Isoperimetric inequalities in high-dimensional convex sets
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Lecture 9: Coupling exponential tilts of log-concave measures

The last two lectures of the semester will be devoted to a proof of the following thin shell bound:

Theorem 9.1. *Let X be an isotropic, log-concave random vector in $\mathbb{R}^n$. Then,*

$$\mathrm{Var}(|X|^2) = \mathbb{E}\left(|X|^2 - n\right)^2 \leq Cn, \tag{1}$$

where $C > 0$ is a universal constant.

Recall from last week that (1) implies that

$$\mathbb{E}\left(|X| - \sqrt{n}\right)^2 \leq C,$$

and hence most of the mass of the random vector X is located in a thin-spherical shell of radius $\sqrt{n}$ and width C. Theorem 9.1 is tight, up to the value of the universal constant. Indeed, if X is a standard Gaussian random vector in $\mathbb{R}^n$ or if X is distributed uniformly in the cube $[-\sqrt{3}, \sqrt{3}]^n \subseteq \mathbb{R}^n$, then X is isotropic and log-concave with

$$\mathrm{Var}(|X|^2) = Cn,$$

where $C = 2$ in the Gaussian case and $C = 4/5$ in the case of the cube.

As we saw last week, Theorem 9.1 implies an affirmative answer to *Bourgain's slicing problem*, since it implies that for any convex body $K \subseteq \mathbb{R}^n$,

$$L_K = \left(\frac{\det \mathrm{Cov}(K)}{\mathrm{Vol}_n^2(K)}\right)^{1/(2n)} \in \left[\frac{1}{\sqrt{2\pi e}}, C\right]. \tag{2}$$

Before proceeding with the proof of Theorem 9.1, let us discuss a geometric consequence of the boundness of the isotropic constant.

Corollary 9.2 ("Corrected Busemann-Petty conjecture"). *Let $n \geq 2$ and let $K, T \subseteq \mathbb{R}^n$ be centered convex bodies such that for any hyperplane $H \subseteq \mathbb{R}^n$ through the origin,*

$$\mathrm{Vol}_{n-1}(K \cap H) \leq \mathrm{Vol}_{n-1}(T \cap H). \tag{3}$$

Then

$$\mathrm{Vol}_n(K) \leq C \cdot \mathrm{Vol}_n(T) \tag{4}$$

for a universal constant $C > 0$.

Proof. Recall from Lecture 6 that for any $\theta \in S^{n-1}$,

$$\frac{\mathrm{Vol}_{n-1}(K \cap \theta^{\perp})}{\mathrm{Vol}_n(K)} \cdot \sqrt{\mathrm{Cov}(K)\theta \cdot \theta} \sim 1.$$

We thus deduce from (3) that

$$\mathrm{Vol}^{-2}(T) \cdot \mathrm{Cov}(T) \leq C \cdot \mathrm{Vol}^{-2}(K) \cdot \mathrm{Cov}(K).$$

Hence,

$$\mathrm{Vol}^{-2n}(T) \cdot \det \mathrm{Cov}(T) \leq C^n \mathrm{Vol}^{-2n}(K) \cdot \det \mathrm{Cov}(K).$$

We will use (2) in order to replace $\det \mathrm{Cov}$ by L_K and L_T. This gives

$$\mathrm{Vol}^{2-2n}(T) \cdot L_T^{2n} \leq C^n \mathrm{Vol}^{2-2n}(K) \cdot L_K^{2n}.$$

We thus obtain that

$$\mathrm{Vol}_n(K) \leq C^{\frac{n}{2n-2}} \cdot \left(\frac{L_K}{L_T}\right)^{\frac{n}{n-1}} \cdot \mathrm{Vol}_n(T) \leq \tilde{C} \cdot \mathrm{Vol}_n(T),$$

where we used (2) in the last passage. □

In the 1950s, Busemann and Petty proved Corollary 9.2 with $C = 1$, in the case where $T = -T$ and where the convex body K is a centered Euclidean ball (or a cross-polytope). In the centrally-symmetric case, the validity of Corollary 9.2 with $C = 1$ turns out to be true if $n \leq 4$ and false if $n \geq 5$, see [2] and references therein. In sufficiently high dimensions, this fails already when K is a cube and T is a Euclidean ball. The example of the cube and ball shows that C from (4) necessarily satisfies

$$C \geq \sqrt{e/2}.$$

Let μ be a probability measure on $\mathbb{R}^n$ with a log-concave density. Recall the $H^{-1}(\mu)$-norm discussed last week, whose geometric meaning is understood through the *infinitesimal transport cost* bound

$$\|f\|_{H^{-1}(\mu)} \leq \liminf_{\varepsilon \to 0^+} \frac{W_2(\mu, (1+\varepsilon f)\mu)}{\varepsilon}, \tag{5}$$

where W_2 is the L^2-Wasserstein distance. The bound(5) is valid under minimal regularity assumptions on f, provided that $\int f d\mu = 0$. Recall the H^{-1}-inequality

$$\mathrm{Var}_\mu(f) \leq \sum_{i=1}^{n} \|\partial^i f\|_{H^{-1}(\mu)}^2 \tag{6}$$

that holds for any smooth function $f \in L^2(\mu)$. In particular, by substituting $f(x) = |x|^2$ in (6) and noting that $\partial^i f = 2x_i$, Theorem 9.1 follows from the following:

Theorem 9.3. *Let μ be an isotropic, log-concave probability measure in $\mathbb{R}^n$. Then,*

$$\sum_{i=1}^{n} \|x_i\|_{H^{-1}(\mu)}^2 \leq Cn,$$

where $C > 0$ is a universal constant.

Write p for the log-concave density of the probability measure μ in $\mathbb{R}^n$. Recall that for $y \in \mathbb{R}^n$ the corresponding *exponential tilt* of μ is the probability measure μ_y with density

$$p_y(x) = e^{x \cdot y - \Lambda(y)} p(x) \qquad\qquad (x \in \mathbb{R}^n), \tag{7}$$

where $\Lambda(y) = \log \int e^{x \cdot y} d\mu(x)$. Observe that for any $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$, as $\varepsilon \to 0$,

$$p_{\varepsilon e_i}(x) = (1 + \varepsilon x_i) p(x) + o(\varepsilon).$$

It is an exercise to modify the proof of (5) from last week and show that when μ is compactly-supported and $i = 1, \ldots, n$,

$$\|x_i\|_{H^{-1}(\mu)} \leq \limsup_{\varepsilon \to 0^+} \frac{W_2(\mu, \mu_{\varepsilon e_i})}{\varepsilon}. \tag{8}$$

Thus, in order to prove Theorem 9.3, it suffices to construct efficient *couplings* of exponential tilts of μ. The specific construction that we use for coupling tilts is related to the theory of non-linear filtering and to Eldan's stochastic localization, which we now describe.

For simplicity, let us assume that the log-concave probability measure μ is compactly-supported. We will add time dependence to the log-Laplace transform to obtain a heat flow. That is, for $t \geq 0$ and $y \in \mathbb{R}^n$ we consider the t-log-Laplace transform

$$\Lambda_t(y) = \log \int_{\mathbb{R}^n} e^{y \cdot x - t|x|^2/2} p(x) dx,$$

and the *t-localized tilts* or *t-Gaussian needles*, which are the probability densities:

$$p_{t,y}(x) = e^{y \cdot x - t|x|^2/2 - \Lambda_t(y)} p(x) = \frac{p(x)\gamma_{1/t}(y/t - x)}{p * \gamma_{1/t}(y/t)}, \tag{9}$$

where

$$\gamma_s(x) = (2\pi s)^{-n/2} \exp(-|x|^2/(2s))$$

is the density of a centered Gaussian in $\mathbb{R}^n$ of covariance $s \cdot \mathrm{Id}$. The main advantage of the $p_{t,y}$ over the exponential tilt p_y is that $p_{t,y}$ is t-uniformly log-concave. In fact, almost everywhere in $\mathbb{R}^n$,

$$\nabla^2(-\log p_{t,y}) \geq t \cdot \mathrm{Id}. \tag{10}$$

Denote by $a_t(y)$ the barycenter of the probability density $p_{t,y}$, namely

$$a_t(y) = \nabla \Lambda_t(y) = \int_{\mathbb{R}^n} x p_{t,y}(x) dx \in \mathbb{R}^n. \tag{11}$$

It is an exercise to show that $a_t : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map, with a Lipschitz constant bounded uniformly in $t \in [0, +\infty)$.

Lemma 9.4. *For any continuous path $w = (w_t)_{t \geq 0}$ in $\mathbb{R}^n$ with $w_0 = 0$ and for any initial condition $\theta_0 \in \mathbb{R}^n$, there exists a unique solution $(\theta_t)_{t \geq 0}$ to the integral equation*

$$\theta_t = \theta_0 + w_t + \int_0^t a_s(\theta_s) ds, \qquad t \geq 0. \tag{12}$$

The solution $\theta_t = \theta_t(x)$ is continuous in $(t, x) \in [0, \infty) \times \mathbb{R}^n$ and is smooth in $x \in \mathbb{R}^n$ for any fixed $t \geq 0$. We denote

$$G_{t,w}(\theta_0) = \theta_t.$$

Thanks to the Lipschitz property of the map $a_t : \mathbb{R}^n \to \mathbb{R}^n$, Lemma 9.4 follows from standard Ordinary Differential Equations (ODE) theory; see [1], also for the standard fact that the map

$$G_{t,w} : \mathbb{R}^n \to \mathbb{R}^n \tag{13}$$

is a diffeomorphism. Consider a standard Brownian motion in $\mathbb{R}^n$

$$(W_t)_{t \geq 0},$$

with $W_0 = 0$. We will use the continuous Brownian path $W = (W_t)_{t \geq 0}$ in Lemma 9.4. Abbreviate

$$G_t(y) = G_{t,W}(y).$$

Proposition 9.5. *For any $y \in \mathbb{R}^n$, the stochastic process $(G_t(y))_{t \geq 0}$ has the same law as the process*

$$(y + tX_y + W_t)_{t \geq 0}$$

where X_y is a random vector with law μ_y which is independent of the Brownian motion $(W_t)_{t \geq 0}$.

Proposition 9.5 is part of the theory of non-linear filtering.

Corollary 9.6. *For any $y \in \mathbb{R}^n$, almost surely, the limit*

$$\lim_{t \to \infty} \frac{G_t(y)}{t} \tag{14}$$

exists, and has law μ_y.

Indeed, thanks to Proposition 9.5, Corollary 9.6 follows from the fact that

$$\lim_{t\to\infty} \frac{y + tX_y + W_t}{t} = X_y + \lim_{t\to\infty} \frac{W_t}{t} = X_y,$$

which is a random vector with law μ_y. Thus the limit in (14), usually denoted by $a_\infty(y)$, provides *simultaneous coupling* of all of the tilts $(\mu_y)_{y\in\mathbb{R}^n}$.

Proof of Proposition 9.5. Our proof requires some familiarity with stochastic processes.

Step 1. Observe that it suffices to prove the proposition for $y = 0$, since switching from X_0 to X_y amounts to replacing the function

$$a_s(\theta)$$

by

$$a_s(\theta + y).$$

We may thus assume that $y = 0$ and abbreviate $X = X_0$. For $t \geq 0$ define

$$\theta_t = tX + W_t. \tag{15}$$

The random vector $\theta_t/t = X + W_t/t$ is a noisy observation of X, which typically gets more and more accurate as t increases. Since W_t/t is a centered Gaussian random vector of covariance Id/t, the density of θ_t/t equals

$$p * \gamma_{1/t}.$$

Our goal is to prove that $(\theta_t)_{t\geq 0}$ coincides in law with $(G_{t,B}(0))_{t\geq 0}$ for a standard Brownian motion $(B_t))_{t\geq 0}$ in $\mathbb{R}^n$ with $B_0 = 0$.

Step 2. What is the conditional law of X given θ_t? It follows from (15) that the joint density of $(X, \theta_t/t)$ in $\mathbb{R}^n \times \mathbb{R}^n$ is

$$(x, z) \mapsto p(x)\gamma_{1/t}(z - x) \qquad\qquad (x, z) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{16}$$

Hence the density of the conditional law of X given θ_t/t is the normalized fiber of the joint density in (16), namely the probability density

$$\frac{p(x)\gamma_{1/t}(\theta_t/t - x)}{p * \gamma_{1/t}(\theta_t/t)} = p_{t,\theta_t}(x),$$

where we used (9) in the last passage. Thus p_{t,θ_t} is the density of the conditional law of X given the observation θ_t (or given the observation θ_t/t). In fact, p_{t,θ_t} is also the density of the conditional law of X given all past observations $(\theta_s)_{0\le s\le t}$. Indeed, it is an exercise to show that the time reversal

$$B_t = tW_{1/t} \qquad\qquad (t>0)$$

is also a standard Brownian motion in $\mathbb{R}^n$, with $W_t = tB_{1/t}$. Thus,

$$\begin{aligned}
&\text{Law}\,(X|(sX+W_s)_{0<s\le t}) = \text{Law}\,\big(X|(X+sW_{1/s})_{s\ge 1/t}\big) \\
&= \text{Law}\,\big(X|(X+B_s)_{s\ge 1/t}\big) = \text{Law}\,\big(X|X+B_{1/t} \text{ and } (B_s-B_{1/t})_{s>1/t}\big) \\
&= \text{Law}\,\big(X|X+B_{1/t}\big) = \text{Law}\,(X|tX+W_t)\,,
\end{aligned}$$

since $B_s - B_{1/t}$ is independent of X and $X+B_{1/t}$. Writing $\mathcal{F}_t$ for the σ-algebra generated by $(\theta_s)_{0\le s\le t}$, we conclude that

$$\mathbb{E}[X|\mathcal{F}_t] = \int_{\mathbb{R}^n} xp_{t,\theta_t}(x)dx = a_t(\theta_t). \tag{17}$$

Step 3. For $t\ge 0$ define $B_t\in\mathbb{R}^n$ via the equation

$$\theta_t = B_t + \int_0^t a_s(\theta_s)ds. \tag{18}$$

Thus, for $t\ge 0$,

$$\theta_t = G_{t,B}(0),$$

where $B=(B_t)_{t\ge 0}$. It remains to prove that the *innovation process* $(B_t)_{t\ge 0}$ is a standard Brownian motion in $\mathbb{R}^n$. From (18) we see that

$$B_t = W_t + tX - \int_0^t a_s(\theta_s)ds = W_t + \int_0^t \left(X - \mathbb{E}[X|\mathcal{F}_s]\right)ds. \tag{19}$$

By (18) the random vector B_t is measurable with respect to $\mathcal{F}_t$. Consequently,

$$B_t = W_t + \int_0^t v_s ds \tag{20}$$

where

$$\mathbb{E}[v_t|(B_s)_{0\le s\le t}] = 0,$$

and where for $s<t$ the increment $W_t - W_s$ is a centered, Gaussian random vector of covariance $(t-s)\text{Id}$ that is independent of $(B_r)_{0\le r\le s}$ and of $(v_r)_{0\le r\le s}$. We see from (20) that $(B_t)_{t\ge 0}$ is a martingale whose quadratic variation is that of a standard Brownian motion. Hence it is a Brownian motion, by Lévy's characterization. □

Thus far we have shown that for any $x, y \in \mathbb{R}^n$,

$$W_2(\mu_x, \mu_y) \leq \sqrt{\mathbb{E} \left| \lim_{t \to \infty} \frac{G_t(x)}{t} - \lim_{t \to \infty} \frac{G_t(y)}{t} \right|^2}, \tag{21}$$

because the first limit in (21) has law μ_x while the second has law μ_y. The next proposition refines (21) by allowing to stop the processes at a finite time.

Lemma 9.7. *For $x, y \in \mathbb{R}^n$ and $t > 0$,*

$$W_2(\mu_x, \mu_y) \leq \frac{1}{t} \cdot \sqrt{\mathbb{E} \left| G_t(x) - G_t(y) \right|^2}.$$

Proof. For $t \geq 0$ and $y \in \mathbb{R}^n$ we denote by $A_t(y)$ the covariance matrix of the probability density $p_{t,y}$, that is,

$$A_t(y) = \nabla^2 \Lambda_t(y) = \int_{\mathbb{R}^n} x \otimes x \, p_{t,y}(x) dx - a_t(y) \otimes a_t(y) \in \mathbb{R}^{n \times n}. \tag{22}$$

Recall from (10) that $p_{t,y}$ is uniformly log-concave. Thus by the log-concave Lichnerowicz inequality,

$$A_t(y) = \nabla^2 \Lambda_t(y) \leq \frac{1}{t} \cdot \mathrm{Id}.$$

This concavity property implies contraction properties of the time-dependent stochastic gradient ascent from Lemma 9.4. That is, for $y_1, y_2 \in \mathbb{R}^n$,

$$\begin{aligned} &\langle a_t(y_1) - a_t(y_2), y_1 - y_2 \rangle = \langle \nabla \Lambda_t(y_1) - \nabla \Lambda_t(y_2), y_1 - y_2 \rangle \qquad (23) \\ &= \int_0^1 \langle \nabla^2 \Lambda_t(s y_1 + (1-s) y_2)(y_1 - y_2), y_1 - y_2 \rangle \, ds \leq \frac{1}{t} \cdot |y_1 - y_2|^2. \end{aligned}$$

By Lemma 9.4,

$$G_t(x) - G_t(y) = x - y + \int_0^t \left[a_s(G_s(x)) - a_s(G_s(y)) \right] ds.$$

Hence by (23),

$$\begin{aligned} \frac{d}{dt} |G_t(x) - G_t(y)|^2 &= 2 \langle a_t(G_t(x)) - a_t(G_t(y)), G_t(x) - G_t(y) \rangle \\ &\leq \frac{2}{t} |G_t(x) - G_t(y)|^2. \end{aligned}$$

Equivalently,

$$\frac{d}{dt} \frac{|G_t(x) - G_t(y)|^2}{t^2} \leq 0.$$

Hence

$$\frac{|G_t(x) - G_t(y)|^2}{t^2} \geq \limsup_{s\to\infty} \frac{|G_s(x) - G_s(y)|^2}{s^2} = \left| \lim_{s\to\infty} \frac{G_s(x) - G_s(y)}{s} \right|^2,$$

where the limit exists almost surely. The conclusion now follows from (21). □

Recall that $G_t : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth diffeomorphism. Denote

$$M_t = G_t'(0) \in \mathbb{R}^{n\times n},$$

i.e., $M_t v = \partial_v G_t(0)$ for any $v \in \mathbb{R}^n$. We write $|M_t|^2$ for the sum of the squares of the n^2 entries of the matrix M_t.

Corollary 9.8. *For any centered, compactly-supported, log-concave probability measure μ and $t > 0$,*

$$\sum_{i=1}^n \|x_i\|^2_{H^{-1}(\mu)} \leq \frac{1}{t^2} \cdot \mathbb{E}|M_t|^2.$$

Proof. It follows from Lemma 9.7 that

$$\limsup_{\varepsilon\to 0} \frac{W_2^2(\mu, \mu_{\varepsilon e_i})}{\varepsilon^2} \leq \frac{1}{t^2} \limsup_{\varepsilon\to 0} \mathbb{E} \left| \frac{G_t(0) - G_t(\varepsilon e_i)}{\varepsilon} \right|^2.$$

It is explained in [1] that the dominated convergence theorem allows us to replace expectation and limit, and obtain that

$$\limsup_{\varepsilon\to 0} \frac{W_2^2(\mu, \mu_{\varepsilon e_i})}{\varepsilon^2} \leq \frac{1}{t^2} \mathbb{E} \left| \lim_{\varepsilon\to 0} \frac{G_t(0) - G_t(\varepsilon e_i)}{\varepsilon} \right|^2 = \frac{1}{t^2} \cdot \mathbb{E}|G_t'(0)e_i|^2.$$

Thus, by (8),

$$\sum_{i=1}^n \|x_i\|^2_{H^{-1}(\mu)} \leq \frac{1}{t^2} \sum_{i=1}^n \mathbb{E}|G_t'(0)e_i|^2 = \frac{1}{t^2} \cdot \mathbb{E}|M_t|^2.$$

□

Thus, in order to prove Theorem 9.3, we should understand the matrix-valued process $(M_t)_{t\geq 0}$ of the derivative at zero of the random diffeomorphism G_t. Recall from (22) that we denote

$$A_t(y) = \nabla^2 \Lambda_t(y) = \mathrm{Cov}(p_{t,y})$$

and let us further abbreviate

$$A_t = A_t(G_t(0)).$$

The integral equation of Lemma 9.4 states that

$$G_t(y) = y + W_t + \int_0^t \nabla \Lambda_s(G_s(y)))ds.$$

By differentiating with respect y (see [1] for justification) we see that

$$G_t'(0) = \mathrm{Id} + \int_0^t \nabla^2 \Lambda_s(G_s(0)))G_s'(0)ds = \mathrm{Id} + \int_0^t A_s M_s ds.$$

Consequently, we have the *product integral equation*

$$\begin{cases} M_0 = \mathrm{Id} \\ \frac{d}{dt} M_t = A_t M_t \end{cases} \tag{24}$$

The following lemma is a non-probabilistic bound for the solution of the product integral equation. Denote the eigenvalues of A_t, repeated according to their multiplicity, by

$$\lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_n(t) > 0.$$

Lemma 9.9. *For any* $t > 0$*,*

$$|M_t|^2 \leq \sum_{i=1}^n \exp\left(2\int_0^t \lambda_i(s)ds\right). \tag{25}$$

It is straightforward to verify that for $n = 1$, equality holds in (25). Rather than proving Lemma 9.9 along the lines of [1], we will prove the lemma by using the Hardy-Littlewood-Polya inequality (see e.g. [3]). This inequality states that when $b_1 \geq b_2 \geq \ldots \geq b_m$ are real numbers and $c_1, \ldots, c_n \in \mathbb{R}$ are such that

$$\sum_{i=1}^k b_i \leq \sum_{i=1}^k c_i \qquad (k = 1, \ldots, n), \tag{26}$$

then for any convex, increasing function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\sum_{i=1}^n \varphi(b_i) \leq \sum_{i=1}^n \varphi(c_i). \tag{27}$$

Denote the singular values of M_t by

$$e^{b_1(t)} \geq \ldots \geq e^{b_n(t)}. \tag{28}$$

The numbers $e^{2b_1(t)}, \ldots, e^{2b_n(t)}$ are the eigenvalues of $M_t^* M_t$. These are absolutely-continuous functions of t. The proof of Lemma 9.9 relies on the following:

Lemma 9.10. *For $k = 1, \dots, n$ and for almost any $t > 0$,*

$$\frac{d}{dt} \sum_{i=1}^{k} b_i(t) \le \sum_{i=1}^{k} \lambda_i(t).$$

Proof. Fix $t > 0$ at which $b_1(t), \dots, b_n(t)$ are differentiable, which happens almost everywhere. By an approximation argument it suffices to prove the lemma under the additional assumption that the inequalities in (28) are strict. Since A_t is a symmetric matrix, it follows from (24) that

$$\frac{d}{dt} M_t^* M_t = 2 M_t^* A_t M_t. \tag{29}$$

From the singular value decomposition of the matrix M_t, there exists orthonormal bases $u_1, \dots, u_n \in \mathbb{R}^n$ and $v_1, \dots, v_n \in \mathbb{R}^n$ such that

$$M_t u_i = e^{b_i(t)} v_i \qquad\qquad (i = 1, \dots, n).$$

In particular $M_t^* M_t u_i = e^{2b_i(t)} u_i$. According to (29) and the Hadamard perturbation lemma,

$$\frac{d}{dt} e^{2b_i(t)} = 2 M_t^* A_t M_t u_i \cdot u_i \qquad\qquad (i = 1, \dots, n).$$

Thus

$$2 e^{2b_i(t)} \frac{d}{dt} b_i(t) = 2 \langle A_t M_t u_i, M_t u_i \rangle = 2 e^{2b_i(t)} \langle A_t v_i, v_i \rangle.$$

In particular,

$$\frac{d}{dt} \sum_{i=1}^{k} b_i(t) = \sum_{i=1}^{k} \langle A_t v_i, v_i \rangle \le \sum_{i=1}^{k} \lambda_i(t),$$

by the min-max characterization of the eigenvalues of the symmetric matrix A_t. □

Proof of Lemma 9.9. Since $b_i(0) = 0$ for all i, we learn from Lemma 9.10 that for $k = 1, \dots, n$,

$$\sum_{i=1}^{k} b_i(t) \le \sum_{i=1}^{k} \int_0^t \lambda_i(s) ds. \tag{30}$$

Denote $b_i = b_i(t)$ and $c_i = \int_0^t \lambda_i(s) ds$. Then $b_1 \ge \ldots \ge b_n$, while condition (26) holds true thanks to (30). Set $\varphi(t) = e^{2t}$, a convex increasing function. According to (27),

$$\sum_{i=1}^{n} e^{2b_i(t)} \le \sum_{i=1}^{n} \exp\left(2 \int_0^t \lambda_i(s) ds \right).$$

Recalling that $e^{2b_1(t)}, \ldots, e^{2b_i(t)}$ are the eigenvalues of $M_t^* M_t$, the lemma follows. $\square$

We summarize this lecture with the following corollary, which will be the starting point of the next lecture:

Corollary 9.11. *For any centered, compactly-supported, log-concave probability measure μ and $t > 0$,*

$$\mathrm{Var}_\mu(|x|^2) \leq \sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2 \leq \frac{1}{t^2} \cdot \sum_{i=1}^n \mathbb{E} \exp\left(2 \int_0^t \lambda_i(s) ds\right),$$

where

$$\frac{1}{t} \geq \lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_n(t) > 0$$

are the eigenvalues of the covariance matrix A_t of the probability density

$$p_t = p_{t, G_t(0)}.$$

Exercises.

1. Modify the proof of (5) from last week and prove (8).
2. Let $(W_t)_{t \geq 0}$ be a standard Brownian motion with $W_0 = 0$. Set $B_t = t W_{1/t}$ for $t > 0$ and $B_0 = 0$. Prove that $(B_t)_{t \geq 0}$ is again a standard Brownian motion in $\mathbb{R}^n$.
3. Let μ be an absolutely-continuous, compactly-supported probability measure with density p in $\mathbb{R}^n$. Consider the vector $a_t(y)$ $(t \geq 0, y \in \mathbb{R}^n)$ defined in (11) above. Prove that $a_t : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map, with a Lipschitz constant bounded uniformly in $t \in [0, +\infty)$.
4. Recall the proof of the Hadamard perturbation lemma and of the Hardy-Littlewood-Polya inequality.

References

[1] Klartag, B., Lehec, J., *Thin-shell bounds via parallel coupling*. arXiv:2507.15495

[2] Koldobsky, A., *Fourier analysis in convex geometry*. American Mathematical Society, 2005.

[3] Polya, G., *Remark on Weyl's note "Inequalities between the two kinds of eigenvalues of a linear transformation."* Proc. Nat. Acad. Sci. U. S. A., Vol. 36, (1950), 49–51.

Request. Please e-mail me at `boaz.klartag@weizmann.ac.il` with any comments, corrections, or suggestions regarding these lecture notes. In addition, if you are able to produce *tikz* figures similar to (or better than) those drawn on the blackboard in class, I would be glad to include them in the notes.

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