + 2B_E).$$

Therefore,

$$|Proj_E K + B_E|^{\frac{1}{k}} \leq N^{\frac{1}{k}} |2B_E| \leq C_\lambda |B_E|. \tag{25}$$

From (23) and (25) we see that $Proj_E K$ is in M-position with constant depending only on λ (see exercise 2 below). Since K° is in M-position, we thus conclude that also $Proj_E K^\circ$ is in M-position with constant depending only on λ. This implies that also

$$(Proj_E K^\circ)^\circ = K \cap E$$

is in M-position with constant depending only on λ. □

Lemma 7.16. *Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body in M-position. Let $0 < \lambda < 1$ and set $k = \lceil \lambda n \rceil$. Let $E \subseteq \mathbb{R}^n$ be a random k-dimensional subspace, distributed uniformly in $G_{n,k}$. Then, with probability of at least $1 - Ce^{-cn}$,*

$$K \cap E \subseteq C_\lambda B_E \tag{26}$$

and

$$Proj_E K \supseteq c_\lambda B_E, \tag{27}$$

where $C_\lambda, c_\lambda > 0$ depend solely on λ.

Proof. Recall the diameter/volume balance theorem from Lecture 5, which asserts that with probability of at least $1 - Ce^{-cn}$,

$$\mathrm{diam}(K \cap E)^{1-\lambda} v.rad.(K \cap E)^{\lambda} \leq Cv.rad.(K), \tag{28}$$

where $\mathrm{diam}(A) = \sup_{x,y \in A} |x-y|$ is the diameter of A. Recall that with probability one, by Lemma 7.15,

$$v.rad.(K \cap E) \geq c_\lambda.$$

Since $v.rad.(K) \sim 1$ as K is in M-position, we deduce (26) from (28). Since K° is in M-position, we conclude from (26) that with high probability,

$$K^\circ \cap E \subseteq C_\lambda B_E$$

and by dualizing,

$$Proj_E K = (K^\circ \cap E)^\circ \supseteq \frac{1}{C_\lambda} B_E.$$

This proves (27). □

Proof of Theorem 7.13. By Lemma 7.16, with high probablity,

$$Proj_E(K) \supseteq cB_E, \tag{29}$$

while $Proj_E(K)$ is in M-position according to Lemma 7.15. We may thus apply Lemma 7.16 again and conclude that with high probability,

$$Proj_E(K) \cap E \subseteq CB_F. \tag{30}$$

From (29) and (30) we deduce (21). The proof of (22) is analogous. □

Exercises.

1. Prove that equality holds in (4) in the case where K is strictly-convex.

2. Let $\alpha > 0$. Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body and let $\mathcal{E} \subseteq \mathbb{R}^n$ be a centrally-symmetric ellipsoid with

$$|K + \mathcal{E}|^{1/n} \leq \alpha |K|^{1/n} \qquad \text{and} \qquad |K|^{1/n} \leq \alpha |\mathcal{E}|^{1/n}.$$

Prove that $\mathcal{E}$ is an M-ellipsoid of K with constant C_α, where $C_\alpha > 0$ depends solely on α.

3. Improve the constant 4^n in Lemma 7.6.

4. Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body, and let $E \subseteq \mathbb{R}^n$ be a subspace. Prove the Rogers-Shephard for sections and projections, namely that

$$|K| \leq |K \cap E| \cdot |Proj_{E^\perp} K| \leq 4^n |K|.$$

Can you get a better constant than 4^n?

5. Let $K \subseteq \mathbb{R}^n$ be a centrally-symmetric convex body in M-position, and let $U \in O(n)$ be a random rotation, distributed uniformly.

   (a) Prove that if $x, y \in \mathbb{R}^n \setminus RB^n$, for $R > 10$, then with probability of at least $1 - (C/R)^n$,

   $$(x + B^n) \cap (Uy + B^n) = \emptyset.$$

   (b) Use the above covering estimates and the union bound, and conclude that with probability of at least $1 - \tilde{C}e^{-\tilde{c}n}$,

   $$K \cap UK \subseteq CB^n.$$

   (c) Dualize and prove that with probability of at least $1 - \tilde{C}e^{-\tilde{c}n}$,

   $$K + UK \supseteq cB^n. \tag{31}$$

   (d) Set $K = [-1/\sqrt{n}, 1/\sqrt{n}]^n$. Conclude that there exists $U \in O(n)$ such that (31) holds true. It is an open problem to find an explicit construction of such U.

Request. Please e-mail me at `boaz.klartag@weizmann.ac.il` with any comments, corrections, or suggestions regarding these lecture notes. In addition, if you are able to produce *tikz* figures similar to (or better than) those drawn on the blackboard in class, I would be glad to include them in the notes.

November 11, 2025