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A Multiplicative Version of the Brunn-Minkowski Inequality in the Plane

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To Tzah, Sari and Hagar

Abstract

In recent years, several connections were discovered between three conjectures in convex geometry. The first conjecture is the logarithmic Brunn-Minkowski inequality, which is inspired by an extension of a work of Firey from 1962. This conjecture is a stability result for the familiar Brunn-Minkowski inequality. The second conjecture is that the cone volume measure of a convex body is unique. This was first raised in a paper by Firey in 1974. Gage proved in 1993 the uniqueness property for smooth and centrally-symmetric convex bodies in $\mathbb{R}^2$. In a paper from 2013, Böröczky, Lutwak, Yang and Zhang showed how to extend his ideas, in a way that implies the logarithmic Brunn-Minkowski inequality for the same family of convex bodies.

The third question arose around 2002, when results concerning the Gaussian measure led to the conjecture that it has the (B) property, and this was proved in 2004 by Cordero-Erausquin, Fradelizi and Maurey. This theorem in turn led to the conjecture that all even and log-concave measures have the (B) property. A result of Saroglou from 2013 shows that this last conjecture is equivalent to the logarithmic Brunn-Minkowski conjecture, and that from what is known about it, we can infer that uniform, even and convex measures on $\mathbb{R}^2$ have the (B) property.

This dissertation consists of two parts. In the first part, I will review these results and present most of the proofs succinctly. In the second part, I will present a novel approach, which gives a direct proof of the (B) property for the same family of measures.

The second part is derived from a paper which will appear in the 2011–2013 installment of Geometric Aspects of Functional Analysis. It is mostly independent of the first part in content, and only refers to the subsection "Property (B)".

Contents

1. Background

Convex sets. A subset $K \subset \mathbb{R}^n$ of Euclidean space which is convex, compact, and has a non-empty interior, is a *convex body*. This is denoted $K \in \mathscr{K}^n$. If the origin is an interior point of K, we write $K \in \mathscr{K}_0^n$. If $K = -K$, we say that K is *centrally symmetric*, and write $K \in \mathscr{K}_{\mathrm{e}}^n$ ("e" stands for *even*). Centrally symmetric convex bodies are the unit balls of norms on $\mathbb{R}^n$.

The *support function* of the convex body $K \in \mathscr{K}^n$ is

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. This satisfies $h_K(tx) = t h_K(x)$ for $t \geq 0$, $h_K(x+y) \leq h_K(x) + h_K(y)$, and $h_K(x) + h_K(-x) > 0$ for $x \neq 0$. Any function from $\mathbb{R}^n$ to $\mathbb{R}$ with these properties is the support function of a unique convex body.

Minkowski combinations. Linear combinations of support functions define combinations of convex bodies. For $K, T \in \mathscr{K}^n$ and $s, t > 0$, the body $sK + tT \in \mathscr{K}^n$ is the body with support function $s\, h_K + t\, h_T$, or

$$sK + tT = \{sx + ty : x \in K, y \in T\}.$$

These linear combinations lead to the Brunn-Minkowski inequality [6], [15] and to the theory of mixed volumes, [17]. We shall often regard support functions as functions from $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ to $\mathbb{R}$. The metric of $C\big(S^{n-1}\big)$ induces the Hausdorff metric on $\mathscr{K}^n$.

Surface area measure. Another concept we need is the *surface area measure* of a convex body. For $K \in \mathscr{K}^n$ we define a positive Borel measure S_K on S^{n-1} by the following equivalent properties (see [17]):

- Let H_{n-1} be the $(n-1)$-dimensional measure in $\mathbb{R}^n$. Let ν_K take $x \in \partial K$ to $u \in S^{n-1}$ such that u is an outer normal at x (or $h_K(u) = \langle x, u\rangle$). This u is unique H_{n-1}-almost everywhere. For any $A \subset S^{n-1}$,
$$S_K(A) = H_{n-1}(\{x \in \partial K : \nu_K(x) \in A\}).$$
- For any $L \in \mathscr{K}^n$,
$$V(L, K, \ldots, K) = \frac{1}{n} \int_{S^{n-1}} h_L(u)\, dS_K(u).$$

If $K_1, K_2, \ldots \to K$ is a convergent sequence of convex bodies, their surface area measures converge weakly: $S_{K_1}, S_{K_2}, \ldots \overset{w}{\to} S_K$.

From the second definition, we have

$$|K| = \frac{1}{n} \int_{S^{n-1}} h_K(u)\, dS_K(u).$$

The *cone-volume measure* of $K \in \mathscr{K}_0^n$ is

$$V_K(A) := \frac{1}{n} \int_A h_K(u) dS_K(u).$$

L_p **combinations.** In [9], the linear combination of support functions was replaced with other power-law formulas. That is, for $p \geq 1$, $K, T \in \mathscr{K}_0^n$ and $s, t > 0$, we define $s \cdot K +_p t \cdot T$ as the body whose support function is $h(x) = (s\, h_K(x)^p + t\, h_T(x)^p)^{1/p}$. This is indeed a support function, as the functions h_K, h_T, and $f(t) = t^p$ are convex:

$$\begin{aligned} h(x+y) &= (s\, h_K(x+y)^p + t\, h_T(x+y)^p)^{1/p} \\ &\leq (s\, h_K(x)^p + t\, h_T(x)^p + s\, h_K(y)^p + t\, h_T(y)^p)^{1/p} \\ &\leq (s\, h_K(x)^p + t\, h_T(x)^p)^{1/p} + (s\, h_K(y)^p + t\, h_T(y)^p)^{1/p} = h(x) + h(y)\,. \end{aligned}$$

This property fails for $p < 1$. We follow [4], and define in this case the combination $s \cdot K +_p t \cdot T$ to be the body whose support function is the maximal among those satisfying $h(x) \leq (s\, h_K(x)^p + t\, h_T(x)^p)^{1/p}$, or $h(x) \leq h_K(x)^s\, h_T(x)^t$ for all $x \in S^{n-1}$ for $p = 0$. This construction is called the Aleksandrov body of the function $(s\, h_K(x)^p + t\, h_T(x)^p)^{1/p}$.

Aleksandrov bodies. For any everywhere positive function $f \in C\left(S^{n-1}\right)$, its Aleksandrov body is $\left\{x \in \mathbb{R}^n : \langle x, u\rangle \leq f(u) \;\; \forall u \in S^{n-1}\right\}$. The support function of this body is $h(x) = \sup\{h'(x) : h' \leq f\}$. If we denote by f^* the Legendre transform of the homogeneous extension of f to $\mathbb{R}^n$, then $h = f^{**}$.

We shall need some properties of Aleksandrov bodies from [1]. For convenience, they are reproduced here.

Lemma 1. *The Aleksandrov body of a function changes continuously with the function.*

Proof. The mapping from a function to its Aleksandrov body respects multiplication by a positive constant, and is monotone. Since the domain of definition contains only functions bounded away from zero, the mapping must be continuous. □

Lemma 2. *If K is the Aleksandrov body of f, then*

$$\left\{u \in S^{n-1} : h_K(u) \neq f(u)\right\}$$

is an S_K-null set.

Proof. Take $x \in \partial K$ and $u \in S^{n-1}$. The difference $f(u) - \langle u, x\rangle$ is non-negative and continuous in u and x. It must vanish for some $u \in S^{n-1}$, for if it is above ε for all u, K must contain a ball of radius ε around x. All the $u \in S^{n-1}$ for which $f(u) = \langle u, x\rangle$ are outward normals at x. For S_K-almost all $x \in \partial K$, there is a unique outward normal n, which must then satisfy $f(n) = \langle n, x\rangle = h_K(n)$. □

Lemma 3. *Let $f \in C\left([0,\varepsilon] \times S^{n-1}\right)$, positive everywhere, and assume that $\frac{f(t,u)-f(0,u)}{t}$ converges uniformly as $t \searrow 0$. Denote by K_t the Aleksandrov body of $f(t,\cdot)$. Then the one-sided derivative $\frac{d}{dt}\left|K_t\right|_{t=0}$ exists, and*

$$\frac{d}{dt}\left|K_t\right|_{t=0} = \int_{S^{n-1}} \frac{\partial f}{\partial t}(0,u)\, dS_{K_0}(u)\,.$$

Proof. We first consider the mixed volume $V_1(K,T) = \frac{1}{n}\int_{S^{n-1}} h_T(u)\,dS_K(u)$. For this, we have

$$\begin{aligned}
\limsup_{t\searrow 0}\frac{V_1(K_0,K_t)-|K_0|}{t} &= \limsup_{t\searrow 0}\frac{\frac{1}{n}\int_{S^{n-1}} h_{K_t}dS_{K_0} - \frac{1}{n}\int_{S^{n-1}} h_{K_0}dS_{K_0}}{t} \\
&\leq \limsup_{t\searrow 0}\frac{\frac{1}{n}\int_{S^{n-1}} f(t,\cdot)\,dS_{K_0} - \frac{1}{n}\int_{S^{n-1}} f(0,\cdot)\,dS_{K_0}}{t} \\
&= \frac{1}{n}\limsup_{t\searrow 0}\int_{S^{n-1}}\frac{f(t,\cdot)-f(0,\cdot)}{t}dS_{K_0} = \frac{1}{n}\int_{S^{n-1}}\frac{\partial f}{\partial t}(0,\cdot)\,dS_{K_0},
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{t\searrow 0}\frac{|K_t|-V_1(K_t,K_0)}{t} &= \liminf_{t\searrow 0}\frac{\frac{1}{n}\int_{S^{n-1}} h_{K_t}dS_{K_t} - \frac{1}{n}\int_{S^{n-1}} h_{K_0}dS_{K_t}}{t} \\
&\geq \liminf_{t\searrow 0}\frac{\frac{1}{n}\int_{S^{n-1}} f(t,\cdot)\,dS_{K_t} - \frac{1}{n}\int_{S^{n-1}} f(0,\cdot)\,dS_{K_t}}{t} \\
&= \frac{1}{n}\liminf_{t\searrow 0}\int_{S^{n-1}}\frac{f(t,\cdot)-f(0,\cdot)}{t}dS_{K_t} = \frac{1}{n}\int_{S^{n-1}}\frac{\partial f}{\partial t}(0,\cdot)\,dS_{K_0}.
\end{aligned}$$

Minkowski's first inequality is $V_1(K,T) \geq |K|^{(n-1)/n}|T|^{1/n}$. Applying it twice, we get

$$\begin{aligned}
\frac{1}{n}\int_{S^{n-1}}\frac{\partial f}{\partial t}(0,\cdot)\,dS_{K_0} &\leq \liminf_{t\searrow 0}\frac{|K_t|-V_1(K_t,K_0)}{t} \\
&\leq \liminf_{t\searrow 0}\frac{|K_t|-|K_t|^{(n-1)/n}|K_0|^{1/n}}{t} \\
&\leq |K_0|^{(n-1)/n}\limsup_{t\searrow 0}\frac{|K_t|^{1/n}-|K_0|^{1/n}}{t} \\
&\leq \limsup_{t\searrow 0}\frac{V_1(K_0,K_t)-|K_t|}{t} \\
&\leq \frac{1}{n}\int_{S^{n-1}}\frac{\partial f}{\partial t}(0,\cdot)\,dS_{K_0}.
\end{aligned}$$

Hence $\frac{|K_t|^{1/n}-|K_0|^{1/n}}{t}$ converges to $\frac{1}{n}\int_{S^{n-1}}\frac{\partial f}{\partial t}(0,u)\,dS_{K_0}(u)$ as $t\searrow 0$. The assertion of the lemma now follows by simple calculus. □

L_p**-Brunn-Minkowski inequality.** If $p \neq 1$, the volume of a combination is not a polynomial in the coefficients, so there is no theory of mixed volumes. However, it is possible to define "quermassintegrals".

We shall also be interested in equivalents of the Brunn-Minkowski inequality. This gets stronger as p decreases: if $K,T \in \mathscr{K}_0^n$ and $0 \leq \lambda \leq 1$, then $\left((1-\lambda)h_K^p + \lambda h_T^p\right)^{1/p}$ increases everywhere with p. Therefore, if $p < 1$,

$$|(1-\lambda)\cdot K +_p \lambda\cdot T| \geq |K|^{1-\lambda}|T|^{\lambda}$$

is stronger than the Brunn-Minkowski inequality, and can be seen as a stability result for it.

If $p<0$, this inequality is reversed, even for K,T that are homothetic, so the best result possible is in the case $p=0$. In some cases it still holds that $|(1-\lambda)\cdot K+_0\lambda\cdot T|<|K|^{1-\lambda}|T|^{\lambda}$, so some kind of symmetry condition is necessary. Central symmetry, $K=-K$, appears to be a sufficient condition. Some of the results below concern bodies in the class $\mathscr{K}_0^n$. They also apply for bodies in $\mathscr{K}_{\mathrm{e}}^n$, with the same proofs.

Like the Brunn-Minkowski inequality, the L_p-Brunn-Minkowski inequality can be cast in additive form.

Lemma 4. *Let $p>0$. The following are equivalent:*

- *For every $K,T\in\mathscr{K}_0^n$, and $0\le\lambda\le 1$,*
$$|(1-\lambda)\cdot K+_p\lambda\cdot T|\ge|K|^{1-\lambda}|T|^{\lambda}.$$
- *For every $K,T\in\mathscr{K}_0^n$, and $0\le\lambda\le 1$,*
$$|(1-\lambda)\cdot K+_p\lambda\cdot T|^{p/n}\ge(1-\lambda)|K|^{p/n}+\lambda|T|^{p/n}.$$

Proof. The multiplicative version follows from the additive version due to the arithmetic-geometric means inequality.

If we assume the multiplicative version, we can derive the additive version. Indeed, let $K,T\in\mathscr{K}_0^n$ and $0\le\lambda\le 1$, and define
$$\alpha=(1-\lambda)|K|^{p/n}+\lambda|T|^{p/n},\qquad\beta=\frac{\lambda|T|^{p/n}}{\alpha}.$$

Then
$$(1-\lambda)\cdot K+_p\lambda\cdot T=\alpha^{1/p}\left[(1-\beta)\cdot\frac{K}{|K|^{1/n}}+_p\beta\cdot\frac{T}{|T|^{1/n}}\right],$$
so
$$\begin{aligned}|(1-\lambda)\cdot K+_p\lambda\cdot T|&=\alpha\left|(1-\beta)\cdot\frac{K}{|K|^{1/n}}+_p\beta\cdot\frac{T}{|T|^{1/n}}\right|\\&\ge\alpha=(1-\lambda)|K|^{p/n}+\lambda|T|^{p/n}.\end{aligned}$$

□

L_p **mixed volume.** The first derivative of the Brunn-Minkowski inequality is Minkowski's inequality. This has an analog in the L_p theory.

Let $K,T\in\mathscr{K}_0^n$. Their interpolation $K_\lambda=(1-\lambda)\cdot K+_p\lambda\cdot T$ is the Aleksandrov body of $\left((1-\lambda)h_K^p+\lambda h_T^p\right)^{1/p}$. This function is continuously differentiable in λ, so the conditions of Lemma 3 apply, and we have
$$\frac{d}{d\lambda}|(1-\lambda)\cdot K+_p\lambda\cdot T|_{\lambda=0}=\int_{S^{n-1}}\frac{h_T^p h_K^{1-p}-h_K}{p}dS_K.$$

Analogously to the case $p=1$, we define the *first mixed volume* as
$$\begin{aligned}V_p(K,T)&=\frac{p}{n}\frac{d}{d\lambda}|(1-\lambda)\cdot K+_p\lambda\cdot T|_{\lambda=0}+|K|\\&=\int_{S^{n-1}}\left(\frac{h_T}{h_K}\right)^p dV_K.\end{aligned}$$

For $p=0$, we have

$$\frac{d}{d\lambda}\left|(1-\lambda)\cdot K+_0\lambda\cdot T\right|_{\lambda=0}=\int_{S^{n-1}}\log\frac{h_T}{h_K}h_K dS_K.$$

In order to have a formula that includes the case $p=0$ as a limit case, we follow [4] and define the *normalized first mixed volume* as

$$\begin{aligned}\bar{V}_p(K,T) &= \left(\frac{1}{|K|}\int_{S^{n-1}}\left(\frac{h_T}{h_K}\right)^p dV_K\right)^{1/p},\\ \bar{V}_0(K,T) &= \exp\left(\frac{1}{|K|}\int_{S^{n-1}}\log\frac{h_T}{h_K}dV_K\right).\end{aligned}$$

We have

Theorem 5. *Let $p\geq 0$ and $n\geq 1$. The following are equivalent:*

- *The L_p-Brunn-Minkowski inequality: For every $K,T\in\mathscr{K}_0^n$ and $0\leq\lambda\leq 1$,*

$$\left|(1-\lambda)\cdot K+_p\lambda\cdot T\right|\geq|K|^{1-\lambda}|T|^{\lambda}.$$

- *The L_p-Minkowski inequality: For every $K,T\in\mathscr{K}_0^n$,*

$$\bar{V}_p(K,T)\geq\left(\frac{|T|}{|K|}\right)^{1/n}.$$

Proof. Assume that the L_p-Brunn-Minkowski inequality holds. For $p=0$ this means that the graph of the function $\lambda\mapsto\log\left|(1-\lambda)\cdot K+_0\lambda\cdot T\right|$ lies above the secant connecting $(0,\log|K|)$ and $(1,\log|T|)$, while for $p>0$ this means that the graph of the function $\lambda\mapsto\left|(1-\lambda)\cdot K+_p\lambda\cdot T\right|^{p/n}$ lies above the corresponding secant. The derivatives of these functions at $\lambda=0$ are $n\log\bar{V}_0(K,T)$ and $\left(\bar{V}_p(K,T)^p-1\right)|K|^{p/n}$, respectively. Comparing these to the slope of the secant gives the L_p-Minkowski inequality $\bar{V}_p(K,T)\geq\left(\frac{|T|}{|K|}\right)^{1/n}$.

In the other direction, we assume that the L_p-Minkowski inequality holds. Denote $K_\lambda=(1-\lambda)\cdot K+_p\lambda\cdot T$. In the case $p=0$, $\log\frac{h_K^{1-\lambda}h_T^{\lambda}}{h_{K_\lambda}}=0$ almost everywhere with respect to S_{K_λ}, so we have

$$\begin{aligned}1&=\exp\left(\frac{1}{n|K_\lambda|}\int_{S^{n-1}}\log\frac{h_K^{1-\lambda}h_T^{\lambda}}{h_{K_\lambda}}h_{K_\lambda}dS_{K_\lambda}\right)\\ &=\exp\left(\frac{1-\lambda}{n|K_\lambda|}\int_{S^{n-1}}\log\frac{h_K}{h_{K_\lambda}}h_{K_\lambda}dS_{K_\lambda}+\frac{\lambda}{n|K_\lambda|}\int_{S^{n-1}}\log\frac{h_T}{h_{K_\lambda}}h_{K_\lambda}dS_{K_\lambda}\right)\\ &=\bar{V}_0(K_\lambda,K)^{1-\lambda}\bar{V}_0(K_\lambda,T)^{\lambda}\geq\left(\frac{|K|^{1-\lambda}|T|^{\lambda}}{|K_\lambda|}\right)^{1/n}.\end{aligned}$$

In the case $p>0$, $h_{K_\lambda}=\left(\left(1-\lambda\right)h_K^p+\lambda h_T^p\right)^{1/p}$ almost everywhere with respect to S_{K_λ}, so

$$\begin{aligned}|K_\lambda| &= \frac{1}{n}\int_{S^{n-1}} h_{K_\lambda} dS_{K_\lambda} = \frac{1}{n}\int_{S^{n-1}} \left(\left(1-\lambda\right)h_K^p+\lambda h_T^p\right) h_{K_\lambda}^{1-p} dS_{K_\lambda} \\ &= \left(1-\lambda\right) V_p\left(K_\lambda, K\right)+\lambda V_p\left(K_\lambda, T\right) \\ &= \left(1-\lambda\right)|K_\lambda| \bar{V}_p\left(K_\lambda, K\right)^p+\lambda|K_\lambda| \bar{V}_p\left(K_\lambda, T\right)^p \\ &\geq \left(1-\lambda\right)|K_\lambda|\left(\frac{|K|}{|K_\lambda|}\right)^{p/n}+\lambda|K_\lambda|\left(\frac{|T|}{|K_\lambda|}\right)^{p/n} \\ &\geq |K_\lambda|\left(\frac{|K|^{1-\lambda}|T|^\lambda}{|K_\lambda|}\right)^{p/n},\end{aligned}$$

or $|K_\lambda|\geq|K|^{1-\lambda}|T|^\lambda$. □

When $p>1$, the L_p-Brunn-Minkowski inequality clearly holds. For equality, it is necessary that the inclusion $\left(1-\lambda\right)\cdot K+_p\lambda\cdot T\supset\left(1-\lambda\right)K+\lambda T$ be not proper, which forces $K=T$. Now consider the L_p-Minkowski inequality. We have $\bar{V}_p\left(K,T\right)=\left(\int_{S^{n-1}}\left(\frac{h_T}{h_K}\right)^p\frac{dV_K}{|K|}\right)^{1/p}$, which is $\geq\frac{1}{|K|}V_1\left(K,L\right)\geq\left(\frac{|L|}{|K|}\right)^{1/n}$ by Jensen's inequality. The condition for equality in Jensen's inequality is that $\frac{h_T}{h_K}$ be a constant function, so K and T must be homothetic. This is a sufficient condition.

These equality conditions are formally equivalent for $p<1$.

Lemma 6. *Let $p>0$ and $n\geq1$. The following are equivalent:*

- *For every $K,T\in\mathscr{K}_0^n$ and $0<\lambda<1$,*
$$\left|\left(1-\lambda\right)\cdot K+_p\lambda\cdot T\right|\geq|K|^{1-\lambda}|T|^\lambda,$$
with equality only if $K=T$.
- *For every $K,T\in\mathscr{K}_0^n$, $\bar{V}_p\left(K,T\right)\geq\left(\frac{|T|}{|K|}\right)^{1/n}$, with equality only if K and T are homothetic.*

Proof. Assume the L_p-Brunn-Minkowski inequality together with the equality conditions. If $\bar{V}_p\left(K,T\right)=\left(\frac{|T|}{|K|}\right)^{1/n}$, take $T'=\left(\frac{|K|}{|T|}\right)^{1/n}T$, and then $\bar{V}_p\left(K,T'\right)=1$. The function $\lambda\mapsto\left|\left(1-\lambda\right)\cdot K+_p\lambda\cdot T'\right|^{p/n}$ is concave, attains the value $|K|$ when $\lambda=0,1$, and its derivative at $\lambda=0$ is

$$\left(\bar{V}_p\left(K,T'\right)^p-1\right)|K|^{p/n}=0.$$

Hence it is constant, and the equality conditions of the L_p-Brunn-Minkowski inequality yield $K=T'$, hence K,T are homothetic.

Now assume that the L_p-Minkowski inequality holds, with the assumed equality condition. In the second part of the proof of Theorem 5, we used the inequalities $\bar{V}_p\left(K_\lambda,K\right)\geq\left(\frac{|K|}{|K_\lambda|}\right)^{1/n}$ and $\bar{V}_p\left(K_\lambda,T\right)\geq\left(\frac{|T|}{|K_\lambda|}\right)^{1/n}$ in our derivation. If equality holds in the Brunn-Minkowski inequality, then equalities must also hold in these steps. This shows that K and T are homothetic (to K_λ and hence to one another). However, if $T = aK$ then

$$(1-\lambda)\cdot K +_p \lambda \cdot T = [(1-\lambda) + \lambda a^p]^{1/p} K,$$

and it is easy to see that $[(1-\lambda) + \lambda a^p]^{n/p} |K| > |K|^{1-\lambda} (a^n |K|)^\lambda$ if $a \neq 0$ and $\lambda \in (0,1)$. ☐

These are the conjectured equality conditions for $0 < p < 1$. For $p = 0$, they are somewhat different. For instance, if D a diagonal linear transformation, K is the unit cube $K = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$, and $T = DK$, then

$$\bar{V}_0(K,T) = (\det D)^{1/n} = \left(\frac{|T|}{|K|}\right)^{1/n},$$

even though the shapes are not homothetic. The conjectured conditions for $p = 0$ are

Lemma 7. *Two convex bodies $K, T \in \mathscr{K}_0^n$ are* similar *if there is an invertible linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ and bodies $K_1, \ldots, K_m$ and scalars $\alpha_1, \ldots, \alpha_m$ (for some integer $m \geq 1$), such that $K = L\big(K_1 \times \ldots \times K_m\big)$ and $T = L\big(\alpha_1 K_1 \times \ldots \times \alpha_m K_m\big)$.*

For each $n \geq 1$, the following are equivalent:

- *For every $K, T \in \mathscr{K}_0^n$ and $0 < \lambda < 1$,*
  $$|(1-\lambda)\cdot K +_0 \lambda \cdot T| \geq |K|^{1-\lambda} |T|^\lambda,$$
  with equality only K and if K and T are similar.
- *For every $K, T \in \mathscr{K}_0^n$, $\bar{V}_p(K,T) \geq \left(\frac{|T|}{|K|}\right)^{1/n}$, with equality only if K and T are similar.*

Proof. Similar to above. ☐

L_p-Minkowski problem. The measure $h_K^{1-p} dS_K$ determines the functional $V_p(K, \cdot)$. The Minkowski problem is to identify which measures on S^{n-1} arise this way. The result for $p = 1$ is well-known: a positive Borel measure μ on S^{n-1} is S_K for some $K \in \mathscr{K}^n$ if and only if it is not concentrated on a linear subspace, and if $\int_{S^{n-1}} x \, d\mu(x) = 0$. The techniques we use for $p \neq 1$ are limited to the centrally symmetric case. We follow the proof from [14], inspired by the approach of [1]. For $p > 1$, there are results in the non-even case, attained by using different techniques.

Theorem 8. *Let μ be a positive Borel measure on S^{n-1}, which is not supported on a linear subspace, and is even: $\mu(A) = \mu(-A)$ for all measurable sets $A \subset S^{n-1}$. If $p \in (0,n) \cup (n,\infty)$, then there is a body $K \in \mathscr{K}_{\mathrm{e}}^n$ such that $d\mu = h_K^{1-p} dS_K$.*

Remark. *It is clear that the non-degenerate support condition is necessary. The value $p = n$ is excluded for homogeneity reasons: homothety does not change $h_K^{1-n} dS_K$. If we allow multiplication of the measure by a scalar, the theorem will hold in this case as well.*

Proof. Denote by $C_{\mathrm{e}}^{+}\big(S^{n-1}\big)$ the set of continuous, positive, even functions $(f(x) = f(-x))$ on S^{n-1}. The volume of $f \in C_{\mathrm{e}}^{+}\big(S^{n-1}\big)$ is defined as $|f| = |K|$, where $K \in \mathscr{K}_{\mathrm{e}}^{n}$ is the Aleksandrov body of f. We investigate the problem of minimizing

$$\Phi(f) = |f|^{-p/n} \int_{S^{n-1}} f(x)^p \, d\mu(x)$$

over functions $f \in C_{\mathrm{e}}^{+}\big(S^{n-1}\big)$.

Note that Φ is 0-homogeneous: $\Phi(t f) = \Phi(f)$ for $t > 0$. We also note that Φ is continuous. The infimum of Φ is reached through support functions of convex bodies, since if $f \in C_{\mathrm{e}}^{+}\big(S^{n-1}\big)$, and K is the Aleksandrov body of f, then $h_K \le f$, and so $\Phi(h_K) \le \Phi(f)$.

The infimum of Φ, then, is reached as the limit of a sequence $\Phi\big(h_{K_j}\big)$, where $K_j \in \mathscr{K}_{\mathrm{e}}^{n}$, and $|K_j| = 1$ and $\Phi\big(h_{K_j}\big)$ decreases with $j \ge 1$. We will now show that the diameters of the bodies K_j are bounded. If $x_j \in K_j$ is a sequence of points, then $[-x_j, x_j] \subset K_j$, and $h_{K_j}(u) \ge |x_j \cdot u|$ for all $u \in S^{n-1}$. We have

$$\begin{aligned} |x_j|^p \cdot \int_{S^{n-1}} |v_j \cdot u|^p \, d\mu(u) &= \int_{S^{n-1}} |x_j \cdot u|^p \, d\mu(u) \\ &\le \int_{S^{n-1}} h_{K_n}^p d\mu = \Phi(h_{K_n}) \le \Phi(h_{K_1}), \end{aligned}$$

where $v_j = x_j / |x_j|$. The integral $\int_{S^{n-1}} |v \cdot u|^p \, d\mu(u)$ is a continuous, positive function of u and v, so it is bounded away from 0, and therefore the sequence $|x_j|$ is bounded.

By the Blaschke selection theorem, some subsequence K_{j_r} has a limit $K \in \mathscr{K}_{\mathrm{e}}^{n}$, and this has $|K| = 1$ and the minimum of Φ is attained on h_K. Our next step is to show that $d\mu = h_K^{1-p} dS_K$.

Let $\chi \in C\big(S^{n-1}\big)$ be an even function. Consider the variation

$$f(t, u) = \left(h_K(u)^p + t\,\chi(u)\right)^{1/p}.$$

The function $t \mapsto \Phi(f(t, \cdot))$ is differentiable, and attains a minimum at 0. Let K_t be the Aleksandrov body of $f(t, \cdot)$. By Lemma 3,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \Phi(f(t, \cdot)) &= \frac{d}{dt}\left[\left|\left(h_K^p + t\chi\right)^{1/p}\right|^{-p/n} \int_{S^{n-1}} \left(h_K^p + t\chi\right) d\mu\right]_{t=0} \\ &= -\frac{1}{n}\left(\int_{S^{n-1}} \chi\, h_K^{1-p} dS_K\right) \int_{S^{n-1}} h_K^p d\mu + \int_{S^{n-1}} \chi \, d\mu. \end{aligned}$$

The derivative must vanish at a minimum, therefore

$$\int_{S^{n-1}} \chi \, d\mu = \frac{1}{n} \int_{S^{n-1}} h_K^p d\mu \cdot \int_{S^{n-1}} \chi\, h_K^{1-p} dS_K.$$

Scaling K by some constant gives $\int_{S^{n-1}} \chi \, d\mu = \int_{S^{n-1}} \chi\, h_K^{1-p} dS_K$ for every even $\chi \in C\big(S^{n-1}\big)$. This means that $d\mu = h_K^{1-p} dS_K$. □

In the $p = 0$ case, there is a similar result, but the non-degeneracy condition is different, see [3] for details. In this case, the measure $h_K dS_K$ also has a geometric meaning: it is a constant multiple of the cone-volume measure.

It is worth mentioning that there is an important difference between the case $p = 1$ and the other cases, which is obscured by the symmetry condition. The measure S_K is not sensitive to translations, so the measure identifies the body only up to n degrees of freedom, and this is reflected in the n conditions for existence $\int x\, d\mu = 0$. These are not necessary for $p > 1$ (see [7]), and are conjectured not to be necessary for $0 \leq p < 1$.

Uniqueness. A related question is whether two different bodies K_1, K_2 can give rise to the same measures: $h_{K_1}^{1-p} dS_{K_1} = h_{K_2}^{1-p} dS_{K_2}$. For $p = 1$, the answer is that this happens if and only if K_1, K_2 are translates of each other.

For $p \neq 1$, the first result is [10], asserting that among centrally symmetric sets, the ones with uniform cone-volume measure are balls.

This problem is related to the Brunn-Minkowski and Minkowski equalities, for L_p combinations with the same p. The equality conditions are very significant for this problem.

Theorem 9. *Let $p > 0$ and $n \geq 1$. The following are equivalent:*

- *For every $K, T \in \mathscr{K}_0^n$, $\bar{V}_p(K,T) \geq \left(\frac{|T|}{|K|}\right)^{1/n}$, with equality if and only if K and T are homothetic.*
- *If $K, T \in \mathscr{K}_0^n$ and $h_K^{1-p} dS_K = h_T^{1-p} dS_T$, then $K = T$.*

Proof. In one direction, we assume the L_p-Minkowski inequality and that $h_K^{1-p} dS_K = h_T^{1-p} dS_T$. Then

$$|K| = \frac{1}{n} \int h_K^p h_K^{1-p} dS_K = \frac{1}{n} \int h_K^p h_T^{1-p} dS_T = |T| \, \bar{V}_p(T,K)^p \geq |T|^{1-\frac{p}{n}} \, |K|^{p/n},$$

and therefore $|K| \geq |T|$. The same holds in the opposite direction, so $|K| = |T|$. But this means that equality holds in the L_p-Minkowski inequality used above, so actually $K = T$.

In the other direction, we assume uniqueness in the L_p-Minkowski problem, and consider the values of $\bar{V}_p(K,T)$ among the convex bodies $T \in \mathscr{K}_0^n$ with $|T| = |K|$. Since

$$\bar{V}_p(K,T) = \left(\frac{1}{n\,|K|} \int_{S^{n-1}} h_T^p h_K^{1-p} dS_K \right)^{1/p},$$

the considerations from the proof of Theorem 8 apply. In particular, minima exist, and correspond to bodies with $h_T^{1-p} dS_T = h_K^{1-p} dS_K$, which means that $K = T$. Therefore $\bar{V}_p(K,T) > 1$ whenever $|K| = |T|$ and $K \neq T$. Since the functional $\bar{V}(K, \cdot)$ is positive-homogeneous, the L_p-Minkowski inequality with the equality condition follows for general T. $\square$

It is conjectured that both these properties hold. As before, for $p = 0$, it is conjectured that they hold after a correction for the parallelepiped case. With this correction, a similar proof shows the equivalence of the L_0-Minkowski inequality to the uniqueness of the solution to the L_0-Minkowski problem.

Results in $\mathbb{R}^2$. It is not known whether the L_p-Brunn-Minkowski inequality holds in general for any $p < 1$. However, in [4] it is shown that for bodies in $\mathscr{K}_e^2$, the solution of the L_0-Minkowski problem is unique (and thus the L_0-inequalities hold). The idea of the proof first appeared in [12], where the theorem was proved for bodies with smooth boundaries, and was adapted to polygons in [18].

Theorem 10. *If $K, T \in \mathscr{K}_e^2$ and $V_K = V_T$, then either $K = T$, or K and T are parallelograms with parallel sides.*

Remark. *It follows that the L_p-Brunn-Minkowski and L_p-Minkowski inequalities hold, for any $p \geq 0$ and bodies in $\mathscr{K}_e^2$, and the equality cases are as conjectured. It also follows that the solution for the L_p-Minkowski problem is unique (for even measures on S^1).*

In order to prove this result, we need a version of Bonessen's inequality [2]. For completeness, a simplified statement and its proof are produced here (the proof follows [11]).

Lemma 11. *Let $K, T \in \mathscr{K}_e^2$ such that $\partial K \cap \partial T \neq \emptyset$. Then*

$$|K| - 2V(K,T) + |T| \leq 0,$$

and equality holds only if there is a segment $L \subset \mathbb{R}^2$ such that $K = T + L$ or $T = K + L$.

Proof. Volume and mixed volume vary continuously with the bodies. Therefore, it is enough to verify the inequality for a dense subset. We may assume then that K and T are polygons whose edges have normals $\pm u_1, \dots, \pm u_N$. For $x \in \mathbb{R}^2$, let $f(x)$ be the cardinality of the set $(\partial K + x) \cap \partial T$. If $x \notin K + T$, then $K + x$ and T are disjoint, so $f(x) = 0$. However, if $x \in K + T$, then $f(x) > 0$: from $\partial K \cap \partial T \neq \emptyset$ it follows that the widths $h_K(u) + h_K(-u) = h_T(u) + h_T(-u)$ are equal for some $u \in S^1$, and hence it is impossible for a translation of K to be contained in the interior of T or vice versa. It is easy to see that $f(x) \neq 1, \infty$ almost everywhere, therefore

$$\iint_{\mathbb{R}^2} f(x)\,dx \geq 2|K+T| = 2|K| + 4V(K,T) + 2|T|.$$

Let T_i be the edge of T with outer normal u_i. The contribution to $\iint_{\mathbb{R}^2} f(x)\,dx$ from intersections inside T_i comes from $x \in \partial K + T_i$, and is equal to the area $|\partial K + T_i| = (h_K(u_i) + h_K(-u_i)) \cdot 2\,\mathrm{length}(T_i)$. Summing these up, we get

$$\iint_{\mathbb{R}^2} f(x)\,dx = 2\sum_{i=1}^{N}\left(h_K(u_i)+h_K(-u_i)\right)\cdot 2\,\mathrm{length}(T_i)$$
$$= \int_{S^1} 4h_K(u)\,dS_L(u) = 8V(K,T).$$

Combining the estimate with this calculation, we see that

$$\begin{aligned} 8V(K,T) &\geq 2|K| + 4V(K,T) + 2|T| \\ \implies 0 &\geq |K| - 2V(K,T) + |T|. \end{aligned}$$

Let us find conditions for equality. If neither of K, T is inscribed in the other, there are $v_1, v_2 \in S^1$ such that $K \cap \mathbb{R}v_1 \subsetneq T \cap \mathbb{R}v_1$ and $K \cap \mathbb{R}v_2 \supsetneq T \cap \mathbb{R}v_2$, so there are at least 4 points in $|\partial K \cap \partial T|$. Slight perturbation of K does not change this condition, and therefore $f(x) > 2$ on a set of positive measure, so there cannot be equality.

If one set is contained in the other, say $K \subset T$, we denote by L the maximal set that satisfies $K + L \subset T$, or $L = \left\{x \in \mathbb{R}^2 : K + x \subset T\right\}$. The set $K + L$ satisfies the condition of this lemma, so $|K| - 2V(K, K+L) + |K+L| \leq 0$, or $|L| \leq 0$, and hence L is a segment or $\{0\}$. For any $u \in S^1$, if $h_K(u) = h_T(u)$, then L is perpendicular to u. The mixed volume $V(K,L)$ is $h_K(u)\cdot \mathrm{length}(L)$, so $V(K,L) = V(T,L) = V(K+L,L)$.

Since T does not contain a Minkowski sum of $K+L$ with any set beside $\{0\}$, there are some $v_1 \neq \pm v_2 \in \mathrm{supp}(S_T)$ such that $h_T(v_1) = h_{K+L}(v_1)$ and $h_T(v_2) = h_{K+L}(v_2)$. If $K + L \neq T$, then there is some $u \in (v_1, v_2)$ with $h_T(u) > h_{K+L}(u)$.

Take $x \in \partial T$ with outer normal u. Then for y in some neighborhood of $-\frac{h_T(u)-h_{K+L}(u)}{2}u$, we have $x+y, -x+y \in T\backslash(K+L)$ but if $z \in \partial T \cap \partial(K+L)$ has outer normal v_1 or v_2, then $z+y, -z+y \in K+L$. Therefore, there is a set of positive measure where $|\partial(T+y) \cap \partial(K+L)| \geq 4$, and thus

$$|K| - 2V(K,T) + |T| = |K+L| - 2V(K+L,T) + |T| < 0.$$

Hence, if $|K| - 2V(K,T) + |T| = 0$, then $K + L = T$.

$\square$

We can now prove the theorem.

Proof of Theorem 10. For any $u \in S^1$, the bodies K and $\frac{h_K(u)}{h_T(u)}T$ satisfy the condition of Lemma 11, so

$$|K| - 2\frac{h_K(u)}{h_T(u)}V(K,T) + \frac{h_K(u)^2}{h_T(u)^2}|T| \leq 0,$$

and integrating with respect to the measure $h_T dS_K$ yields

$$\int_{S^1} \frac{h_K(u)^2}{h_T(u)} dS_K(u) \leq 2\frac{|K|}{|T|}V(K,T).$$

Since $V_K = V_T$, we have in particular $|K| = |T|$, so we can rewrite the above as

$$\int_{S^1} \frac{h_K}{h_T} dV_K \le \int_{S^1} \frac{h_T}{h_K} dV_K.$$

Applying this repeatedly, we have

$$\int_{S^1} \frac{h_K}{h_T} dV_K \le \int_{S^1} \frac{h_T}{h_K} dV_K = \int_{S^1} \frac{h_T}{h_K} dV_T \le \int_{S^1} \frac{h_K}{h_T} dV_T = \int_{S^1} \frac{h_K}{h_T} dV_K,$$

so equality each time we apply Lemma 11. Thus for every $u \in \operatorname{supp}(S_K)$,

$$|K| - 2\frac{h_K(u)}{h_T(u)} V(K,T) + \frac{h_K(u)^2}{h_T(u)^2} |T| = 0.$$

If $K \neq T$, one of $K, \frac{h_K(u)}{h_T(u)} T$ is inscribed in the other for every such u, and therefore $\frac{h_K(u)}{h_T(u)}$ takes only two values for $u \in \operatorname{supp}(S_K)$. We can write $K = a_1 T + L_1$, $T = a_2 K + L_2$ for $L_1, L_2 \subset \mathbb{R}^2$ segments. If follows that $K = \frac{L_1 + a_1 L_2}{1 - a_1 a_2}$ and $T = \frac{a_2 L_1 + L_2}{1 - a_1 a_2}$, two parallelograms with parallel edges. □

Property (B). It was asked by Banaszczyk (see [13]) whether for any $K \in \mathscr{K}_{\mathrm{e}}^n$, $\lambda \in [0,1]$ and $s, t > 0$,

$$\mu\left(s^{1-\lambda} t^{\lambda} K\right) \ge \mu(sK)^{1-\lambda} \mu(tK)^{\lambda},$$

where μ is the Gaussian measure

$$\mu(A) = \frac{1}{(2\pi)^{n/2}} \int_A e^{-\frac{1}{2}|x|^2} dx.$$

A strong version of this property was proved in [8]:

Definition 12. *Let μ be a positive Borel measure on $\mathbb{R}^n$.*

- *If for every $K \in \mathscr{K}_{\mathrm{e}}^n$, the function $f(t) = \mu\left(e^t K\right)$ is log-concave, we say that μ has the weak* (B) *property.*
- *If for every $K \in \mathscr{K}_{\mathrm{e}}^n$, the function*

$$f(t_1, \ldots, t_n) = \mu\big(D\left(e^{t_1}, \ldots, e^{t_n}\right) K\big)$$

is log-concave, where $D(a_1, \ldots, a_n)$ is a diagonal matrix with these eigenvalues, we say that μ has the strong (B) *property.*

Theorem (Theorem 1 in [8])**.** *For any $n \ge 1$, the standard Gaussian measure on $\mathbb{R}^n$ has the strong* (B) *property.*

In addition, several special cases of the (B) property were proved, for some measures and subclasses of $\mathscr{K}_{\mathrm{e}}^n$.

Theorem 13 (Propositions 7 and 10 in [8])**.** *Let μ be a log-concave measure on $\mathbb{R}^n$.*

- *If n is even, we identify $\mathbb{R}^n = \mathbb{C}^{n/2}$. A measure μ or body K is called circled if it is invariant under phase rotations* (*multiplication by a scalar $e^{i\theta}$ for $\theta \in \mathbb{R}$*)*. If μ and $K \in \mathscr{K}^n$ are circled, then $f(t) = \mu\left(e^t K\right)$ is log-concave.*

- *A measure μ or body K is called unconditional if it is invariant under reflection of each coordinate. If $K, T \subset \mathbb{R}^n$ are closed and unconditional, define for $\lambda \in [0,1]$*

  $$K^{1-\lambda}T^{\lambda} = \left\{ w \in \mathbb{R}^n : \exists x \in K, y \in T.\ |w_j| = |x_j|^{1-\lambda} |y_j|^{\lambda} \quad \forall j = 1, \dots, n \right\}.$$

  Then the function $f(\lambda) = \mu\left(K^{1-\lambda}T^{\lambda}\right)$ is log-concave.

Remark. *Circled bodies are the unit balls of norms in finite-dimensional complex normed spaces. The first result is proved using the theory of interpolation of Banach spaces. The second result is a direct consequence of the logarithmic Prékopa-Leindler inequality.*

If the bodies K, L are unconditional and convex, the combination $K^{1-\lambda}T^{\lambda}$ is contained in $(1-\lambda)\cdot K +_0 \lambda \cdot T$: if $u \in S^{n-1}$,

$$\begin{aligned} h_{K^{1-\lambda}T^{\lambda}}(u) &= \max_{w \in K^{1-\lambda}T^{\lambda}} \sum_{j=1}^{n} w_j u_j \leq \max_{x \in K, y \in T} \sum_{j=1}^{n} |x_j u_j|^{1-\lambda} |y_j u_j|^{\lambda} \\ &\leq h_K(u)^{1-\lambda} h_T(u)^{\lambda}. \end{aligned}$$

In particular, if μ is the Lebesgue measure, the L_0-Brunn-Minkowski inequality for unconditional bodies follows from Theorem 13.

Equivalence. As hinted above, the (B) property and the L_0-Brunn-Minkwoski inequality are closely connected. In [16], the following was proved.

Theorem 14 (Theorems 1.5 and 1.6 in [16])**.** *Let $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ be the n-dimensional centered unit cube.*

If the L_0-Brunn-Minkowski inequality holds for n-dimensional bodies, then:

- *The uniform measure on Q_n has the strong* (B) *property.*
- *The uniform measure on any $K \in \mathscr{K}_{\mathrm{e}}^n$ has the weak* (B) *property.*
- *Any centrally-symmetric log-concave measure has the weak* (B) *property.*

If for every $n \geq 1$ the uniform measure on Q_n has the strong (B) *property, then the L_0-Brunn-Minkowski inequality is true for centrally-symmetric bodies in all dimensions.*

From Theorem 14, together with Theorem 10, Lemma 7, and the L_0 equivalent of Theorem 9 , it follows that the L_0-Brunn-Minkowski inequality holds in two dimensions. Therefore, symmetric uniform measures on $\mathbb{R}^2$ have the weak (B) property, and the uniform measure on the square Q_2 has the strong (B) property.

2. New Results

In this chapter, I give a direct proof of the weak (B) property of uniform measures in two dimensions, explain how the same method can prove the strong (B) property, and show a version with a different symmetry condition.

(B) property in $\mathbb{R}^2$. We shall prove the

Theorem 15. *Let $K, L \subset \mathbb{R}^2$ be centrally-symmetric convex shapes. Then the function $f_{K,L}(t) = |e^t K \cap L|$ is log-concave.*

Obviously, it suffices to show log-concavity near $t = 0$.

If we consider the space of centrally-symmetric convex shapes in the plane, equipped with the Hausdorff metric d_H, then the operations $K, L \mapsto K \cap L$ and $K \mapsto |K|$ are continuous. This means that the correspondence $K, L \mapsto f_{K,L}$ is continuous as well. Since the condition of log-concavity in the vicinity of a point is a closed condition in the space $C(\mathbb{R})$ of bounded continuous functions, the class of pairs of centrally-symmetric shapes $K, L \subset \mathbb{R}^2$ for which $f_{K,L}(t)$ is log-concave near $t = 0$ is closed with respect to Hausdorff distance. Thus in order to prove Theorem 15 it suffices to prove that $f_{K,L}(t)$ is a log-concave function near $t = 0$ for a dense set in the space of pairs of centrally-symmetric convex shapes.

As a dense subset, we shall pick the class of transversely-intersecting convex polygons. This class will be denoted by $\mathcal{F}$. The elements of $\mathcal{F}$ are pairs (K, L) of shapes $K, L \subset \mathbb{R}^2$ that satisfy:

- (K, L) is a pair of centrally-symmetric convex polygons in $\mathbb{R}^2$.
- The intersection $\partial K \cap \partial L$ is finite.
- None of the points $x \in \partial K \cap \partial L$ are vertices of K or of L. That is, there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \cap \partial K$ and $B(x, \varepsilon) \cap \partial L$ are line segments.
- For every $x \in \partial K \cap \partial L$, $\nu_K(x) \neq \nu_L(x)$.

Claim 16. *The class $\mathcal{F}$ is dense in the space of pairs of centrally-symmetric convex shapes (with respect to the Hausdorff metric).*

Hence, in order to prove Theorem 15, it is enough to consider polygons with transversal intersection.

Deriving a concrete inequality.

Lemma 17. *If $(K, L) \in \mathcal{F}$, then $f_{K,L}(t)$ is twice differentiable in some neighbourhood of $t = 0$.*

Remark. *In this case, log-concavity near $t = 0$ amounts to the inequality*

$$\frac{d^2}{dt^2} \log f(t)\Big|_{t=0} \leq 0,$$

i.e.

$$f(0) \cdot f''(0) \leq f'(0)^2. \tag{1}$$

Proof. The area of the intersection is

$$|aK \cap L| = \int_0^a dr \int_{x \in r\partial K \cap L} h_K(\nu_K(\tfrac{x}{r}))d\ell,$$

where $d\ell$ is the length element.

Denote

$$g_{K,L}(r) = \int_{x \in r\partial K \cap L} h_K(\nu_K(\tfrac{x}{r}))d\ell.$$

The transversality of the intersection implies that $g_{K,L}(r)$ is continuous near $r = 1$. Therefore, $a \mapsto |aK \cap L|$ is continuously differentiable near $a = 1$.

The contour $r\partial K \cap L$ is a finite union of segments in $\mathbb{R}^2$. Transversality implies that the number of connected components does not change with r in a small neighbourhood of $r = 1$. The beginning and end points of each component are smooth functions of r, also in some neighbourhood of $r = 1$. Therefore, $g_{K,L}(r)$ is differentiable, as claimed. □

Note that in such a neighbourhood of $r = 1$, the function $g_{K,L}(r)$ only depends on the parts of K and L that are close to $\partial K \cap L$, and is in fact a sum of contributions from each of the connected components.

Writing (1) in terms of $g(r)$, we get the following condition:

Definition 18. *For convex shapes $(K, L) \in \mathcal{F}$, we say that K and L satisfy property B, or that $B(K, L)$, if*

$$|K \cap L| \cdot [g_{K,L}(1) + g'_{K,L}(1)] \leq g_{K,L}(1)^2. \tag{2}$$

Transversal intersection is an open condition on the space of polygons, and therefore, if $(K, L) \in \mathcal{F}$ then $(K, rL) \in \mathcal{F}$ for every r in some neighbourhood of $r = 1$. If $B(K, rL)$ holds for every r in such a neighbourhood, then $f_{K,L}(t)$ is log-concave in some neighbourhood of $t = 0$, as

$$f_{K,L}(t_0 + t) = e^{2t_0} f_{K,e^{-t_0}L}(t).$$

Therefore, verifying (2) for all pairs $(K, L) \in \mathcal{F}$ will prove Theorem 15.

Reduction to parallelograms. Given two polygons $(K, L) \in \mathcal{F}$, the intersection $\partial K \cap L$ consists of a finite number of connected components. Due to central symmetry, they come in opposite pairs. We denote these components by $S_1, \ldots, S_{2n}$, and $S_{i+n} = \{-x : x \in S_i\}$.

We define a pair of convex shapes $K^{(i)}, L^{(i)}$ for each $1 \leq i \leq n$ via the following properties.

- The shape $K^{(i)}$ is the largest convex set whose boundary contains $S_i \cup S_{i+n}$. Equivalently, denoting by x_1, x_2 the endpoints of S_i, and by x the solution of the equations

$$\begin{cases} \langle \nu_K(x_1), x \rangle = h_K(\nu_K(x_1)), \\ \langle \nu_K(x_2), x \rangle = -h_K(\nu_K(x_2)), \end{cases}$$

  we have $K^{(i)} = \operatorname{conv}(S_i \cup S_{i+n} \cup \{x, -x\})$.

- The shape $L^{(i)}$ is the parallelogram defined by the four lines

$$\langle \nu_L(x_1), x\rangle = \pm h_L(\nu_L(x_1)) \quad , \quad \langle \nu_L(x_2), x\rangle = \pm h_L(\nu_L(x_2))$$

See Figure 1 for examples.

If S_i is a segment, then $K^{(i)}$ described above is an infinite strip, and if $\nu_L(x_1) = \nu_L(x_2)$, then $L^{(i)}$ is an infinite strip. We would like to work with compact shapes, thus we apply a procedure to modify $K^{(i)}, L^{(i)}$ to become bounded without changing their significant properties. Transversality implies that the intersection $K^{(i)} \cap L^{(i)}$ is bounded, even if both sets are strips. For each $1 \leq i \leq n$ we pick a centrally-symmetric strip $A \subset \mathbb{R}^2$ such that $A \cap K^{(i)}$ and $A \cap L^{(i)}$ are both bounded, and which contains K and L, and whichever of $K^{(i)}, L^{(i)}$ that is bounded. From now on we replace $K^{(i)}$ and $L^{(i)}$ by their intersection with A.

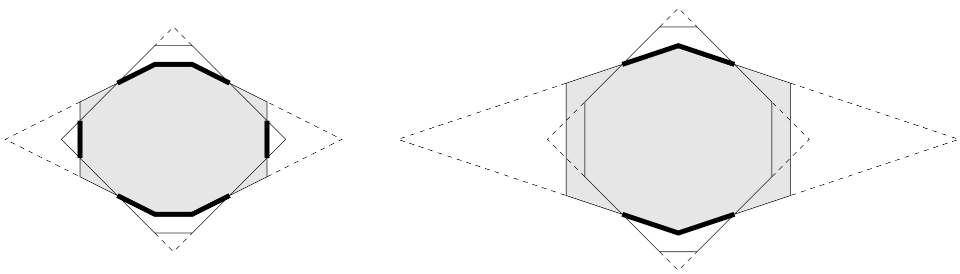

FIGURE 1. Two examples of the extension $K, L \implies K^{(i)}, L^{(i)}$. The shaded shape in each diagram is K and the white shape with a solid boundary line is the corresponding L.

Remark. *Note that the sets grow in the described process: $K \subset K^{(i)}$ and $L \subset L^{(i)}$ for all $i = 1, \ldots, n$. They satisfy $\partial K^{(i)} \cap L^{(i)} = S_i \cup S_{i+n}$. Also note that if K is a parallelogram, then so is $K^{(i)}$, for every i. It is clear that $(K^{(i)}, L^{(i)}) \in \mathcal{F}$ when $(K, L) \in \mathcal{F}$.*

Lemma 19. *If $B(K^{(i)}, L^{(i)})$ for all $i = 1, \ldots, n$, then $B(K, L)$.*

Proof. The function $g_{K,L}(r)$ takes non-negative values for $r > 0$. In addition, its value is the sum of contributions from the different connected components of $r\partial K \cap L$. Thanks to transversality, these components vary continuously around $r = 1$, hence $g'_{K,L}(1)$ is also a sum of values coming from the different components. Therefore, we can write

$$|K\cap L| \cdot [g_{K,L}(1) + g'_{K,L}(1)] = |K \cap L| \cdot \sum_{i=1}^{n} \left[g_{K^{(i)},L^{(i)}}(1) + g'_{K^{(i)},L^{(i)}}(1)\right]$$

$$\leq \sum_{i=1}^{n} |K^{(i)} \cap L^{(i)}| \cdot \left[g_{K^{(i)},L^{(i)}}(1) + g'_{K^{(i)},L^{(i)}}(1)\right]$$

$$\underset{\text{by} B(K^{(i)},L^{(i)})}{\leq} \sum_{i=1}^{n} g_{K^{(i)},L^{(i)}}(1)^2 \leq \left(\sum_{i=1}^{n} g_{K^{(i)},L^{(i)}}(1)\right)^2 = g_{K,L}(1)^2. \quad \square$$

Lemma 20. *If $B(K, L)$ holds for all pairs of parallelograms $(K, L) \in \mathcal{F}$, then Theorem 15 follows.*

Proof. Let $(K, L) \in \mathcal{F}$ be any polygons. Construct the sequence of pairs $K^{(i)}, L^{(i)}$ from K, L. The shape $L^{(i)}$ is a parallelogram for every i. Then construct the pairs $\left(L^{(i)}\right)^{(j)}, \left(K^{(i)}\right)^{(j)}$ from $L^{(i)}, K^{(i)}$, for all i. The shapes $\left(L^{(i)}\right)^{(j)}$ and $\left(K^{(i)}\right)^{(j)}$ will be parallelograms for every i, j. Under our assumption, we have $B\left(\left(L^{(i)}\right)^{(j)}, \left(K^{(i)}\right)^{(j)}\right)$. From this and the previous lemma, $B(L^{(i)}, K^{(i)})$ follows.

The property B is symmetric in the shapes. That is, $B(S, T) \Longleftrightarrow B(T, S)$ for all $(S, T) \in \mathcal{F}$. This is since $f_{S,T}$ and $f_{T,S}$ differ by a log-linear factor:

$$f_{S,T}(t) = |e^t S \cap T| = e^{2t} f_{T,S}(-t)$$

This means that we have $B(K^{(i)}, L^{(i)})$ as well. Applying the previous lemma again gives $B(K, L)$. $\square$

All that remains in order to deduce Theorem 15 is to analyze the case of centrally-symmetric parallelograms.

If K, L are parallelograms and $K = TQ$, where T is an invertible linear map and $Q = [-1, 1] \times [-1, 1]$, then

$$f_{K,L} = \det T \cdot f_{Q, T^{-1}L}.$$

Therefore, we can take one of the parallelograms to be a square. In other words, establishing $B(Q, L)$ where Q is the unit square and L is a parallelogram, and $(Q, L) \in \mathcal{F}$, will imply Theorem 15.

In fact, we may place additional geometric constraints on the square and the parallelogram.

If neither Q nor L contains a vertex of the other quadrilateral in its interior, then $\partial Q \cap L$ has four connected components. Applying the reduction above to Q, L gives $Q^{(i)}, L^{(i)}$ with $i = 1, 2$, and the intersection $\partial Q^{(i)} \cap L^{(i)}$ has only two connected components, as remarked above.

Since the shapes are convex, if all the vertices of one shape are contained in the other, we have $Q \subset L$ or $L \subset Q$, and then (2) holds trivially. If L contains vertices of Q, but Q does not contain vertices of L, we swap them.

These arguments leave two cases to be considered:

(1) Q contains two vertices of L, and L does not contain vertices of Q. In this case the intersection $\partial Q \cap L$ is contained in two opposite edges of Q.
This case is proved in Lemma 21 below.

(2) Q contains two vertices of L, and L contains two vertices of Q. In this case the intersection $\partial Q \cap L$ is a subset of the edges around these vertices of Q.
This case is proved in Lemma 22 below.

Computation of the special cases. These cases are defined by 4 real parameters – the coordinates of the vertices of L. A symbolic expression for $f(t)$ can be derived, and (2) will be a polynomial inequality in these parameters. The geometric conditions given above are also polynomial inequalities in these parameters. Thus each of the two cases can be expressed by a universally-quantified formula in the language of real closed fields. By Tarski's theorem [19], this first-order theory has a decision procedure. This is implemented in the QEPCAD B computer program [5]. Relevant computer files, for generation of the symbolic condition and for running the logic solver, for one of the two cases above, are available at `http://www.tau.ac.il/~livnebaron/files/bconj_201311/bconj_corners.mac` and `http://www.tau.ac.il/~livnebaron/files/bconj_201311/bconj_qelim.txt`.

A human-readable proof of both cases is included here as well.

Lemma 21. *If L is a centrally-symmetric parallelogram such that $(Q, L) \in \mathcal{F}$, and if L crosses Q only inside the vertical edges of Q, then $B(Q, L)$.*

Proof. Let α, β, c, d be as in Figure 2.

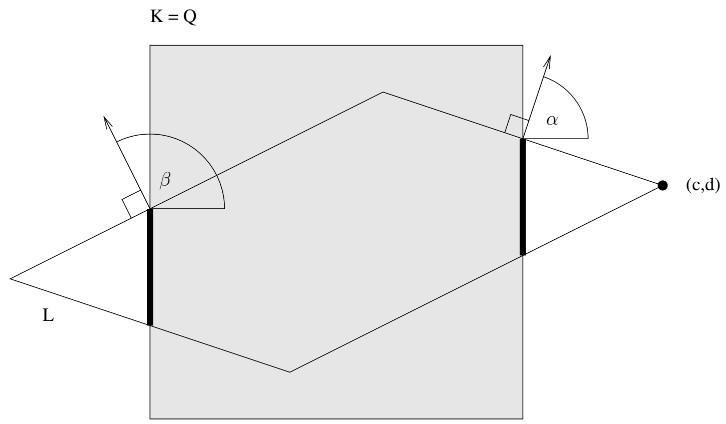

FIGURE 2

The equations for the edges of L are

$$\begin{cases} x\cos\alpha + y\sin\alpha &= \pm(c\cos\alpha + d\sin\alpha), \\ x\cos\beta + y\sin\beta &= \pm(c\cos\beta + d\sin\beta). \end{cases}$$

Relevant parameters are computed as follows:

$$\begin{aligned} \partial Q \cap \partial L &= \left\{\pm\big(1, (c-1)\cot\alpha + d\big), \pm\big(1, (c-1)\cot\beta + d\big)\right\}, \\ g_{Q,L}(1) &= 2(c-1)(\cot\alpha - \cot\beta), \\ g'_{Q,L}(1) &= -2(\cot\alpha - \cot\beta), \\ g_{Q,L}(1) + g'_{Q,L}(1) &= (2c-4)(\cot\alpha - \cot\beta). \end{aligned}$$

The area of L is comprised of $Q \cap L$ and of two triangles. The area of the triangles is $\frac{1}{2}g(1) \cdot (c-1)$, so

$$|Q \cap L| = |L| - (c-1)^2(\cot\alpha - \cot\beta).$$

Note that $0 < \alpha < \frac{\pi}{2} < \beta < \pi$, so $\cot\alpha - \cot\beta$ is a positive quantity, and that if $c < 2$ the value of $g(1)+g'(1)$ is negative, so inequality (2) is satisfied immediately.

Assume $c \geq 2$ from now on. What we need to prove is

$$(2c-4)(\cot\alpha-\cot\beta)\cdot\left[|L|-(c-1)^2(\cot\alpha-\cot\beta)\right] \leq 4(c-1)^2(\cot\alpha-\cot\beta)^2.$$

Or, equivalently,

$$(2c-4)|L| \leq (c-1)^2(\cot\alpha - \cot\beta)\cdot(4+2c-4),$$

or still

$$|L| \leq \left(1+\tfrac{2}{c-2}\right)\cdot\tfrac{1}{2}(c-1)g(1).$$

The amount $\frac{1}{2}(c-1)g(1)$ is the area of the triangles $L\backslash Q$. If each of the triangles in $L\backslash Q$ is blown up by a factor of $\left(\frac{c}{c-1}\right)^2$, until it touches the origin, together they cover L. Therefore, the area of L is bounded by $\left(\frac{c}{c-1}\right)^2\cdot\frac{1}{2}(c-1)g(1)$. It remains to verify that for $c \geq 2$, $\frac{c^2}{(c-1)^2} \leq 1+\frac{2}{c-2}$. This is a simple exercise in algebra:

$$\begin{aligned}\frac{c^2}{(c-1)^2} &= 1+\frac{2c-1}{(c-1)^2} = 1+\tfrac{2}{c-2}\cdot\frac{(c-\frac{1}{2})(c-2)}{(c-1)^2}\\ &= 1+\tfrac{2}{c-2}\left[1-\frac{c/2}{(c-1)^2}\right] < 1+\tfrac{2}{c-2}.\end{aligned}$$

□

Lemma 22. *If L is a centrally-symmetric parallelogram such that $(Q,L)\in\mathcal{F}$, and if each of Q, L contains two vertices of the other, then $B(Q,L)$.*

Proof. Let a and b be as in Figure 3, and let S stand for the area $S = |Q\cap L|$. The numbers a and b are in the range $0 < a, b < 2$, and α and β satisfy $\frac{1}{2}\pi < \alpha < \beta < \pi$. The area S is in the range $4 - ab < S < 4$.

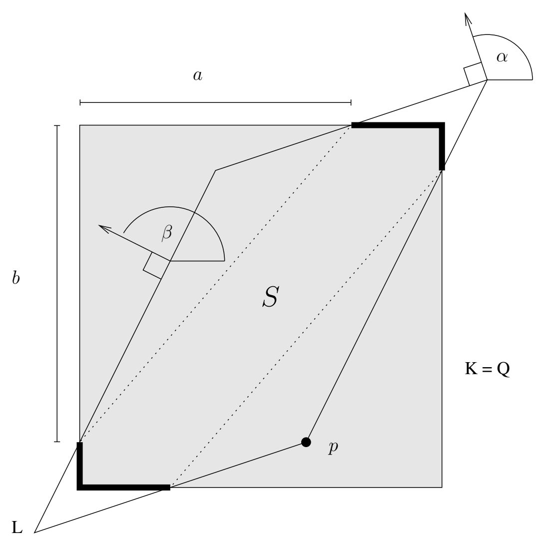

FIGURE 3

The quantity $g(1)$ is simply $8-2a-2b$, and $g'(1)$ will soon be shown to be bounded by

$$g'(1) \leq -8\frac{S-(4-ab)}{(4-S)+\frac{1}{2}(a-b)^2}.$$

This gives an inequality in the three variables a, b, S, which will be proved for values in the prescribed ranges.

The length of each dotted line in Figure 3 is $(a^2+b^2)^{1/2}$. Denoting the height of the triangle (the distance between p and the closest dotted line) by h, the area is

$$S = (4-ab) + 2\cdot\frac{1}{2}h\cdot(a^2+b^2)^{1/2},$$

so

$$h = \frac{S-(4-ab)}{(a^2+b^2)^{1/2}}.$$

The formula for $g'(1)$ in terms of the angles α, β is

$$g'(1) = 4 + 2\tan\alpha + 2\cot\beta.$$

Denote $c = \beta - \alpha$. Holding c fixed, the function

$$\alpha \mapsto g'(1) = 4 + 2\tan\alpha + 2\cot(\alpha + c)$$

is concave and takes the same value for α as for $\frac{3}{2}\pi - c - \alpha$. Therefore its maximum is attained at $\alpha = \frac{3}{4}\pi - \frac{1}{2}c$. This gives a bound for $g'(1)$ for a given $c = \beta - \alpha$:

$$g'(1) \leq 4 + 2\tan\left(\tfrac{3}{4}\pi - \tfrac{1}{2}c\right) + 2\cot\left(\tfrac{3}{4}\pi + \tfrac{1}{2}c\right).$$

This bound is stronger for higher values of c, since tan is an increasing function and cot is a decreasing function.

The angle between the edges of L meeting at p is $\pi-(\beta-\alpha) = \pi-c$. When a, b, and h are kept fixed, the position of p gives a bound for $g'(1)$. This bound is the weakest when the angle $\pi - c$ is the largest. Simple geometric considerations show that in a family of triangles with the same base and height, the apex angle is largest when the triangle is isosceles, so we will pursue the case where the triangle formed by p and the nearest dotted line is isosceles.

The value of c in this case is $c = 2\tan^{-1}\frac{\frac{1}{2}(a^2+b^2)^{1/2}}{h}$, and we get

$$\begin{aligned} g'(1) &\leq 4 + 2\tan\left(\tfrac{3}{4}\pi - \tfrac{1}{2}\pi + \tan^{-1}\frac{\frac{1}{2}(a^2+b^2)^{1/2}}{h}\right) \\ &\qquad + 2\cot\left(\tfrac{3}{4}\pi + \tfrac{1}{2}\pi - \tan^{-1}\frac{\frac{1}{2}(a^2+b^2)^{1/2}}{h}\right) \\ &= 4 + 4\tan\left(\tfrac{1}{4}\pi + \tan^{-1}\frac{1}{2}\frac{a^2+b^2}{S-(4-ab)}\right) \\ &= 4 + 4\cdot\frac{1+\frac{1}{2}\frac{a^2+b^2}{S-(4-ab)}}{1-\frac{1}{2}\frac{a^2+b^2}{S-(4-ab)}} = \frac{8}{1-\frac{1}{2}\frac{a^2+b^2}{S-(4-ab)}} = -8\frac{S-(4-ab)}{(4-S)+\frac{1}{2}(a-b)^2}, \end{aligned}$$

which proves the aforementioned bound for $g'(1)$.

Therefore, to prove (2) it is enough to show that

$$S \cdot \left(8 - 2a - 2b - 8 \frac{S - (4 - ab)}{(4 - S) + \frac{1}{2}(a - b)^2} \right) \leq (8 - 2a - 2b)^2.$$

Rearranging and taking into account that $S < 4$, this is equivalent to

$$\underbrace{(8 - 2a - 2b)(8 - 2a - 2b - S)\left((4 - S) + \tfrac{1}{2}(a - b)^2\right) + 8S(S - (4 - ab))}_{E} \geq 0.$$

When a and b are held fixed, this is a second-degree condition on S. Since $0 < a, b < 2$, the value and the first two derivatives in the point $S = 4 - ab$ are positive:

$$\begin{aligned} E|_{S=4-ab} &= (8 - 2a - 2b)(2 - a)(2 - b) \cdot \tfrac{1}{2}(a^2 + b^2) > 0, \\ \left.\frac{\partial E}{\partial S}\right|_{S=4-ab} &= (a + b)\left((5 - a - b)^2 - 1\right) + 2(a - b)^2 > 0, \\ \left.\frac{\partial^2 E}{\partial S^2}\right|_{S=4-ab} &= 16 + 2(8 - 2a - 2b) > 0. \end{aligned}$$

This means that the condition stays true for all $S > 4-ab$, as required. □

Strong (B) property. A similar technique can be used to prove the strong (B) property of the uniform measure on the square. A sketch of the proof follows.

Recall the meaning of the strong (B) property: for every $K \in \mathscr{K}_{\mathrm{e}}^n$, the function

$$f_K(t_1, t_2) = \left| Q \cap \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{t_2} \end{pmatrix} K \right|$$

is log-concave, where $Q = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ is the unit square.

As before, it is enough to verify log-concavity around $(t_1, t_2) = (0, 0)$, and we can multiply by a log-linear factor in order to study the function

$$\tilde{f}_K(t_1, t_2) = \left| K \cap [-\tfrac{1}{2}e^{t_1}, \tfrac{1}{2}e^{t_1}] \times [-\tfrac{1}{2}e^{t_2}, \tfrac{1}{2}e^{t_2}] \right| = e^{t_1 + t_2} f_K(-t_1, -t_2)$$

instead. We assume again that K and Q only intersect transversely.

The function $\tilde{f}_K$ is then differentiable in a neighbourhood of the origin, so it is log-concave if and only if the Hessian $\nabla^2(-\log \tilde{f}_K)$ is positive semi-definite at $(t_1, t_2) = (0, 0)$.

We write

$$\tilde{g}_K(a, b) = \left| K \cap [-\tfrac{1}{2}a, \tfrac{1}{2}a] \times [-\tfrac{1}{2}b, \tfrac{1}{2}b] \right|.$$

If we express $\nabla^2(-\log \tilde{f}_K)$ in terms of the partial derivatives of $\tilde{g}$, we get

$$\begin{aligned} \nabla^2(-\log \tilde{f}) &= \frac{1}{\tilde{f}^2} \begin{pmatrix} (\partial_1 \tilde{f})^2 - (\partial_{11} \tilde{f})\tilde{f} & (\partial_1 \tilde{f})(\partial_2 \tilde{f}) - (\partial_{12} \tilde{f})\tilde{f} \\ (\partial_1 \tilde{f})(\partial_2 \tilde{f}) - (\partial_{12} \tilde{f})\tilde{f} & (\partial_2 \tilde{f})^2 - (\partial_{22} \tilde{f})\tilde{f} \end{pmatrix} \\ &= \frac{1}{\tilde{f}^2} \begin{pmatrix} (\partial_1 \tilde{g})^2 - (\partial_1 \tilde{g} + \partial_{11} \tilde{g})\tilde{g} & (\partial_1 \tilde{g})(\partial_2 \tilde{g}) - (\partial_{12} \tilde{g})\tilde{g} \\ (\partial_1 \tilde{g})(\partial_2 \tilde{g}) - (\partial_{12} \tilde{g})\tilde{g} & (\partial_2 \tilde{g})^2 - (\partial_2 \tilde{g} + \partial_{22} \tilde{g})\tilde{g} \end{pmatrix}, \end{aligned}$$

where $\tilde{f}$ is evaluated at (t_1, t_2) and $\tilde{g}$ is evaluated at $(a, b) = (e^{t_1}, e^{t_2})$.

According to Sylvester's criterion, a matrix is positive semi-definite if and only if all its leading principal minors are non-negative. In our case, this means

$$\begin{aligned} &(\partial_1\tilde{g})^2 - |K\cap Q|(\partial_1\tilde{g} + \partial_{11}\tilde{g}) \geq 0 \quad \text{and} \\ &\big[(\partial_1\tilde{g})^2 - |K\cap Q|(\partial_1\tilde{g} + \partial_{11}\tilde{g})\big] \cdot \big[(\partial_2\tilde{g})^2 - |K\cap Q|(\partial_2\tilde{g} + \partial_{22}\tilde{g})\big] \\ &\qquad - \left[(\partial_1\tilde{g})(\partial_2\tilde{g}) - |K\cap Q|(\partial_{12}\tilde{g})\right]^2 \geq 0, \end{aligned}$$

evaluated at $(1,1)$. Since K and Q intersect transversely, $\partial_{12}\tilde{g}(1,1) = 0$. Using this, we can simplify these inequalities to

$$\begin{aligned} &|K\cap Q|\,(\partial_1\tilde{g} + \partial_{11}\tilde{g}) \leq (\partial_1\tilde{g})^2, \\ &(\partial_2\tilde{g})^2(\partial_1\tilde{g} + \partial_{11}\tilde{g}) + (\partial_1\tilde{g})^2(\partial_2\tilde{g} + \partial_{22}\tilde{g}) \\ &\qquad\qquad \leq |K\cap Q|\,(\partial_1\tilde{g} + \partial_{11}\tilde{g})(\partial_2\tilde{g} + \partial_{22}\tilde{g}). \end{aligned} \tag{3}$$

The first of these is the weak (B) property $B(K\cap L_1, L_2)$, where

$$\begin{aligned} L_1 &= [-M, M] \times [-\frac{1}{2}, \frac{1}{2}], \\ L_2 &= [-\frac{1}{2}, \frac{1}{2}] \times [-M, M], \end{aligned}$$

and $M > 0$ is such that $K \subset M\,Q$.

The inequality (3) does not enjoy the monotonicity or separability properties used in the reduction above. However, it depends only on the volume $|K\cap Q|$, and on the first and second derivatives of $\tilde{g}$ at $(1,1)$. As before, the first derivatives $\partial_1\tilde{g}(1,1)$ and $\partial_2\tilde{g}(1,1)$ only depend on the intersection $\partial Q\cap K$, and the second derivatives $\partial_{11}\tilde{g}(1,1)$ and $\partial_{22}\tilde{g}(1,1)$ only depend on the angles at the intersections $\partial Q\cap\partial K$.

We separate to cases according to the space of $\partial Q\cap K$:

- If $\partial Q\cap K = \partial Q$, then $Q\subset K$, and inequality (3) is $0\leq 0$.
- If $\partial Q\cap K$ contains two opposite vertices of Q, there is a parallelogram $K' \in \mathscr{K}_{\mathrm{e}}^2$ with $\partial Q\cap K = \partial Q\cap K'$, with the same intersection angles, and with $Q\cap K \subset Q\cap K'$.
  In this case, we denote the vertices of K' by $\pm(a_1, a_2), \pm(a_3, a_4)$, and let $a_5 = |Q\cap K'| - |Q\cap K|$.
- If $\partial Q\cap K$ consists of four line segments, there is an octagon $K' \in \mathscr{K}_{\mathrm{e}}^2$ with $\partial Q\cap K = \partial Q\cap K'$, with the same intersection angles, and with $Q\cap K \subset Q\cap K'$.
  In this case, we denote the vertices of K' by $\pm(a_1, a_2), \dots, \pm(a_7, a_8)$, and let $a_9 = |Q\cap K'| - |Q\cap K|$.
- If $\partial Q\cap K$ consists of two opposite line segments, there is a hexagon $K' \in \mathscr{K}_{\mathrm{e}}^2$ with $\partial Q\cap K = \partial Q\cap K'$, with the same intersection angles, and with $Q\cap K \subset Q\cap K'$.
  In this case, we denote the vertices of K' by $\pm(a_1, a_2), \dots, \pm(a_5, a_6)$, and let $a_7 = |Q\cap K'| - |Q\cap K|$.
- If $\partial Q\cap K$ is empty, then $K\subset Q$, and inequality (3) is $0\leq 0$.

In each case, the geometrically realizable values of the parameters $a_1, \ldots$ are given by polynomial inequalities, and (3) is also a polynomial inequality on these parameters. Such problems are algorithmically decidable by Tarski's theorem, and in most cases, not too hard to solve by hand. Therefore, checking out all the cases will verify the strong (B) property of the uniform measure on the square Q, independently of the results of [16].

Dihedral symmetry. For an integer $n \geq 2$, let D_n be the group of symmetries of $\mathbb{R}^2$ that is generated by two reflections, one across the axis $\mathrm{Span}\{(1,0)\}$ and the other across the axis $\mathrm{Span}\{(\cos\frac{\pi}{n}, \sin\frac{\pi}{n})\}$. The dihedral group D_n contains $2n$ transformations. A D_n-symmetric shape $A \subset \mathbb{R}^2$ is one invariant under the action of D_n.

Theorem 23. *Let $n \geq 2$ be an integer, and let $K, L \subset \mathbb{R}^2$ be D_n-symmetric convex shapes. Then $t \mapsto |e^t K \cap L|$ is a log-concave function.*

For $n = 2$ the group D_n is generated by reflections across the standard axes. This corresponds to unconditional sets and functions, and Theorem 13 solves this case.

The proof for $n \geq 3$ is by reduction to the unconditional case.

A *smooth strongly-convex* shape $K \subset \mathbb{R}^2$ is one whose boundary is a smooth curve with strictly positive curvature everywhere. The radial function ρ_K of a smooth strongly-convex shape $K \subset \mathbb{R}^2$ is a smooth function. The boundary ∂K is the curve

$$\gamma_K(\theta) = (\rho_K(\theta)\cos\theta, \rho_K(\theta)\sin\theta)\,.$$

The convexity of K is reflected in the sign of the curvature of γ_K. Positivity of the curvature can be written as a condition on the radial function:

$$\rho(\theta)^2 + 2\rho'(\theta)^2 - \rho(\theta)\rho''(\theta) > 0. \tag{4}$$

Proof of Theorem 23. For any D_n-symmetric convex shape $K \subset \mathbb{R}^2$ there is a sequence of D_n-symmetric convex shapes whose boundaries are smooth and strongly convex curves, and whose Hausdorff limit is K. By the continuity argument from the previous section, the general case follows from the smooth case. From here on, K and L are smooth D_n-symmetric shapes.

D_n-symmetric shapes correspond to radial functions that are even and have period $\frac{2\pi}{n}$. These shapes are completely determined by their intersection with the sector

$$G_n = \left\{(r\cos\theta, r\sin\theta) : r \geq 0, \theta \in [0, \tfrac{\pi}{n}]\right\}.$$

Given two such shapes K and L, the area function is

$$f_{K,L}(t) = |e^t K \cap L| = 2n f_{K\cap G_n, L\cap G_n}(t).$$

Let $K \subset \mathbb{R}^2$ be a D_n-symmetric strongly convex shape, and consider the function $\tilde{\rho}(\theta) = \rho_K(\frac{2}{n}\theta)$. This is an even function with period π. It also satisfies (4):

$$\tilde{\rho}(\theta)^2+\tilde{\rho}'(\theta)^2-\tilde{\rho}(\theta)\tilde{\rho}''(\theta)=$$
$$\tfrac{4}{n^2}\left(\rho_K(\tfrac{2}{n}\theta)^2+2\rho_K'(\tfrac{2}{n}\theta)^2-\rho_K(\tfrac{2}{n}\theta)\rho_K''(\tfrac{2}{n}\theta)\right)+(1-\tfrac{4}{n^2})\rho_K(\tfrac{2}{n}\theta)^2>0.$$

This means that $\tilde{\rho}(\theta)$ is the radial function of some D_2-symmetric (unconditional) strongly convex shape. We denote this by $w(K)$: the unique shape that satisfies $\rho_{w(K)}(\theta)=\rho_K(\frac{2}{n}\theta)$.

The following function, also named w, is defined on G_n:

$$w\begin{pmatrix}r\cos\theta\\ r\sin\theta\end{pmatrix}=\begin{pmatrix}r\cos\frac{n}{2}\theta\\ r\sin\frac{n}{2}\theta\end{pmatrix}\qquad\left(\text{for } r\geq 0,\ \theta\in[0,\tfrac{\pi}{n}]\right).$$

The point function w is a bijection between G_n and G_2. It is related to the shape function w by the formula

$$\{w(x):x\in K\cap G_n\}=w(K)\cap G_2.$$

The point function w is differentiable inside G_n, and has a constant Jacobian determinant $\frac{n}{2}$.

Hence

$$f_{K,L}(t)=2nf_{K\cap G_n,L\cap G_n}(t)=4f_{w(K)\cap G_2,w(L)\cap G_2}(t)=f_{w(K),w(L)}(t),$$

and the theorem follows from Theorem 13.

□

References

[1] Aleksandr D. Alexandroff. “Zur Theorie der gemischten Volumina von konvexen Körpern. III. Die Erweiterung zweier Lehrsätze Minkowskis über die konvexen Polyeder auf die beliebigen konvexen Körper”. Russian. *Mat. Sbornik N.S.* 3.45 (1938), pp. 27–46.

[2] W. Blaschke. *Vorlesungen über Integralgeometrie*. 1. B. G. Teubner, 1936.

[3] Károly Böröczky, Erwin Lutwak, Deane Yang, and Gaoyong Zhang. “The logarithmic Minkowski problem”. *Journal of the American Mathematical Society* 26.3 (2013), pp. 831–852.

[4] Károly Böröczky, Erwin Lutwak, Deane Yang, and Gaoyong Zhang. “The log-Brunn-Minkowski inequality”. *Advances in Mathematics* 231.3–4 (2012), pp. 1974–1997.

[5] Christopher W. Brown. “QEPCAD B: A program for computing with semi-algebraic sets using CADs”. *SIGSAM Bulletin* 37 (2003), pp. 97–108.

[6] Hermann Brunn. *Über Ovale und Eiflächen, Inaugural-Dissertation.* München, 1887.

[7] Kai-Seng Chou and Xu-Jia Wang. “The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry”. *Advances in Mathematics* 205.1 (2006), pp. 33–83.

[8] Dario Cordero-Erausquin, Matthieu Fradelizi, and Bernard Maurey. "The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems". *Journal of Functional Analysis* 214.2 (2004), pp. 410–427.

[9] William J. Firey. "p-Means of Convex Bodies". *Mathematica Scandinavica* 10 (1962), pp. 17–24.

[10] William J. Firey. "Shapes of worn stones". *Mathematika* 21 (1974), pp. 1–11.

[11] Harley Flanders. "A Proof of Minkowski's Inequality for Convex Curves". *The American Mathematical Monthly* 75.6 (1968), pp. 581–593.

[12] Michael E. Gage. "Evolving plane curves by curvature in relative geometries". *Duke Mathematical Journal* 72.2 (1993), pp. 441–466.

[13] Rafał Latała. "On some inequalities for Gaussian measures". *Proceedings of the ICM* 2 (2002), pp. 813–822.

[14] Erwin Lutwak. "The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem". *Journal of Differential Geometry* 38.1 (1993), pp. 131–150.

[15] Hermann Minkowski. "Volumen und Oberfläche". *Mathematische Annalen* 57 (1903), pp. 447–495.

[16] Christos Saroglou. "Remarks on the conjectured log-Brunn-Minkowski inequality". *ArXiv e-prints* (2013). arXiv: `1311.4954 [math.FA]`.

[17] Rolf Schneider. *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press, 1993.

[18] Alina Stancu. *Uniqueness of Self-similar Solutions for a Crystalline Flow.* University of Rochester Department of Mathematics, 1996.

[19] Alfred Tarski. *A decision method for elementary algebra and geometry.* University of California Press, 1951.

תקציר

בשנים האחרונות התגלו קשרים בין שלוש בעיות בגיאומטריה קמורה. האחת היא השערת אי-שוויון ברון-מינקובסקי הלוגריתמי, שמגיעה מהרחבה של עבודה של Firey משנת 1962. השערה זו היא תוצאת יציבות עבור אי-שוויון ברון-מינקובסקי המוכר. השניה היא שאלת היחידות של מידת החרוטים (cone volume measure) של קבוצות קמורות, שנשאלה לראשונה במאמר של Firey משנת 1974. Gage הוכיח בשנת 1993 את תכונת היחידות למשפחה רחבה יחסית של גופים קמורים, הגופים הקמורים החלקים ב-$\mathbb{R}^2$ שסימטריים להיפוך ביחס לראשית הצירים. במאמר משנת 2013, Böröczky, Lutwak, Yang ו-Zhang הראו כיצד אפשר להרחיב את הרעיונות שלו ולהסיק מתוכם את אי-שוויון ברון-מינקובסקי הלוגריתמי לאותה משפחת גופים.

הבעיה השלישית עלתה בסביבות 2002, כאשר תוצאות הנוגעות למידה הגאוסית הובילו להשערה שהיא מקיימת את תכונה (B), והשערה זו הוכחה ב-2004 על-ידי Cordero-Erausquin, Fradelizi ו-Maurey. משפט זה הוביל להשערה שתכונה (B) מתקיימת בכל המידות הלוג-קעורות הזוגיות. תוצאה של Saroglou משנת 2013 מראה שהשערה זו שקולה לאי-שוויון ברון-מינקובסקי הלוגריתמי, ושעל פי הידוע עליו ניתן להסיק את תכונה (B) עבור מידות אחידות, קעורות וזוגיות, על $\mathbb{R}^2$.

לעבודה הנוכחית שני חלקים: בחלק הראשון אסקור את התוצאות האלה ואציג את רוב ההוכחות בקיצור, ובחלק השני אראה גישה חדשה באמצעותה אפשר להוכיח ישירות את תכונה (B) עבור אותה משפחת מידות.

החלק השני עתיד להופיע כמאמר בכרך 2011–2013 של כתב העת Geometric Aspects of Functional Analysis. שני חלקי העבודה כמעט בלתי תלויים זה בזה, וניתן להבין את החלק השני לאחר קריאת תת-הפרק Property (B) בלבד, בלי לקרוא את שאר החלק הראשון.

לצח, שרי והגר

שלמי תודה

אני אסיר תודה למנחה שלי, פרופסור בועז קלרטג, על הדרכתו, ועל העזרה והעידוד שנתן לי במהלך כל שלבי העבודה והמחקר.

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אוניברסיטת תל-אביב
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ביה"ס למדעי המתמטיקה

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מוסמך למדעים (M.Sc.) במתמטיקה עיונית

החוג למתמטיקה עיונית

מוגש ע"י
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בהנחיית פרופ' בועז קלרטג

נובמבר 2014