Isoperimetric inequalities in high-dimensional convex sets
Boaz Klartag, ETH Zurich 2025

Lecture 8: From thin-shell to slicing convex bodies and the Bochner method

8.1. Thin shell bound for the isotropic constant

In Lecture 3 we discussed the thin-shell theorem, asserting that under the isotropic normalization, random vectors whose mass is concentrated in a *thin spherical shell* admit approximately Gaussian marginals. The thin-shell constant for log-concave distributions in $\mathbb{R}^n$ is defined as

$$\sigma_n = \sup_X \sqrt{\frac{\mathrm{Var}(|X|^2)}{n}}$$

where the supremum ranges over all isotropic, log-concave random vectors X in $\mathbb{R}^n$. Clearly, for any isotropic, log-concave random vector X in $\mathbb{R}^n$,

$$\mathrm{Var}(|X|) \leq \mathbb{E}(|X| - \sqrt{n})^2 \leq \frac{1}{n}\mathbb{E}(|X|^2 - n)^2 = \frac{1}{n}\mathrm{Var}(|X|^2) \leq \sigma_n^2, \qquad (1)$$

and hence σ_n is an upper bound for the maximal width of the thin spherical shell that captures most of the mass of an isotropic, log-concave random vector. We remark that by reverse Hölder inequalities for polynomials, the chain of inequalities in (1) may be reversed up to a multiplicative universal constant. Later on we will show that

$$\sigma_n < C \qquad (2)$$

for a universal constant $C > 0$. Thus the thin spherical shell that captures most of the mass of an isotropic, log-concave random vector has radius $\sqrt{n}$ and has width at most C. Another application of the bound (2) is the following:

Theorem 8.1 ("thin-shell bound for the isotropic constant"). *For any convex body $K \subseteq \mathbb{R}^n$ with $K = -K$,*

$$L_K \leq C\sigma_n,$$

where $C > 0$ is a universal constant.

In fact, it is shown in [3] that $L_X \leq C\sigma_n$ for any log-concave random vector X in $\mathbb{R}^n$, but for simplicity we confine ourselves to the centrally-symmetric convex body case.

The proof of Theorem 8.1 relies on the logarithmic Laplace transform. Let X be a random vector X in $\mathbb{R}^n$ with a log-concave density p. Recall that the logarithm Laplace transform

$$\Lambda(y) = \log \mathbb{E} e^{X \cdot y}$$

is smooth in the open convex set $\Omega = \{\Lambda < \infty\}$. For $y \in \Omega$ we write X_y for the log-concave random vector with density

$$p_y(x) = e^{x \cdot y - \Lambda(y)} p(x), \tag{3}$$

referred to as an *exponential tilt* of the random vector X. Recall that the derivatives of Λ at a point $y \in \Omega$ are given by

$$\nabla \Lambda(y) = \mathbb{E} X_y, \quad \nabla^2 \Lambda(y) = \mathrm{Cov}(X_y), \quad \nabla^3 \Lambda(y) = \mathbb{E}(X_y - a_y)^{\otimes 3}, \tag{4}$$

where $a_y = \mathbb{E} X_y$. Let us consider the function

$$F(y) = \log \det \nabla^2 \Lambda(y) = \log \det \mathrm{Cov}(X_y).$$

This function measures how the determinant of the covariance matrix changes when we tilt the given distribution.

Lemma 8.2. *Let X be an absolutely-continuous, log-concave random vector in $\mathbb{R}^n$. Then the following bound holds pointwise in all of Ω:*

$$(\nabla^2 \Lambda)^{-1} \nabla F \cdot \nabla F \leq n \sigma_n^2. \tag{5}$$

Proof. Let us first prove (5) at the point $y = 0$, and under the additional assumption that X is isotropic. By isotropicity,

$$\nabla^2 \Lambda(0) = \mathrm{Cov}(X) = \mathrm{Id}.$$

Recalling how we differentiate a determinant, we see that for any unit vector $v \in S^{n-1}$,

$$\begin{aligned} \partial_v F(0) &= \mathrm{Tr}\left[(\nabla^2 \Lambda)^{-1}(0) \cdot \partial_v \nabla^2 \Lambda(0) \right] = \mathbb{E}(X \cdot v)|X|^2 \\ &\leq \sqrt{\mathbb{E}(X \cdot v)^2 \cdot \mathrm{Var}(|X|^2)} \leq \sqrt{n} \sigma_n. \end{aligned}$$

By considering the supremum over all $v \in S^{n-1}$, we see that $|\nabla F(0)| \leq \sqrt{n} \sigma_n$. This completes the proof of (5) at the point $y = 0$.

In order to obtain the bound (5) at any point $y \in \Omega$ and without the assumption that X is isotropic we need to think invariantly, as we now explain.

Define a Riemannian metric on Ω via the Hessian of the log-Laplace transform Λ. That is, consider the Riemannian manifold (Ω, g), where the scalar product of two tangent vectors $u, v \in T_x \Omega = T_x \mathbb{R}^n \cong \mathbb{R}^n$ is defined as

$$g_x(u, v) = \nabla^2 \Lambda(x) u \cdot v.$$

The main observation is that the expression on the left-hand side of (5) is precisely the squared Riemannian length of the Riemannian gradient of the function $F : \Omega \to \mathbb{R}$.

We view F as a function that is defined only up to an additive constant. Write $[F]$ for the equivalence class of F under the equivalence relation "F is equivalent to G if and only if $F - G$ is a constant function". Let us refer to the triplet

$$\mathcal{M}_X = (\Omega, g, [F])$$

as the "Riemannian package" associated with X. This means that (Ω, g) is a Riemannian manifold and that F is a function on Ω modulo an additive constant. An *isomorphism* between two Riemannian packages is a bijective map which is a Riemannian isometry and which transforms correctly the function modulo the additive constant.

What happens to the Riemannian package associated with X when we perform various operations?

- Translation: When we switch from X to $X - v$, for a fixed $v \in \mathbb{R}^n$, the Riemannian metric stays exactly the same, as well as the function F. Thus we get the same Riemannian package.
- Tilting: When switching from X to X_y, we obtain an isomorphism of the two Riemannian packages by *translation* by y. We translate Ω, g and $[F]$ by the vector $y \in \Omega$.
- Linear transformation: Applying an invertible linear transformation to X induces an isomorphism of the Riemannian packages.[1] We apply a linear transformation and push forward Ω, g and $[F]$.

By the first and last items, we actually proved (5) at the point $y = 0$ for any log-concave random vector (not necessarily centered or isotropic). By the middle item, we proved (5) also at all other points of Ω. $\square$

What can we say about balls in this Riemannian manifold? Write $B_g(x_0, r)$ for the open Riemannian ball of radius r centered at the point $x_0 \in \Omega$.

[1]We may thus think of X as a random vector defined on an abstract affine space, rather than on $\mathbb{R}^n$, and observe that the Riemannian manifold (Ω, g) is well-defined, as well as the function $F : \Omega \to \mathbb{R}$ modulo additive constants.

Lemma 8.3. *Assume that X is a centered, log-concave random vector in $\mathbb{R}^n$. Then for any $r > 0$,*

$$\frac{1}{2} \cdot \{\Lambda \leq r\} \subseteq B_g(0, \sqrt{r}).$$

Proof. Let $y \in \Omega$ satisfy $\Lambda(2y) \leq r$. We need to find a curve from 0 to y whose Riemannian length is at most r. Let us try a line segment:

$$\begin{aligned}
Length_g([0,y]) &= \int_0^1 \sqrt{\nabla^2 \Lambda(ty) y \cdot y} dt = \int_0^1 \sqrt{\frac{d^2}{dt^2} \Lambda(ty)} dt \\
&\leq \sqrt{\int_0^2 (2-t) \frac{d^2}{dt^2} \Lambda(ty) dt \cdot \int_0^1 \frac{1}{2-t} dt} \\
&= \sqrt{\log 2} \cdot \sqrt{\Lambda(2y) - [\Lambda(0) + \nabla \Lambda(0) \cdot (2y)]} \\
&= \sqrt{\log 2} \cdot \sqrt{\Lambda(2y)} \leq \sqrt{r}.
\end{aligned}$$

□

Proof of Theorem 8.1. Let X be an isotropic random vector in $\mathbb{R}^n$, distributed uniformly in a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$. We would need two estimates for the proof of Theorem 8.1:

(i) First, we need to show that for $r = n/\sigma_n^2$,

$$Vol_n(K) \geq e^{-n} \cdot Vol_n(B_g(0, \sqrt{r})),$$

the Euclidean volume of the Riemannian ball. This is related to mass transport in a simple case.

(ii) Second, we need to show that

$$Vol_n(\{\Lambda \leq r\})^{1/n} \geq c \frac{r}{n} L_K.$$

This is related to the Bourgain-Milman inequality.

Since X is isotropic and log-concave, by (i), (ii) and Lemma 8.3,

$$\begin{aligned}
L_K = Vol_n(K)^{-1/n} &\leq C \cdot Vol_n(B_g(0, \sqrt{r}))^{-1/n} \\
&\leq 2C \cdot Vol_n(\{\Lambda \leq r\})^{-1/n} \leq C' \frac{n}{r L_K} = C' \frac{\sigma_n^2}{L_K}.
\end{aligned}$$

Thus $L_K \leq C'' \cdot \sigma_n$, and the proof is complete once we establish (i) and (ii).

In order to prove estimate (i), we note that the function F vanishes at the origin. By Lemma 8.2, this function is a Riemannian Lipschitz function with Lipschitz constant at most $\sqrt{n}\sigma_n$. Hence,

$$|F| \leq n \qquad \text{in} \qquad B_g(0, \sqrt{r}).$$

Consequently, for any $y \in B_g(0, \sqrt{r})$,

$$e^{-n} \leq \det \nabla^2 \Lambda(y) \leq e^n.$$

We will use the fact that $\nabla \Lambda(y) = \mathbb{E} X_y \in K$ and that $y \mapsto \nabla \Lambda(y)$ is one-to-one. Changing variables, we obtain

$$\begin{aligned} Vol_n(K) \geq Vol_n\left(\nabla \Lambda(B_g(0, \sqrt{r}))\right) &= \int_{B_g(0,\sqrt{r})} \det \nabla^2 \Lambda(y)\, dy \\ &\geq e^{-n} \cdot Vol_n(B_g(0, \sqrt{r})). \end{aligned}$$

This proves (i). We move on to the proof of estimate (ii). For any $y \in rK^\circ$,

$$\Lambda(y) = \log \mathbb{E} e^{y \cdot X} \leq \log(e^r) = r.$$

Therefore,

$$rK^\circ \subseteq \{\Lambda \leq r\}.$$

By the Bourgain-Milman inequality,

$$Vol_n(\{\Lambda \leq r\})^{1/n} \geq Vol_n(rK^\circ)^{1/n} \geq c\frac{r}{n} Vol_n(K)^{-1/n} = c\frac{r}{n} L_K.$$

This completes the proof of (ii). □

8.2. Bochner identities and curvature

For a deeper investigation of isoperimetry and thin-shell of log-concave probability measures, we would need to add a Gaussian factor to the exponential tilts, and make contact with the heat flow in $\mathbb{R}^n$. Rather than studying the tilts defined in (3), we will discuss the t-log-Laplace transform

$$\Lambda_t(\theta) = \log \int_{\mathbb{R}^n} e^{\theta \cdot x - t|x|^2/2} p(x) dx,$$

and the t-localized tilts

$$p_{t,\theta}(x) = e^{\theta \cdot x - t|x|^2/2 - \Lambda_t(\theta)} p(x).$$

The t-localized tilts are clearly log-concave, but in fact more is true. The t-localized tilts are *t-uniformly log-concave*, since for almost any $x \in \mathbb{R}^n$ with $p(x) > 0$,

$$-\nabla^2 \log p_{t,\theta}(x) \geq \nabla^2(t|x|^2/2) = t \cdot \mathrm{Id}.$$

In order to exploit this uniform log-concavity, let us now discuss a technique that originated in Riemannian Geometry and connects the Poincaré inequality and Curvature.

The approach was developed in the works of Bochner in the 1940s and also Lichnerowicz in the 1950s, and it fits well with convex bodies and log-concave measures in high dimension. In a nutshell, the idea is to make local computations involving something like curvature, as well as integrations by parts, and then dualize and obtain Poincaré-type inequalities. This may sound pretty vague, let us explain what we mean.

Suppose that μ is an absolutely-continuous, log-concave probability measure in $\mathbb{R}^n$. The measure μ is supported in some open, convex set $K \subseteq \mathbb{R}^n$ (possibly $K = \mathbb{R}^n$), and it has a positive, log-concave density

$$p = e^{-\psi}$$

in K. We will measure distances using the Euclidean metric in $\mathbb{R}^n$, but we will measure volumes using the measure μ. We thus look at the *weighted Riemannian manifold* or the *metric-measure space*

$$(K, |\cdot|, \mu).$$

We define the Dirichlet energy of a smooth function $f : K \to \mathbb{R}$ as

$$\|f\|^2_{\dot{H}^1(\mu)} = \int_K |\nabla f|^2 d\mu.$$

Indeed, we measure the length of the gradient with respect to the Euclidean metric, while we integrate with respect to the measure μ. The *Poincaré constant* of μ, denoted by $C_P(\mu)$, is the minimal number $A > 0$ such that for all μ-integrable, locally-Lipschitz functions $f : K \to \mathbb{R}$ with $\int_K f d\mu = 0$,

$$\int_K f^2 d\mu \leq A \cdot \int_K |\nabla f|^2 d\mu.$$

The Poincaré constant is finite and non-zero (see [2]), and it is a geometric characteristic of the measure μ that is closely related to the isoperimetric inequality. The Poincaré constant of the standard Gaussian measure, for instance, equals one. The inequality

$$\mathrm{Var}_\mu(f) \leq C_P(\mu) \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

where $\mathrm{Var}_\mu(f) = \int f^2 d\mu - (\int f d\mu)^2$, is referred to as the *Poincaré inequality*.

The Laplace-type operator associated with our measure-metric space is defined, initially for $u \in C_c^\infty(K)$, via

$$Lu = L_\mu u = \Delta u - \nabla\psi \cdot \nabla u = e^\psi div(e^{-\psi}\nabla u). \tag{6}$$

Here, $C_c^\infty(K)$ is the space of smooth functions that are compactly-supported in K. The reason for the definition (6) is that for any smooth functions $u, v : \mathbb{R}^n \to \mathbb{R}$, with one of them compactly-supported in K,

$$\int_{\mathbb{R}^n} (Lu) v d\mu = \int_{\mathbb{R}^n} div(e^{-\psi}\nabla u)v = -\int_{\mathbb{R}^n} [\nabla u \cdot \nabla v] e^{-\psi} = -\int_{\mathbb{R}^n} [\nabla u \cdot \nabla v]\, d\mu.$$

In particular

$$\langle -Lu, u\rangle_{L^2(\mu)} = \int_{\mathbb{R}^n} |\nabla u|^2 d\mu.$$

Thus L is a symmetric operator in $L^2(\mu)$, defined initially for $u \in C_c^\infty(K)$. It can have more than one self-adjoint extension, for example corresponding to the Dirichlet or Neumann boundary conditions when K is bounded.[2]

It will be convenient to make an (inessential) regularity assumption on the measure μ, in order to avoid all boundary terms in all integrations by parts. We say that μ is a *regular, log-concave* measure in $\mathbb{R}^n$ if its density, denoted by $e^{-\psi}$, is smooth and positive in $\mathbb{R}^n$ and the following two requirements hold:

(i) Log-concavity amounts to ψ being convex, so $\nabla^2\psi \geq 0$ everywhere in $\mathbb{R}^n$. We require a bit more, that there exists $\varepsilon > 0$ such that for all $x \in \mathbb{R}^n$,

$$\varepsilon \cdot \mathrm{Id} \leq \nabla^2\psi(x) \leq \frac{1}{\varepsilon} \cdot \mathrm{Id}. \tag{7}$$

(ii) The function ψ, as well as each of its partial derivatives of any order, grows at most polynomially at infinity.

According to an exercise below, any log-concave probability measure may be approximated arbitrary well by a regular one.

From now on, we assume that our probability measure μ is a regular, log-concave measure. It turns out that in this case, the operator L, initially defined on $C_c^\infty(\mathbb{R}^n)$, is

[2]When discussing the Bochner technique, it is possible to find ways to circumvent spectral theory of the operator L. Still, spectral theory helps us understand and form intuition, and we will at least quote the relevant spectral theory.

essentially self-adjoint, positive semi-definite operator in $L^2(\mu)$ with a discrete spectrum. Its eigenfunctions $1 \equiv \varphi_0, \varphi_1, \dots$ constitute an orthonormal basis, and the eigenvalues of $-L$ are

$$0 = \lambda_0(L) < \lambda_1(L) = \frac{1}{C_P(\mu)} \leq \lambda_2(L) \leq \dots$$

with the eigenfunction corresponding to the trivial eigenvalue 0 being the constant function. The eigenfunctions are smooth functions in $\mathbb{R}^n$ that do not grow too fast at infinity: each function

$$\varphi_j e^{-\psi/2}$$

decays exponentially at infinity. Also $(\partial^\alpha \varphi_j) e^{-\psi/2}$ decays exponentially at infinity for any partial derivative α. This follows from known results on exponential decay of eigenfunctions of Schrödinger operators. The eigenvalues are given by the following infimum of Rayleigh quotients

$$\lambda_k(L) = \inf_{f \perp \varphi_0, \dots, \varphi_{k-1}} \frac{\int_{\mathbb{R}^n} |\nabla f|^2 d\mu}{\int_{\mathbb{R}^n} f^2 d\mu}$$

where the infimum runs over all (say) locally-Lipschitz functions $f \in L^2(\mu)$. Since $\varphi_0 \equiv 1$, we indeed see that the first eigenfunction φ_1 saturates the Poincaré inequality for μ. The linear space

$$\{a + Lu\,;\, a \in \mathbb{R},\ u \in C_c^\infty(\mathbb{R}^n)\}$$

is dense in $L^2(\mu)$. For proofs of these spectral theoretic facts, see references in [4].

Let us return to Geometry. In Riemannian geometry, the Ricci curvature appears when we commute the Laplacian and the gradient. Analogously, here we have the easily-verified commutation relation

$$\nabla(Lu) = L(\nabla u) - (\nabla^2 \psi)(\nabla u),$$

where $L(\nabla u) = (L(\partial^1 u), \dots, L(\partial^n u))$. Hence the matrix $\nabla^2 \psi$ corresponds to a curvature term, analogous to the Ricci curvature.

Proposition 8.4 (Integrated Bochner's formula)**.** *For any $u \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (Lu)^2 d\mu = \int_{\mathbb{R}^n} \left(\nabla^2 \psi\right) \nabla u \cdot \nabla u \, d\mu + \int_{\mathbb{R}^n} \|\nabla^2 u\|_{HS}^2 \, d\mu,$$

where $\|\nabla^2 u\|_{HS}^2 = \sum_{i=1}^n |\nabla \partial_i u|^2$.

Proof. Integration by parts gives

$$\begin{aligned}\int_{\mathbb{R}^n} (Lu)^2 d\mu &= -\int_{\mathbb{R}^n} \nabla(Lu)\cdot\nabla u \ d\mu \\ &= -\int_{\mathbb{R}^n} L(\nabla u)\cdot\nabla u \ d\mu + \int_{\mathbb{R}^n} \left[(\nabla^2\psi)\nabla u\cdot\nabla u\right] d\mu \\ &= \sum_{i=1}^n \int_{\mathbb{R}^n} |\nabla\partial_i u|^2 d\mu + \int_{\mathbb{R}^n} \left(\nabla^2\psi\right)\nabla u\cdot\nabla u \ d\mu.\end{aligned}$$

□

The assumption that u is compactly-supported was used in order to discard the boundary terms when integrating by parts. In fact, it suffices to know that u is μ-tempered. We say that u is μ-tempered if it is a smooth function, and $(\partial^\alpha u)e^{-\psi/2}$ decays exponentially at infinity for any partial derivative $\partial^\alpha u$. Any eigenfunction of L is μ-tempered. If f is μ-tempered, then so is Lf. The following inequality is concerned with distributions that are *uniformly* log-concave.

Theorem 8.5 (improved log-concave Lichnerowicz inequality)**.** *Let $t > 0$ and assume that $\nabla^2\psi(x) \geq t$ for all $x \in \mathbb{R}^n$. Then,*

$$C_P(\mu) \leq \sqrt{\|\mathrm{Cov}(\mu)\|_{op}\cdot\frac{1}{t}},$$

where $\|A\|_{op}$ is the operator norm of the symmetric matrix $A \in \mathbb{R}^{n\times n}$.

Equality in Theorem 8.5 is attained when μ is a Gaussian measure, with any covariance matrix.

Proof of Theorem 8.5. Denote $f = \varphi_1$, the first eigenfunction, normalized so that $\|f\|_{L^2(\mu)} = 1$. Set $\lambda = 1/C_P(\mu)$. By the Bochner formula and the Poincaré inequality for $\partial^i f$ $(i = 1,\ldots,n)$,

$$\begin{aligned}\lambda^2 &= \int_{\mathbb{R}^n} (Lf)^2 d\mu = \int_{\mathbb{R}^n} [(\nabla^2\psi)\nabla f\cdot\nabla f] d\mu + \int_{\mathbb{R}^n} \|\nabla^2 f\|_{HS}^2 d\mu \\ &\geq t\int_{\mathbb{R}^n} |\nabla f|^2 d\mu + \lambda\left[\int_{\mathbb{R}^n} |\nabla f|^2 d\mu - \left|\int_{\mathbb{R}^n} \nabla f d\mu\right|^2\right] \\ &= (t+\lambda)\cdot\lambda - \lambda\left|\int_{\mathbb{R}^n} \nabla f d\mu\right|^2. \end{aligned} \tag{8}$$

Therefore the first eigenfunction has a “preferred direction”, i.e.,

$$\left| \int_{\mathbb{R}^n} \nabla f d\mu \right|^2 \geq t. \tag{9}$$

Using that the i^{th} coordinate of ∇f is $\nabla f \cdot \nabla x_i$ and integrating by parts we have

$$\int_{\mathbb{R}^n} \nabla f d\mu = -\int_{\mathbb{R}^n} (Lf) x d\mu = \lambda \int_{\mathbb{R}^n} f x d\mu$$

Since $\int f d\mu = 0$, by Cauchy-Schwartz, for some $\theta \in S^{n-1}$,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \nabla f d\mu \right| &= \int_{\mathbb{R}^n} \langle \nabla f, \theta \rangle d\mu = \lambda \int_{\mathbb{R}^n} f(x) \langle x, \theta \rangle \, \mu(dx) \\ &\leq \lambda \|f\|_{L^2(\mu)} \cdot \sqrt{\mathrm{Cov}(\mu)\theta \cdot \theta} \leq \lambda \|\mathrm{Cov}(\mu)\|_{op}. \end{aligned}$$

This expression is at least t, and the theorem follows. □

Observe that by testing the Poincaré inequality with linear functions, we obtain

$$\|\mathrm{Cov}(\mu)\|_{op} \leq C_P(\mu).$$

We thus deduce from Theorem 8.5 that

$$C_P(\mu) \leq \frac{1}{t}. \tag{10}$$

Inequality (10) is sometimes referred to as the log-concave Lichnerowicz inequality.

The Bochner formula states that in the log-concave case, for any $u \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (Lu)^2 d\mu = \int_{\mathbb{R}^n} [(\nabla^2 \psi) \nabla u \cdot \nabla u] d\mu + \int_{\mathbb{R}^n} \|\nabla^2 u\|_{HS}^2 d\mu \geq \int_{\mathbb{R}^n} \|\nabla^2 u\|_{HS}^2 d\mu.$$

Let us dualize this inequality in order to obtain a Poincaré-type inequality. To this end, for $f \in L^2(\mu)$ we define the dual Sobolev norm

$$\|f\|_{H^{-1}(\mu)} = \sup \left\{ \int_{\mathbb{R}^n} f u d\mu \, ; \, \int_{\mathbb{R}^n} |\nabla u|^2 d\mu \leq 1 \; u \in C_c^\infty(\mathbb{R}^n) \right\}.$$

This supremum can be finite only when $\int f d\mu = 0$.

Proposition 8.6. *(H^{-1}-inequality) Let μ be a regular, log-concave probability measure in $\mathbb{R}^n$. Then for $f \in L^2(\mu)$,*

$$Var_\mu(f) \leq \|\nabla f\|_{H^{-1}(\mu)}^2 = \sum_{i=1}^n \|\partial^i f\|_{H^{-1}(\mu)}^2.$$

Proof. We may assume that $\int f d\mu = 0$. By approximation, assume that $f = -Lu$ for $u \in C_c^\infty(\mathbb{R}^n)$. See [1] for the approximation argument. Then,

$$\int_{\mathbb{R}^n} f^2 d\mu = \int_{\mathbb{R}^n} [\nabla f \cdot \nabla u] d\mu \leq \|\nabla f\|_{H^{-1}(\mu)} \sqrt{\int_{\mathbb{R}^n} \|\nabla^2 u\|_{HS}^2 d\mu}$$
$$\leq \|\nabla f\|_{H^{-1}(\mu)} \sqrt{\int_{\mathbb{R}^n} (Lu)^2 d\mu}.$$

The proposition follows. □

The H^{-1}-norm has a geometric interpretation as infinitesimal *transport cost*, which may be roughly expressed by saying that when $\int f d\mu = 0$, as $\varepsilon \to 0$,

$$\|f\|_{H^{-1}(\mu)} \approx \frac{1}{\varepsilon} W_2(\mu, (1 + \varepsilon f)\mu). \tag{11}$$

Let us explain (11). Let μ_1, μ_2 be Borel probability measures on $\mathbb{R}^n$. We say that a Borel probability measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ is a *coupling* of μ_1 and μ_2 if

$$(\pi_i)_* \gamma = \mu_i \qquad\qquad (i = 1, 2),$$

where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. That is, the marginal of γ on the first coordinate is μ_1, and the marginal of γ on the second coordinate is μ_2. The L^2-Wasserstein distance between μ_1, μ_2 is defined as

$$W_2(\mu_1, \mu_2) = \inf_\gamma \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \, d\gamma(x, y) \right)^{1/2},$$

where the infimum runs over all couplings γ of μ_1 and μ_2. In probabilistic notation, we have

$$W_2(\mu_1, \mu_2) = \inf_{(X,Y)} \sqrt{\mathbb{E}|X - Y|^2}$$

where the infimum runs over all *possibly-dependent* random vectors $X, Y \in \mathbb{R}^n$ with X having law μ_1 and Y having law μ_2.

Proposition 8.7 ("bounding the H^{-1}-norm by transport cost")**.** *Let μ be a finite, compactly-supported measure on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded, measurable function with*

$$\int f d\mu = 0.$$

For a sufficiently small $\varepsilon > 0$, let μ_ε be the measure whose density with respect to μ is the non-negative function $1 + \varepsilon f$. Then,

$$\|f\|_{H^{-1}(\mu)} \leq \liminf_{\varepsilon \to 0^+} \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon}.$$

Proof. We need to prove that for any $u \in C_c^\infty(\mathbb{R}^n)$, function $u : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} fud\mu \leq \sqrt{\int_{\mathbb{R}^n} |\nabla u|^2 d\mu} \cdot \liminf_{\varepsilon \to 0^+} \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon}. \tag{12}$$

Fix such a test function $u \in C_c^\infty(\mathbb{R}^n)$. Then the second derivatives of u are bounded on $\mathbb{R}^n$. By Taylor's theorem, there exists a constant $R = R(u)$ with

$$u(y) - u(x) \leq |\nabla u(x)| \cdot |x - y| + R|x - y|^2 \qquad \forall x, y \in \mathbb{R}^n. \tag{13}$$

We may assume that $\sup |f| > 0$ (otherwise, the theorem holds trivially), and let $\varepsilon > 0$ be smaller than $1/\sup |f|$. Then μ_ε is a non-negative measure on $\mathbb{R}^n$. Let γ be any coupling of μ and μ_ε. We see that

$$\int_{\mathbb{R}^n} fud\mu = \frac{1}{\varepsilon} \int_{\mathbb{R}^n} ud\left[\mu_\varepsilon - \mu\right] = \frac{1}{\varepsilon} \int_{\mathbb{R}^n \times \mathbb{R}^n} \left[u(y) - u(x)\right] d\gamma(x, y).$$

Write

$$W_2^\gamma(\mu, \mu_\varepsilon) = \sqrt{\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 d\gamma(x, y)}.$$

According to (13) and to the Cauchy-Schwartz inequality,

$$\begin{aligned}
\int_{\mathbb{R}^n} hud\mu \quad &\leq \quad \frac{1}{\varepsilon} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla u(x)| \cdot |x - y| d\gamma(x, y) + \frac{R}{\varepsilon} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 d\gamma(x, y) \\
&\leq \quad \frac{1}{\varepsilon} \sqrt{\int_{\mathbb{R}^n} |\nabla u(x)|^2 d\mu(x)} \cdot W_2^\gamma(\mu, \mu_\varepsilon) + \frac{R}{\varepsilon} W_2^\gamma(\mu, \mu_\varepsilon)^2.
\end{aligned}$$

By taking the infimum over all couplings γ of μ and μ_ε, we obtain

$$\int_{\mathbb{R}^n} hud\mu \leq \sqrt{\int_{\mathbb{R}^n} |\nabla u|^2 d\mu} \cdot \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon} + R \frac{W_2(\mu, \mu_\varepsilon)^2}{\varepsilon}, \tag{14}$$

with R depending only on u. We may assume that $\liminf_{\varepsilon \to 0^+} W_2(\mu, \mu_\varepsilon)/\varepsilon < \infty$; otherwise, there is nothing to prove. Consequently,

$$\liminf_{\varepsilon \to 0^+} \frac{W_2(\mu, \mu_\varepsilon)^2}{\varepsilon} = \liminf_{\varepsilon \to 0^+} \varepsilon \left(\frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon} \right)^2 = 0.$$

Hence by letting ε tend to zero in (14), we deduce (12). The proof is complete. $\square$

Exercises.

1. Begin with an arbitrary log-concave measure μ on $\mathbb{R}^n$, convolve it by a tiny Gaussian, and then multiply its density by $\exp(-\varepsilon|x|^2)$ for small $\varepsilon > 0$. Show that the resulting measure is regular, log-concave, with approximately the same covariance matrix, and that the Poincaré constant cannot jump down by much under this regularization process.

2. Verify the Poincaré constant of the standard Gaussian measure in $\mathbb{R}^n$ equals one.

3. Let $t > 0$, and let X be a t-uniformly log-concave random vector in $\mathbb{R}^n$, Use the Prékopa-Leindler inequality, and show that for any subspace $E \subseteq \mathbb{R}^n$, also

$$Proj_E X$$

is a t-uniformly log-concave random vector.

4. The Bochner formula also states that for any $u \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (Lu)^2 d\mu \geq \int_{\mathbb{R}^n} [(\nabla^2\psi)\nabla u \cdot \nabla u] d\mu.$$

Dualize this inequality in order to prove the *Brascamp-Lieb inequality*: For any C^1-smooth $f \in L^2(\mu)$,

$$\mathrm{Var}_\mu(f) \leq \int_{\mathbb{R}^n} \left(\nabla^2\psi\right)^{-1} \nabla f \cdot \nabla f \; d\mu(x).$$

Can you find equality cases, other than a constant function f?

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[3] Eldan, R., Klartag, B., *Approximately Gaussian marginals and the hyperplane conjecture.* In Concentration, functional inequalities and isoperimetry, Vol. 545 of Contemp. Math., Amer. Math. Soc., (2011), 55–68.

[4] Klartag, B., *Logarithmic bounds for isoperimetry and slices of convex sets.* Ars Inveniendi Analytica, Paper No. 4, (2023), 17pp.

Request. Please e-mail me at `boaz.klartag@weizmann.ac.il` with any comments, corrections, or suggestions regarding these lecture notes. In addition, if you are able to produce *tikz* figures similar to (or better than) those drawn on the blackboard in class, I would be glad to include them in the notes.

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