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Long lines in subsets of large measure in high dimension

Dor Elboim and Bo'az Klartag

Abstract

We show that for any set $A \subseteq [0,1]^n$ with $\mathrm{Vol}(A) \geq 1/2$ there exists a line ℓ such that the one-dimensional Lebesgue measure of $\ell \cap A$ is at least $\Omega(n^{1/4})$. The exponent $1/4$ is tight. More generally, for a probability measure μ on $\mathbb{R}^n$ and $0 < a < 1$ define

$$L(\mu, a) := \inf_{A;\mu(A)=a} \sup_{\ell \text{ line}} \left|\ell \cap A\right|$$

where $|\cdot|$ stands for the one-dimensional Lebesgue measure. We study the asymptotic behavior of $L(\mu, a)$ when μ is a product measure and when μ is the uniform measure on the ℓ_p ball. We observe a rather unified behavior in a large class of product measures. On the other hand, for ℓ_p balls with $1 \leq p \leq \infty$ we find that there are phase transitions of different types.

1 Introduction

One of the simplest high-dimensional features of the geometry of $\mathbb{R}^n$, for large n, is the fact that rather long segments fit inside the n-dimensional unit cube. In fact, both the unit cube and the Euclidean ball of volume one contain segments of length $c\sqrt{n}$, for a universal constant $c > 0$. More generally, let $K \subseteq \mathbb{R}^n$ be a convex body of volume one. The classical isodiametric inequality states that K necessarily contains a segment of length

$$\left(\sqrt{\frac{2}{\pi e}} + o(1)\right) \cdot \sqrt{n},$$

with the Euclidean ball being the extremal case. Can one avoid these long segments by restricting to a subset of K of volume $1/2$? In order to exclude trivial answers involving removing a dense set of small measure, we slightly modify this question and formulate it precisely as follows: Does there exist a subset $A \subseteq K$ of volume $1/2$ such that for any line ℓ in $\mathbb{R}^n$,

$$|A \cap \ell| < C \tag{1}$$

for a universal constant $C > 0$? Here, $|A \cap \ell|$ is the one-dimensional length measure of the intersection of A with the line ℓ. We may answer this question in the affirmative in the example where K is a Euclidean ball of volume one centered at the origin. In this case, the thin spherical shell

$$A = K \setminus \left(1 - \frac{1}{n}\right) K \tag{2}$$

has a volume of $1 - (1 - 1/n)^n \approx 1 - 1/e > 1/2$. An elementary argument based on the curvature of the sphere shows that this subset A does not contain any long segment, and that (1) holds true with a universal constant $C > 0$. The answer is nearly affirmative, up to logarithmic factors, also in the case of ℓ_p-balls for $1 < p < 2$. That is, when

$$K = B_p^n := \left\{ x \in \mathbb{R}^n \,:\, \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \leq \kappa_{p,n} \right\} \tag{3}$$

where $\kappa_{p,n} = \Gamma(1 + n/p)^{1/n} / (2\Gamma(1 + 1/p))$ is chosen so that K has volume one. The situation changes when one considers the case where $p \notin (1, 2]$, as we explain below. For a Borel probability measure μ on $\mathbb{R}^n$ and a parameter $0 < a < 1$ we define

$$L(\mu, a) := \inf_{A; \mu(A) = a} \sup_{\ell \text{ line}} |\ell \cap A|$$

where the infimum runs over all Borel subsets $A \subseteq \mathbb{R}^n$ with $\mu(A) \geq a$, and the supremum runs over all lines $\ell \subseteq \mathbb{R}^n$. We write λ_K for the uniform probability measure on a convex body $K \subseteq \mathbb{R}^n$ and abbreviate $L(K, a) = L(\lambda_K, a)$. The definition (3) of B_p^n makes sense for all $1 \leq p < \infty$, and by continuity $B_\infty^n := \{x \in \mathbb{R}^n : \forall i, |x_i| \leq 1/2\}$ is a unit cube.

Theorem 1.1. *Let $n \geq 1$ and $1 \leq p \leq \infty$. Then,*

$$L(B_p^n, 1/2) = \begin{cases} \Theta\left(n^{1/4}\right) & p = 1, \infty \\ \Theta_p\left((\log n)^{\frac{2-p}{2p}}\right) & 1 < p \leq 2 \\ \Theta_p\left(n^{\frac{p-2}{4p+2}}\right) & 2 < p < \infty \end{cases}$$

Here, $\Theta(X)$ stands for a quantity Y with $cX \leq Y \leq CX$ for universal constants $c, C > 0$. By $\Theta_p(X)$ we mean that the constants c, C are not universal, but allowed to depend on p solely.

Remark 1.2. The constant $1/2$ in the theorem can be replaced by any other fixed $a \in (0, 1)$. However, the estimates will not be uniform as $a \to 0$ or $a \to 1$.

A somewhat peculiar feature of Theorem 1.1 is the exponent

$$\frac{p-2}{4p+2}$$

in the case $2 < p < \infty$, which interpolates continuously between the values 0 and $1/4$ attained at the endpoints $p = 2, \infty$. There is a discontinuity at $p = 1$, where the exponent jumps to $1/4$.

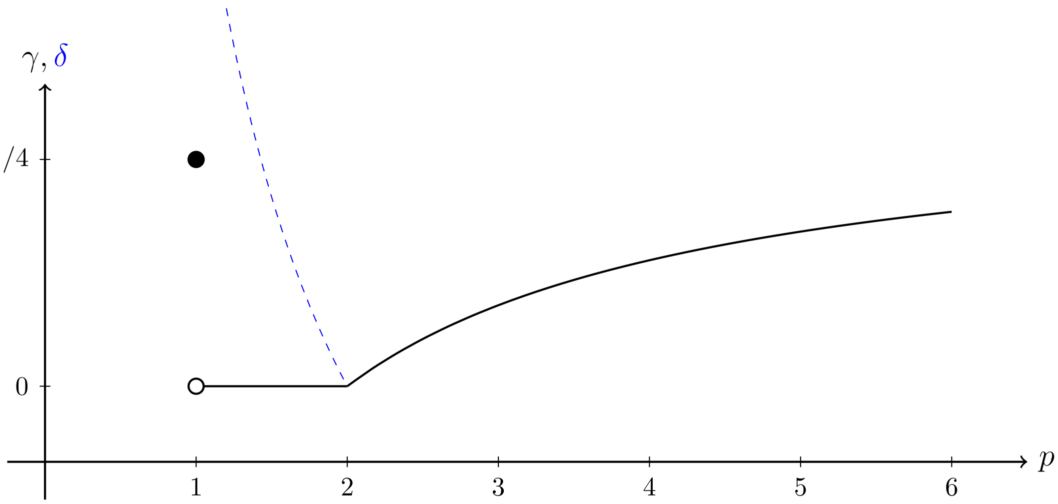

Figure 1: The exponent $\gamma(p)$ appears in black. Theorem 1.1 states that $L(B_p^n, 1/2) = \tilde{\Theta}(n^{\gamma(p)})$. The exponent $\delta(p)$ in blue. Theorem 1.1 states that for $1 < p < 2$, $L(B_p^n, 1/2) = \Theta((\log n)^{\delta(p)})$.

For $p = 1, \infty$, an extremal set can be easily described, it suffices to consider the intrsection of B_p^n with a *Euclidean spherical shell*,

$$A = B_p^n \cap \left\{ x \in \mathbb{R}^n \ : \ r \leq \|x\|_2 \leq r \left(1 + C/\sqrt{n}\right) \right\} \tag{4}$$

for a certain value of $r = \Theta(\sqrt{n})$, where $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$. For p in the range $(1, \infty)$, the Euclidean-norm considerations are less prominent in our construction of an extremal set A.

We move on to a detailed analysis of the case of the unit cube in $\mathbb{R}^n$ with varying $a \in (0,1)$. In fact, our results hold not just for the uniform measure on the unit cube, but also for general product measures μ in $\mathbb{R}^n$, satisfying the following conditions:

(i) The measure μ is absolutely-continuous with density $\prod_{i=1}^n \rho_i(x_i)$, where the function $\rho_i : \mathbb{R} \to [0, \infty)$ is smooth in the interval $(-1/2, 1/2)$, and in this interval the derivatives $(\log \rho)^{(k)}$ for $k = 0, \ldots, 4$ are bounded in absolute value by C.

(ii) Sub-Gaussian tail: $\int_{-\infty}^{\infty} \exp(ct^2) \rho_i(t) dt \leq C$ for all i, for some constants $c, C > 0$.

For example, the standard Gaussian measure in $\mathbb{R}^n$, whose density is $(2\pi)^{-n/2} \exp(-\|x\|_2^2/2)$, satisfies (i) and (ii), as well as the uniform measure on the unit cube B_∞^n.

Theorem 1.3. *Let $n \geq 1, 0 < a < 1$ and let μ be a probability measure on $\mathbb{R}^n$ satisfying conditions (i) and (ii). Then,*

$$L(\mu, a) = \begin{cases} \Theta\left(a \cdot n^{1/4}\right) & e^{-n} \leq a \leq 1/2 \\ \Theta\left(n^{1/4} \cdot |\log(1-a)|^{1/4}\right) & 1/2 \leq a \leq 1 - e^{-n} \end{cases}$$

Here, the implies constants in the $\Theta(..)$ *notation depend solely on the constants from conditions (i) and (ii).*

Thus, unless a is exponentially close to zero or to one, we observe *universality* in the class of product measures. We can determine the value of $L(\mu, a)$ up to a constant factor, no matter what the precise details of the distribution ρ_i are, as long as the tails are sub-Gaussian and the density is somewhat regular near the origin. We suspect that the range $\min\{a, 1-a\} < e^{-n}$ corresponds to the "large deviations" regime where the specifics of ρ_i should matter.

It is possible to view Theorem 1.1 and Theorem 1.3 in the context of the Radon transform. Write $\mathcal{G}$ for the collection of all lines in $\mathbb{R}^n$. For a Borel measurable function $f : \mathbb{R}^n \to \mathbb{R}$ we define

$$Rf(\ell) = \int_\ell f.$$

When f is, say, compactly-supported, the Radon transform Rf is a well-defined bounded function on $\mathcal{G}$. Consider the case where $f = 1_A$, for a measurable subset $A \subseteq \mathbb{R}^n$. Theorem 1.1 and Theorem 1.3 tell us that $\sup Rf$ is large, provided that A has a substantial intersection with an ℓ_p-ball, or that A has a non-neglible mass with respect to a certain product measure μ.

It is also possible to view our results in the context of the lower-dimensional Busemann Petty problem, which is the case where $A = K$ and where the line (or more generally and better, the subspace) is forced to pass through the origin. See Koldobsky [14, Section 5.5] and references therein for information about the lower-dimensional Busemann Petty problem.

In the discrete setting analogous questions have been studied, especially in the field of incidence geometry. In the discrete world, lines with large Lebesgue measure are replaced by lines having large number of points (large number of incidences on the line). For instance, see [17, Theorem 3] for a bound of the number of t-rich lines (these are lines that contains at least t points) in large subsets of points of a block design.

Another discrete result that shares similarities with the problems considered in this paper, is the *density Hales-Jewett theorem*. This theorem states that in a sufficiently high dimension any subset of positive density contains a combinatorial line. The theorem was proved by Furstenberg and Katznelson [9, 10]. See also [16] for an elementary proof. The exact statement is the following. For any $d \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n_0 = n_0(d, \varepsilon)$ such that for all $n \geq n_0$ any subset $A \subseteq \{1, \ldots, d\}^n$ with $|A| \geq \varepsilon d^n$ contains a combinatorial line. A combinatorial line is a set ℓ of size d of the form

$$\ell = \left\{ \left(\varepsilon_1 a_1 + (1-\varepsilon_1)j, \ldots, \varepsilon_n a_n + (1-\varepsilon_n)j \right) \ : \ j \in \{1, \ldots, d\} \right\} \tag{5}$$

where $a_1, \ldots, a_n \in \{1, \ldots, d\}$ and where $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ are not all 1. In other words, in a combinatorial line some coordinates (not all) are fixed, and some change linearly from 1 to d.

The exponent $1/4$ observed in the case of the cube and the cross-polytope in Theorem 1.1 is somewhat of a natural barrier for this problem. We say that a convex body $K \subseteq \mathbb{R}^n$ of volume one is in *isotropic position* if its barycenter $b_K = \int_K x dx$ is at the origin and its covariance matrix

$$Cov(\mu) = \int_K (x \otimes x) dx \in \mathbb{R}^{n \times n}$$

is a scalar matrix. Here, $x \otimes x = (x_i x_j)_{i,j=1,\ldots,n} \in \mathbb{R}^{n \times n}$. For example, the convex body B_p^n is in isotropic position for all p and n.

Proposition 1.4. *Let $n \geq 2$ and let $K \subseteq \mathbb{R}^n$ be a convex body of volume one in isotropic position. Then,*

$$L(\lambda_K, 1/2) \leq n^{1/4+o(1)},$$

where $o(1)$ stands for a function of n that tends to zero as n tends to infinity.

In order to prove Proposition 1.4 we use a construction similar to (4) above, and consider the intersection of K with a thin Euclidean spherical shell,

$$A = K \cap \left\{ x \in \mathbb{R}^n \, : \, r \leq \|x\|_2 \leq r \left(1 + C n^{-1/2+o(1)} \right) \right\}, \tag{6}$$

for a certain value of $r = \Theta(\sqrt{n})$. Indeed, it follows from the recent breakthrough by Chen [4] combined with Eldan and Klartag [8] that the set A captures at least $1/2$ of the mass of K. Yet the subset A cannot intersect any line in a set whose length measure is above $n^{1/4+o(1)}$, as we see from the proof of Corollary 3.7 below.

Remark 1.5. We remark in passing that for any convex body $K \subseteq \mathbb{R}^n$ of volume one, we have

$$L(\lambda_K, 1/2) \geq c, \tag{7}$$

for a universal constant $c > 0$. Our proof of (7) uses convex geometric tools such as localization and needle decomposition, and will be discussed elsewhere.

In a vague sense that we were not able to make precise, we feel that the exponent $1/4$ corresponds to the case of a "generic" convex body in isotropic position. Are there natural probability measures μ on $\mathbb{R}^n$ for which $L(\mu, 1/2)$ is much larger than $n^{1/4}$? Such measures had better be unrelated to convexity and without a product structure of the type considered above.

Proposition 1.6. *Let X, Y be independent, standard Gaussian random vectors in $\mathbb{R}^n$. Let U be a random variable, independent of X and Y, that is distributed uniformly in the interval $[0,1]$. Write μ for the probability measure on $\mathbb{R}^n$ that is the law of the random vector*

$$X + UY.$$

Then

$$L(\mu, 1/2) = \Theta(\sqrt{n}).$$

Unless stated otherwise, throughout this text we use the letters $c, C, \tilde{C}$ etc. to denote positive universal constants, whose value may change from one line to the next.

1.1 Main ideas in the proofs

We think of our main technique as a "Mermin-Wagner type argument in a random direction", or alternatively, as an approximate needle decomposition into segments that are as long as possible. In order to explain this technique we sketch the proof of Theorem 1.3 in the special case where $a = 1/2$ and $\mu = \gamma_n$ is the standard Gaussian measure in $\mathbb{R}^n$.

Suppose that $A \subseteq \mathbb{R}^n$ satisfies $\gamma_n(A) \geq 1/2$. We would like to prove that there exists a line $\ell \subseteq \mathbb{R}^n$ with

$$|A \cap \ell| \geq cn^{1/4} \tag{8}$$

where $|\cdot|$ stands for the one-dimensional Lebesgue measure. Let Z, W be independent standard Gaussian random vectors in $\mathbb{R}^n$. It is well-known (e.g., [18, Chapter 2]) that

$$\mathbb{P}\left(|W| > \sqrt{n}/2\right) > 0.9. \tag{9}$$

Furthermore we claim that there exists $c_1 > 0$ such that for any $r \leq c_1 n^{-1/4}$,

$$d_{TV}(Z, Z + rW) < 0.1 \tag{10}$$

where $d_{TV}(X,Y) = \sup_B |\mathbb{P}(X \in B) - \mathbb{P}(Y \in B)|$ is the total variation distance between X and Y. A neat proof of (10) using Pinsker's inequality can be found in [6]. It follows from (10) and the fact that $\mathbb{P}(Z \in A) = \gamma_n(A) \geq 1/2$ that

$$\mathbb{P}\big(Z + rW \in A\big) \geq 0.4.$$

Since the last inequality holds for any $r \leq c_1 n^{-1/4}$, it holds when replacing r with a random variable U distributed uniformly in the interval $[0, c_1 n^{-1/4}]$ that is independent of Z and W. We obtain using (9) that

$$\mathbb{P}\big(Z + UW \in A,\ |W| \geq \sqrt{n}/2\big) \geq 0.3. \tag{11}$$

It follows that there are realizations $z, w \in \mathbb{R}^n$ with $|w| \geq \sqrt{n}/2$ such that

$$\mathbb{P}\big(z + Uw \in A\big) \geq 0.3. \tag{12}$$

Finally, note that the last probability is exactly the normalized one-dimensional Lebesgue measure of the intersection of A with the line segment $[z, z + c_1 n^{-1/4} w]$. This line segment is of length $c_1 n^{-1/4}|w| \geq cn^{1/4}$, completing the proof of (8).

We may now explain the proof of Proposition 1.6. By the definition of μ, we have $z, w \in \mathbb{R}^n$ with $|w| \geq \sqrt{n}/2$ such that (12) holds, where now U is uniformly distributed in the interval $[0, 1]$. This proves the lower bound for $L(\mu, 1/2)$. The upper bound follows by considering the set A which is a Euclidean ball of radius $5\sqrt{n}$ centered at the origin in $\mathbb{R}^n$.

What we see from the above is that in order to obtain lower bounds for $L(\mu, a)$ it is useful to "push" the measure μ in a random direction. Equation (10) shows that in the Gaussian case one can push the measure to a distance of order $n^{1/4}$ without changing it by much in total variation. In the mathematical physics literature, this technique of pushing a measure was introduced in

[15] and is usually referred to as a Mermin-Wagner type argument. In the convexity literature [7, 11, 12, 13] it is quite common to decompose a measure on an n-dimensional space into one-dimensional needles, as in the approximate decomposition into uniform measures on segments discussed above.

Consider next the case of the uniform measure on the unit cube $[0,1]^n$. Let $X = (X_1, \ldots, X_n)$ be a uniform point in the cube and note that the coordinates X_i are i.i.d. uniform random variables in $[0,1]$. In order to prove Theorem 1.3 in this case, the first attempt would be to use the same perturbation as in the Gaussian case. That is, to perturb each coordinate by $Y_i := X_i + n^{-1/4} Z_i$ where $Z_1, \ldots, Z_n$ are i.i.d. normal random variables. However, it is easy to see that such a perturbation will push the random point outside of the unit cube with high probability and the total variation distance $d_{TV}(X,Y)$ will tend to 1. To overcome this issue we only perturb coordinates which are not too close to 0 or 1. More precisely, we use a perturbation of the form

$$Y_i := X_i + n^{-1/4} \varphi(X_i) Z_i, \tag{13}$$

where φ is a smooth bump function supported on $[1/3, 2/3]$. We show that for a suitable choice of φ, the perturbation in (13) satisfies $d_{TV}(X,Y) \leq 0.1$. This strategy can be used to obtain lower bounds on $L(\mu, a)$ for general product measures μ as long as a does not tend to 0 or 1. When a is small, this strategy fails as the set A can be concentrated around the center of the cube where the density of $Y = (Y_1, \ldots, Y_n)$ can be very small. To obtain tight bounds in this case, instead of perturbing the original measure, we first tilt the measure slightly toward the center of the cube and then perturb it randomly. See Section 2.1 for more details.

In Section 3 we study the case of ℓ_p balls. The idea here is to perturb the coordinates as much as possible without changing the ℓ_p norm by much. It turns out that when $p > 2$ it is better to only perturb coordinates which are close to zero, of order $n^{-1/(2p+1)}$ while for $1 < p < 2$ one should only perturb large coordinates of order $\log^{1/p} n$. Another difference between the two regimes is that when $p > 2$ we perturb each coordinate independently like in (13) but for $1 < p < 2$ such a perturbation will change the p norm by too much. In order to handle this issue we perturb pairs of consecutive coordinates at a time. For each pair, we perturb the first coordinate of the pair randomly and then use the other coordinate of the pair to "correct" the p norm.

1.2 Extensions and open problems

There are a few natural extensions of the results in this paper. Perhaps some of those can solved using the methods developed in our proofs. One such extension is to estimate $L(B_p^n, 1/2)$ uniformly in p. For example, as p tends to 1, at what rate does the behavior change from logarithmic to $n^{1/4}$? It would also be interesting to understand the asymptotic behavior of $L(B_p^n, a)$ as $a \to 0$ or $a \to 1$ like in Theorem 1.3.

Another question, is what can be said when the lines in our main theorems are replaced by higher dimensional planes or by other low degree polynomial curves. Perhaps the most interesting problem which we were not able to solve is to understand $L(K, 1/2)$ for a general convex body K. For example, is there a simple geometric parameter of K that explains the asymptotic behavior of $L(K, 1/2)$?

1.3 Acknowledgements

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2 Product measures

In this section we prove Theorem 1.3. We start with the proof of the lower bound for small a.

2.1 Lower bound for general product measures and $e^{-n} \le a \le 1/2$

Let μ be a measure satisfying requirement (i) from Section 1. The constants hidden in the $O(...)$ notation in this proof may depend on C from requirement (i). Let $n \ge 2$ and fix real numbers R, r with $|R|, |r| \le 1$.

Fix a C^4-smooth, non-negative bump function φ on the real line, supported in $(-1/3, 1/3)$ with $\varphi(0) = 1/100$ and $|\varphi'| < 1$. For concreteness, say that $\varphi(t) = (1-9t^2)^5/100$ for $|t| < 1/3$ and $\varphi(t)$ vanishes for $|t| \ge 1/3$. Define

$$g_i = \frac{(\varphi^2 \rho_i)''}{\rho_i}, \tag{14}$$

which is C^2-smooth in the real line and supported in $(-1/2, 1/2)$ thanks to requirement (i). Let $X_1, \dots, X_n$ be independent random variables, where the density of X_i is

$$\tilde{\rho}_i(t) = \kappa_{i,R} e^{-R^2 g_i(t)} \rho_i(t). \tag{15}$$

Here,

$$\kappa_{i,R} = \left(\int_{-\infty}^{\infty} e^{-R^2 g_i(t)} \rho_i(t) dt \right)^{-1} = 1 + O(R^4), \tag{16}$$

since $\int g_i \rho_i = 0$, with the implied constant in the $O(R^4)$ depending on C from (i). Let $\delta_1, \dots, \delta_n \in \{-1, 1\}$ be independent symmetric Bernoulli variables, independent of the X_i's. Let $U \in [0,1]$ be a uniform random variable, independent of all of the previous random variables. Denote

$$Y_i = X_i + rU\delta_i \varphi(X_i).$$

Proposition 2.1. *The density $f = f_{R,r}$ of the random vector $Y = (Y_1, \dots, Y_n)$ satisfies*

$$f(y) \ge e^{-\tilde{C}(R^4 n + r^4 n)} \cdot e^{\left(r^2/6 - R^2\right) \cdot \sum_{i=1}^n g_i(y)} \cdot \prod_{i=1}^n \rho_i(y_i),$$

where $\tilde{C} > 0$ depends solely on the constant from requirement (i).

In order to prove Proposition 2.1 we denote

$$\tilde{Y}_i = X_i + r\delta_i\varphi(X_i).$$

Observe that the two maps $t \mapsto t \pm r\varphi(t)$ are monotone increasing in $\mathbb{R}$ as $r \leq 1$. We may therefore apply the change of variables formula, and conclude that the density of the random variable $\tilde{Y}_i$ equals

$$f_i(t) = f_i^{(R,r)}(t) = \frac{1}{2}\left[\frac{\tilde{\rho}_i(x_1)}{1 + r\varphi'(x_1)} + \frac{\tilde{\rho}_i(x_2)}{1 - r\varphi'(x_2)}\right] \tag{17}$$

where $x_1 < t < x_2$ are determined by $x_1 + r\varphi(x_1) = t = x_2 - r\varphi(x_2)$. We emphasize that $f_i(t)$ as defined in (17) depends also on the parameters r and R. Its Taylor approximation with respect to these two parameters, which is uniform in t, is given in the following lemma.

Lemma 2.2. *For any $t \in \mathbb{R}$ and $i = 1, \dots, n$,*

$$f_i(t) = \exp\left[\left(r^2/2 - R^2\right) \cdot g_i(t) + O\left(r^4 + R^4\right)\right] \cdot \rho_i(t),$$

where the implicit constant in $O(r^4 + R^4)$ depends only on C from requirement (i) of Section 1.

Proof. Since $|r| \leq 1$, the equation $x_1 + r\varphi(x_1) = t$ implies

$$x_1 = t - r\varphi(t) + r^2\varphi'(t)\varphi(t) + O(|r|^3)$$

and similarly

$$x_2 = t + r\varphi(t) + r^2\varphi'(t)\varphi(t) + O(|r|^3).$$

Thus, from (15), (16) and (17),

$$\begin{aligned} f_i(t) = &\frac{1}{2} \cdot \frac{1 - R^2 g_i(t) + R^2 r g_i'(t)\varphi(t) + O(R^4 + R^2 r^2)}{1 + r\varphi'(t) - r^2\varphi''(t)\varphi(t) + O(|r|^3)} \cdot \rho_i(x_1) \\ &+ \frac{1}{2} \cdot \frac{1 - R^2 g_i(t) - R^2 r g_i'(t)\varphi(t) + O(R^4 + R^2 r^2)}{1 - r\varphi'(t) - r^2\varphi''(t)\varphi(t) + O(|r|^3)} \cdot \rho_i(x_2). \end{aligned}$$

Next,

$$\frac{\rho_i(x_1)}{\rho_i(t)} = 1 - r\varphi(\log\rho_i)' + r^2(\log\rho_i)'\varphi'\varphi + r^2\varphi^2\frac{(\log\rho_i)'' + [(\log\rho_i)']^2}{2} + O(|r|^3), \tag{18}$$

where φ, ρ and their derivatives are evaluated at t. The expression for $\rho_i(x_2)/\rho_i(t)$ is similar to the right-hand side of (18), the only difference is that the coefficient of $r\varphi(\log\rho_i)'$ is $+1$ and not -1. Consequently,

$$\begin{aligned} \frac{f_i(t)}{\rho_i(t)} &= 1 - R^2 g_i + r^2\left[\varphi''\varphi + (\varphi')^2 + 2(\log\rho_i)'\varphi'\varphi + \varphi^2\frac{(\log\rho_i)'' + [(\log\rho_i)']^2}{2}\right] + O(R^4 + |r|^3) \\ &= 1 - R^2 g_i + r^2\frac{(\varphi^2\rho_i)''}{2\rho_i} + O(R^4 + |r|^3). \end{aligned} \tag{19}$$

The function $f_i(t)/\rho_i(t)$ is an even function of r and of R, hence the Taylor expansion in r and R contains only even powers of r and R. Thus one might expect to improve $O(R^4+|r|^3)$ to $O(R^4+r^4)$ in (19). Indeed, since φ and $\log\rho_i$ have bounded C^4-norm, the odd terms in the Taylor expansion vanish as the function is even, and the error in the Taylor approximation is $O(R^4+r^4)$, uniformly in t. To summarize, for any $t \in \mathbb{R}$,

$$\frac{f_i(t)}{\rho_i(t)} = 1 + \left(\frac{r^2}{2} - R^2\right) g_i(t) + O(R^4+r^4).$$

Since $\sup|g_i| < C$ and $|r|, |R| \le 1$, the lemma follows. ◻

Proof of Proposition 2.1. From the definition of f, we have $f(y) = \mathbb{E}_U \prod_{i=1}^n f_i^{(r,Ur)}(y_i)$. Thus, by Lemma 2.2,

$$f(y) = \mathbb{E}_U \exp\left[O\left(r^4 n + R^4 n\right) + \left(U^2 r^2/2 - R^2\right)\cdot \sum_{i=1}^n g_i(y_i)\right]\cdot \prod_{i=1}^n \rho_i(y_i). \qquad (20)$$

From Jensen's inequality,

$$f(y) \ge e^{-\tilde{C}\left(r^4 n + R^4 n\right)} \cdot \exp\left[\mathbb{E}_U \left(U^2 r^2/2 - R^2\right)\cdot \sum_{i=1}^n g_i(y_i) + O\left(r^4+R^4\right)\right]\cdot \prod_{i=1}^n \rho_i(y_i).$$

Since $\mathbb{E}U^2 = 1/3$, the conclusion of the proposition follows. ◻

Proposition 2.3. *Let $n \ge 2$ and let μ be a probability measure on $\mathbb{R}^n$ satisfying requirement (i) from Section 1. Then there exists $\tilde{c} > 0$, depending solely on the constants from (i), such that if $e^{-n} \le a \le 1/2$ then,*

$$L(\mu, a) \ge \tilde{c}\cdot n^{1/4}\cdot a.$$

Proof. Set $r = n^{-1/4}/(2\tilde{C})^{1/4}$ with $\tilde{C}$ from Proposition 2.1. We will consider a mixture of two distributions. Write $Y^{(1)}$ for a random vector with density $f^{(1)} := f_{0,r}$, it has the law of the random vector Y with the parameter $R=0$. Let $Y^{(2)}$ be a random vector with density $f^{(2)} := f_{r,r}$. It has the law of the random vector Y with the parameter $R = n^{-1/4}/(2\tilde{C})^{1/4}$. By Proposition 2.1,

$$f^{(1)}(y) \ge \frac{1}{e}\cdot e^{\frac{1}{6}\cdot\frac{\sum_{i=1}^n g_i(y)}{\sqrt{2\tilde{C}n}}}\cdot \prod_{i=1}^n \rho_i(y_i),$$

while

$$f^{(2)}(y) \ge \frac{1}{e}\cdot e^{-\frac{5}{6}\cdot\frac{\sum_{i=1}^n g_i(y)}{\sqrt{2\tilde{C}n}}}\cdot \prod_{i=1}^n \rho_i(y_i).$$

Consequently,

$$\frac{f^{(1)}(y) + f^{(2)}(y)}{2} \geq \frac{1}{2e} \cdot \prod_{i=1}^{n} \rho_i(y_i) \qquad \text{for all } y \in \mathbb{R}^n. \tag{21}$$

Let $A \subseteq \mathbb{R}^n$ satisfy $\mu(A) \geq a \geq e^{-n}$. According to (21) either for $i = 1$ or for $i = 2$,

$$\int_A f^{(i)} \geq \frac{1}{2e} \cdot a. \tag{22}$$

Let $R = 0$ in case $i = 1$ and $R = r$ in case $i = 2$. Let X be distributed as above with the parameter R, i.e., $X_1, \dots, X_n$ are independent with density given in (15). Denote

$$Z_i = \delta_i \varphi(X_i).$$

Then the random vector $Y = (Y_1, \dots, Y_n)$ defined via $Y_i = X_i + rU\delta_i\varphi(X_i)$ satisfies

$$Y = X + rUZ.$$

We claim that for some $c_1, c_2 > 0$ depending on the constants from condition (i),

$$\mathbb{P}(|Z|^2 > c_1^2 n) \geq 1 - \frac{1}{20} \cdot e^{-n}. \tag{23}$$

Indeed, the random variable Z_i is a bounded, symmetric random variable with $\mathbb{E}Z_i^2 > c$. Hence (23) follows from the Bernstein inequality (e.g. [18, Theorem 2.8.4]), where for small n, inequality (30) follows directly from requirement (i) and the definitions. Inequality (22) means that

$$\mathbb{P}(X + rUZ \in A) \geq \frac{1}{2e} \cdot a. \tag{24}$$

Since $a \geq e^{-n}$, from (23) and (24) we deduce that there exist $x, z \in \mathbb{R}^n$ with $|z| > c_1\sqrt{n}$ such that

$$\mathbb{P}(x + rUz \in A) \geq c'a.$$

This means that

$$\frac{|A \cap [x, x + rz]|}{r|z|} \geq c'a. \tag{25}$$

Since $|z| > c_1\sqrt{n}$ and $r > \tilde{c}n^{-1/4}$, the segment in (25) is of length at least $\tilde{c}n^{1/4}$, completing the proof. □

2.2 Lower bound for $1/2 \leq a \leq 1 - e^{-n}$

We continue with the notation and assumptions of Section 2.1. We use the parameter value $R = 0$, while r will be determined soon. In particular $X_1, \dots, X_n$ are independent random variables, where ρ_i is the law of X_i, and

$$Y_i = X_i + rU\delta_i\varphi(X_i).$$

From (20) we know that the density f of Y satisfies

$$f(y) \le e^{\tilde{C} r^4 n} \cdot \mathbb{E}_U \exp\left[\frac{U^2 r^2}{2} \cdot \sum_{i=1}^n g_i(y_i)\right] \cdot \prod_{i=1}^n \rho_i(y_i), \tag{26}$$

with $\tilde{C}$ depending on the constant from requirement (i). Denote $W_i = g_i(X_i)$, where g_i is defined in (14). The random variables $W_1, \ldots, W_n$ are independent, mean zero random variables. From requirement (i) and from (14), we see that for all i,

$$|W_i| < C'$$

for some C' depending solely on the parameter from requirement (i). Therefore $\sum_{i=1}^n W_i/\sqrt{n}$ is a sub-Gaussian random variable, in the terminology of [18, Section 2.5]. That is,

$$\mathbb{P}\left(\left|\frac{\sum_{i=1}^n W_i}{\sqrt{n}}\right| \ge t\right) \le \tilde{C} \exp(-\tilde{c} t^2) \qquad\qquad (t \in \mathbb{R})$$

for $\tilde{c}, \tilde{C} > 0$ depending on parameter from requirement (i).

Lemma 2.4. *Let $A \subseteq \mathbb{R}^n$ with $\varepsilon := \mathbb{P}(X \in A) \le 1/2$. Then for any $0 < s < \sqrt{|\log \varepsilon|}$,*

$$\mathbb{E} \exp\left(s \frac{\sum_{i=1}^n W_i}{\sqrt{n}}\right) \cdot 1_{\{X \in A\}} \le C_1 \cdot \varepsilon \cdot e^{C_2 s \sqrt{|\log \varepsilon|}}, \tag{27}$$

where $C_1, C_2 > 0$ depend solely on the constant from (i).

Proof. The left-hand side of (27) equals

$$\begin{aligned}
\int_0^\infty \mathbb{P}\left(X \in A, \exp\left(s \frac{\sum_{i=1}^n W_i}{\sqrt{n}}\right) \ge t\right) dt &\le \int_0^\infty \min\left\{\varepsilon, \tilde{C} \exp\left(-\tilde{c} \frac{\log^2 t}{s^2}\right)\right\} dt \\
&\le \varepsilon \cdot e^{\hat{C} s \sqrt{|\log \varepsilon|}} + \tilde{C} \int_{e^{\hat{C} s \sqrt{|\log \varepsilon|}}}^\infty \exp\left(-\tilde{c} \frac{\log^2 t}{s^2}\right) dt \\
&= \varepsilon \cdot e^{\hat{C} s \sqrt{|\log \varepsilon|}} + \tilde{C} s \int_{\hat{C} \sqrt{|\log \varepsilon|}}^\infty \exp\left(sr - \tilde{c} r^2\right) dr \le C_1 \varepsilon \cdot e^{C_2 s \sqrt{|\log \varepsilon|}},
\end{aligned}$$

provided that we choose $\hat{C}$ large enough. ◻

Proposition 2.5. *Let $n \ge 1$ and let μ be a probability measure on $\mathbb{R}^n$ satisfying condition (i) from Section 1. Then there exists $\tilde{c} > 0$ depending solely on the constant from (i), such that if $1/2 \le a \le 1 - e^{-n}$ then for $\varepsilon = 1 - a$,*

$$L(\mu, a) \ge \tilde{c} n^{1/4} |\log \varepsilon|^{1/4}.$$

Proof. Let $A \subseteq \mathbb{R}^n$ satisfy $\mu(A) = 1 - \varepsilon$. Applying Lemma 2.4 for the complement of A we obtain

$$\mathbb{E}_U \int_{A^c} \exp\left[\frac{U^2 r^2}{2} \cdot \sum_{i=1}^n g_i(y_i)\right] \prod_{i=1}^n \rho_i(y_i) dy = \mathbb{E} 1_{\{X \notin A\}} \exp\left[\frac{\sqrt{n} U^2 r^2}{2} \cdot \frac{\sum_{i=1}^n W_i}{\sqrt{n}}\right]$$
$$\leq C_1 \mathbb{E} \varepsilon \cdot e^{C_2 \sqrt{n} U^2 r^2 \sqrt{|\log \varepsilon|}} \leq C_1 \varepsilon \cdot e^{C_2 \sqrt{n} r^2 \sqrt{|\log \varepsilon|}} \leq \sqrt{\varepsilon} \tag{28}$$

provided that $r = c n^{-1/4} |\log \varepsilon|^{1/4}$ for a small enough constant c (depending solely on the parameter from requirement (i)). We select c small enough so that $c^4 \tilde{C} < 1/4$ where $\tilde{C}$ is the constant from (26). From (26) and (28) we conclude that

$$\mathbb{P}(Y \notin A) \leq \sqrt{\varepsilon} \cdot e^{\tilde{C} r^4 n} \leq \sqrt{\varepsilon} \cdot \varepsilon^{-1/4} = \varepsilon^{3/4} \leq 9/10. \tag{29}$$

However, $Y = X + rUZ$, where as above $Z_i = \delta_i \varphi(X_i)$ and by (23),

$$\mathbb{P}(|Z| > \hat{c}_1 \sqrt{n}) > 19/20. \tag{30}$$

Consequently, from (29) and (30) there exist $y, z \in \mathbb{R}^n$ with $|z| > \hat{c}_1 \sqrt{n}$ such that

$$\mathbb{P}(x + rUz \in A) \geq 1/20.$$

Hence at least a 0.05-fraction of the points in the segment $[x, x + rz]$ belong to A, and the length of this segment is at least $c n^{1/4} |\log \varepsilon|^{1/4}$. □

2.3 Upper bounds for $0 < a \leq 1 - e^{-n}$

Let μ be a probability measure on $\mathbb{R}^n$ satisfying condition (i) and (ii) from Section 1. The constants c, C in this section depend solely on those from conditions (i) and (ii). Let $X = (X_1, \ldots, X_n)$ be a random vector with law μ. It follows from condition (i) that

$$E_i := \mathbb{E} X_i^2 > c \qquad \text{for all } i.$$

On the other hand, condition (ii) states that X_i is sub-Gaussian, and hence $X_i^2 - E_i$ is sub-exponential, in the terminology of [18, Section 2.8]. Denote

$$E = \sqrt{\mathbb{E}|X|^2} \in (c\sqrt{n}, C\sqrt{n}). \tag{31}$$

From Bernstein's inequality [18, Theorem 2.8.1], for $t > 0$,

$$\mathbb{P}\left(\left|\frac{|X|^2 - E^2}{\sqrt{n}}\right| \geq t\right) \leq \tilde{C} e^{-\tilde{c} \min\{t^2, t\sqrt{n}\}} \tag{32}$$

Since $||X| - E| \leq ||X|^2 - E^2|/E \leq C||X|^2 - E^2|/\sqrt{n}$, we conclude from (32) that

$$\mathbb{P}\left(E - t \leq |X| \leq E + t\right) \geq 1 - \tilde{C} e^{-\tilde{c} \min\{t^2, t\sqrt{n}\}}. \tag{33}$$

Lemma 2.6. *For any* $1/2 \le a \le 1 - e^{-n}$ *we have* $L(\mu, a) \le \tilde{C} \cdot n^{1/4} \cdot |\log(1-a)|^{1/4}$.

Proof. Define $\varepsilon = 1 - a$ and set

$$A = \left\{ x \in \mathbb{R}^n \,:\, E - \hat{C}\sqrt{|\log \varepsilon|} \le |x| \le E + \hat{C}\sqrt{|\log \varepsilon|} \right\}$$

so that $\mu(A) \ge 1 - \varepsilon$ thanks to (33). We need to show that $|A \cap \ell|$ is at most $Cn^{1/4}|\log \varepsilon|^{1/4}$ for any line ℓ. Indeed, it follows from (31) that for any $x \in A$,

$$E^2 - \tilde{C}\sqrt{n \cdot |\log \varepsilon|} \le |x|^2 \le E^2 + \tilde{C}\sqrt{n \cdot |\log \varepsilon|}.$$

In particular, if $\ell(t) = x + ty$ with $x, y \in \mathbb{R}^n$ and $|y| = 1$, the set of $t \in \mathbb{R}$ for which $\ell(t) \in A$ is contained in the set of $t \in \mathbb{R}$ for which t^2 belongs to an interval of length at most $2\tilde{C}\sqrt{n \cdot |\log \varepsilon|}$. This set is of Lebesgue measure at most $C'n^{1/4}|\log \varepsilon|^{1/4}$, completing the proof. See the proof of Corollary 3.7 below for one more argument. $\square$

Lemma 2.6 proves the upper bound in Theorem 1.3 in the range $1/2 \le a \le 1 - e^{-n}$. An upper bound for the range $a \in [0, 1/2]$ will be obtained from the upper bound in the case $a = 1/2$ and the following super-additivity property:

Lemma 2.7. *Let* μ *be an absolutely continuous measure in* $\mathbb{R}^n$. *Then, for any* $a, b \in (0, 1)$ *with* $a + b < 1$ *we have*

$$L(\mu, a + b) \ge L(\mu, a) + L(\mu, b).$$

Let $L(A) := \sup_\ell |A \cap \ell|$ and note that $L(\mu, A) = \inf_{\mu(A) \ge a} L(A)$. For the proof of the lemma we need the following claims.

Claim 2.8. *The function* $a \mapsto L(\mu, a)$ *is monotone and continuous in* $(0, 1)$.

Proof. The monotonicity of $L(\mu, a)$ is clear. Observe that $L(A \setminus B(x, \varepsilon)) \ge L(A) - 2\varepsilon$. Let $\varepsilon > 0$ and $a \in (0, 1)$. Then there is $\delta > 0$ such that for each set $A \subseteq \mathbb{R}^n$ with $\mu(A) > a/2$ there is $x \in \mathbb{R}^n$ with $\mu(A \cap B(x, \varepsilon)) > \delta$. By considering a near contender for the infimum of $L(\mu, a + \delta/2)$ and removing from it a ball of radius ε we obtain

$$L(\mu, a - \delta/2) \ge L(\mu, a + \delta/2) - 2\varepsilon.$$

This proves the continuity at a. $\square$

Claim 2.9. *For any* $0 < \lambda < 1$ *and* $\epsilon > 0$ *there is a set* $B \subseteq [0, 1]^n$ *with Vol*$(B) = \lambda$ *such that for any line* ℓ *we have* $|\ell \cap B| \le \lambda |\ell \cap [0, 1]^n| + \epsilon$.

Sketch of Proof. For $r \in [0\sqrt{n}]$ denote $Q_r = Q \cap rS^{n-1}$ and observe that the sets Q_r are disjoint and are subsets of spheres. We subdivide $[0, \sqrt{n}]$ into sufficiently small intervals, pick roughly λ-fraction of these small intervals that are roughly uniformly distributed in $[0, \sqrt{n}]$, and define B to be the union of the corresponding subsets of spheres. One may prove that for any line whose intersection with Q has length at least ε, we have the desired property: its intersection with the subsets of spheres consisting of B is roughly of length λ times the total intersection with the cube. Note that the volume of B obtained in this way is not exactly λ. To make the volume exactly λ we add or subtract a small ball in the same way as in the proof of Claim 2.8. ☐

We can now prove Lemma 2.7

Proof of Lemma 2.7. Since any set $A \subseteq \mathbb{R}^n$ contains a compact K with $\mu(A \setminus K) < \varepsilon$, we conclude from Claim 2.8 that

$$L(\mu, a) = \inf_{\mu(K)=a} L(K)$$

where the infimum runs over all compacts $K \subseteq \mathbb{R}^n$ with $\mu(K) = a$. Next, write K_δ for the δ-neighborhood of the compact K, and observe that $L(K_\delta) \longrightarrow L(K)$ as $\delta \to 0^+$. We say that $A \subseteq \mathbb{R}^n$ is elementary if it is a finite unions of cubes, each of the form $Q = \prod_{i=1}^n [a_i, b_i)$. For any compact K and $\delta > 0$ we may find an elementary set contained in K_δ and containining K. It follows that

$$L(\mu, a) = \inf_{\mu(A)=a} L(A) \tag{34}$$

where the infimum runs over all elementary sets $A \subseteq \mathbb{R}^n$ with $\mu(A) = a$.

Next, let $a, b \in (0, 1)$ satisfy $a + b < 1$ and denote $\lambda = a/(a + b)$. Let $\epsilon > 0$ and let A be an elementary set with $\mu(A) = a + b$ such that $L(A) \leq L(\mu, a + b) + \epsilon$. It follows from Claim 2.9 that there is a set $B \subseteq A$ with $\text{Vol}(B) = \lambda \text{Vol}(A) = a$ such that for any line ℓ we have $|\ell \cap B| \leq \lambda |\ell \cap A| + \epsilon$. In particular we have $L(B) \leq \lambda L(A) + \epsilon$. We obtain that

$$L(\mu, a) \leq L(B) \leq \lambda L(A) + \epsilon \leq \lambda L(\mu, a + b) + 2\epsilon. \tag{35}$$

Similarly we have that $L(\mu, b) \leq (1 - \lambda) L(\mu, a + b) + 2\epsilon$ and therefore

$$L(\mu, a) + L(\mu, b) \leq L(\mu, a + b) + 4\epsilon. \tag{36}$$

This finishes the proof of the lemma. ☐

From Lemma 2.6 and the super-additivity we immediately obtain:

Corollary 2.10. *For any* $0 < a \leq 1/2$ *we have* $L(\mu, a) \leq \tilde{C} n^{1/4} \cdot a$.

The proof of Theorem 1.3 is now complete, as the lower bounds follow from Proposition 2.3 and Proposition 2.5, while the upper bounds follow from Lemma 2.6 and Corollary 2.10.

Remark 2.11. The sub-Gaussian assumption in condition (ii) is not really used in the proof of Corollary 2.10, and it may be replaced by weaker conditions such as $\int_{-\infty}^{\infty} t^4 \rho_i(t) dt \leq C$.

3 The case of ℓ_p balls

In this section we prove Theorem 1.1. Throughout this section none of the estimates will be uniform in p. Thus, the constants C, c as well as the O and Θ notations are allowed to depend on p. In the proof we will use the following result from [3]. Recall the definition of B_p^n given in (3) and note that by Stirling's approximation $\kappa_{p,n} := \Gamma(1+n/p)^{1/n}/(2\Gamma(1+1/p))$ is of order $\Theta(n^{1/p})$.

Theorem 3.1. *Let $p > 0$ and $n \geq 1$. Let $g_1, \dots, g_n$ be i.i.d. random variables with density*

$$\frac{1}{2\Gamma(1+1/p)} e^{-|t|^p}, \quad t \in \mathbb{R}$$

and let Z be an independent $\exp(1)$ random variable. Then, the random vector

$$X = (X_1, \dots, X_n) := \frac{\kappa_{p,n}}{\left(\sum_{i=1}^n |g_i|^p + Z\right)^{1/p}} (g_1, \dots, g_n) \tag{37}$$

is uniformly distributed in B_p^n.

The following claims quantify the fact that the coordinates of a uniform point in B_p^n are roughly independent and behave like a constant multiple of the random variables g_i given in Theorem 3.1. To state the claim we let

$$\tilde{X}_i := a_n g_i \quad \text{where} \quad a_n := \frac{\kappa_{p,n} p^{1/p}}{n^{1/p}} = \frac{e^{-1/p}}{2\Gamma(1+1/p)} \Big(1 + O\Big(\frac{\log n}{n}\Big)\Big), \tag{38}$$

where the last equality follows from the definition of $\kappa_{p,n}$ after equation (3) and from Stirling's formula. In the claims we let X be the uniform point in B_p^n given by (37). By symmetry, it suffices to consider the first two coordinates X_1 and X_2 in the claims below. The first claim is Corollary 1 in [2].

Claim 3.2. *We have that $\mathit{Cov}\big(X_1^2, X_2^2\big) \leq 0$.*

Claim 3.3. *We have that $\mathbb{E}\big[(X_1 - \tilde{X}_1)^2\big] \leq C/n$*

Claim 3.4. *Let φ be a compactly supported differentiable function such that φ' is a Lipschitz function. Then, there exist a constant $C > 0$ depending on φ such that for all $n \geq 1$ and $1 \leq R \leq \sqrt{n}$ we have*

1.
$$\mathbb{E}\big[\varphi(RX_1) - \varphi(R\tilde{X}_1)\big] \leq \frac{C}{n}$$

2.
$$\mathit{Cov}\big(\varphi(RX_1), \varphi(RX_2)\big) \leq \frac{C}{nR}$$

The proofs of Claim 3.4 and Claim 3.3 are slightly technical and we postpone the proofs to Appendix A.

3.1 Upper bounds

The next corollary follows from Theorem 5 in [2].

Corollary 3.5. *Let $n \geq 1$, $p \in [1, \infty]$ and let X be a uniform sample from B_p^n. Then,*

$$\mathit{Var}\big(||X||_2^2\big) \leq Cn.$$

Proof. By Claim 3.2 we have

$$\mathrm{Var}\left(||X||_2^2\right) = \mathrm{Var}\left(\sum_{i=1}^n X_i^2\right) \leq \sum_{i=1}^n \mathrm{Var}\left(X_i^2\right) \leq Cn$$

as claimed. ∎

Remark 3.6. In fact a more careful analysis shows that when $p \neq 2$,

$$\mathrm{Var}\left(||X||_2^2\right) = (1 + o(1))\frac{p\Gamma(5/p)\Gamma(1/p) - (p+4)\Gamma(3/p)^2}{p\Gamma(1/p)^2}n.$$

We can now prove the following corollary that gives the right upper bound in the case that p is 1 or ∞.

Corollary 3.7. *For all $p \in [1, \infty]$ we have $L(B_p^n, 1/2) \leq Cn^{1/4}$.*

Proof. By Corollary 3.5 and Chebyshev's inequality, there exists $C_0 > 0$ such that

$$\mathbb{P}\Big(\big||X|^2 - \mathbb{E}[|X|^2]\big| \geq C_0\sqrt{n}\Big) \leq 1/2$$

and therefore the set

$$A := \left\{x \in B_p^n : \big||x|^2 - \mathbb{E}[|X|^2]\big| \leq C_0\sqrt{n}\right\}$$

has volume at least $1/2$. We claim the any line ℓ satisfies $|\ell \cap A| = O(n^{1/4})$. To this end, note that a line can intersect A in at most two intervals. We claim that each of these intervals has length of at most $O(n^{1/4})$. Indeed, let x and $x + y$ be the endpoints of one of these intervals and consider the function

$$f(t) := |x + ty|^2 = \sum_{i=1}^n (x_i + ty_i)^2.$$

For all $0 \leq t \leq 1$ we have that $x + ty \in A$ and therefore $f(t) = \mathbb{E}[|X|^2] + O(\sqrt{n})$. Thus

$$\frac{1}{2}\sum_{i=1}^n y_i^2 = f(1) + f(0) - 2f(1/2) \leq C_p\sqrt{n}.$$

and therefore the length of this interval is $|y| \leq C_p n^{1/4}$. ∎

The proof of the following corollary is similar to the proof of Corollary 3.7 but it uses an ℓ_p spherical shell instead of an ℓ_2 spherical shell.

Corollary 3.8. *For all* $1 < p \leq 2$ *we have that* $L(B_p^n, 1/2) \leq C(\log n)^{\frac{2-p}{2p}}$.

Proof. Define the sets

$$A := \left\{x \in B_p^n : ||x||_p^p \geq \kappa_{p,n}^p - C_0\right\} \quad \text{and} \quad B := \left\{\forall i \leq n,\ |x_i| \leq C_0 \log^{1/p} n\right\}.$$

where $C_0 > 0$ is a sufficiently large constant that will be chosen later. We start by proving that $\mathrm{Vol}(A \cap B) \geq 1/2$. We have

$$\mathrm{Vol}\left(\left\{x \in \mathbb{R}^n : \|x\|_p^p \leq \kappa_{p,n}^p - C_0\right\}\right) = \left(1 - \frac{C_0}{\kappa_{p,n}^p}\right)^{n/p} \leq 1/4,$$

where the last inequality holds as long as C_0 is sufficiently large since $\kappa_{p,n} = \Theta_p(n^{1/p})$. It follows that $\mathrm{Vol}(A) \geq 3/4$.

We turn to bound the volume of B. Let X be the uniform point in B_p^n given in Theorem 3.1. By Bernstein's inequality and the fact that the random variables $|g_i|^p$ from Theorem 3.1 have exponential tails, we have that

$$\mathbb{P}\left(c_1 n \leq \sum_{i=1}^n |g_i|^p \leq C_1 n\right) \geq 1 - Ce^{-cn}. \tag{39}$$

for some $c_1, C_1 > 0$. Moreover, the density of g_i is proportional to $e^{-|t|^p}$ and therefore $\mathbb{P}\big(|g_i| \geq 2\log^{1/p} n\big) \leq Cn^{-2}$. Thus, by (39) and Theorem 3.1, as long as C_0 is sufficiently large we have that $\mathbb{P}\big(|X_i| \geq C_0 \log^{1/p} n\big) \leq Cn^{-2}$. It follows from a union bound that $\mathrm{Vol}(B) \geq 1 - C/n$ and therefore $\mathrm{Vol}(A \cap B) \geq 1/2$.

We turn to show that for any line ℓ, we have that $|\ell \cap A \cap B| = O\big((\log n)^{\frac{2-p}{2p}}\big)$. The set A is the difference of two convex sets and therefore the intersection $\ell \cap A$ is the union of at most two intervals. As in the proof of Corollary 3.7, it suffices to bound the length of the intersection of each of these intervals with B. Let x and $x + y$ be two points inside one of these intervals such that $x \in B$. It suffices to show that $|y| = O\big((\log n)^{\frac{2-p}{2p}}\big)$. To this end, define the functions

$$f_i(t) := (x_i + ty_i)^p \quad \text{and} \quad f(t) := ||x + ty||_p^p = \sum_{i=1}^n f_i(t).$$

Since the interval $[x, x + y]$ is contained in A we have that

$$4C_0 \geq f(0) + f(1) - 2f(1/2) = \sum_{i=1}^n f_i(0) + f_i(1) - 2f_i(1/2). \tag{40}$$

Next, we have that

$$f_i(0) + f_i(1) - 2f_i(1/2) = \int_0^{1/2} s f_i''(s)ds + \int_{1/2}^1 (1-s) f_i''(s)ds$$
$$\geq \frac{1}{4}\int_{1/4}^{1/2} p(p-1)y_i^2(x_i + sy_i)^{p-2}ds \geq c y_i^2\big(\max(|y_i|, |x_i|)\big)^{p-2} \tag{41}$$
$$\geq c\min\left(|y_i|^p, |x_i|^{p-2}y_i^2\right) \geq c\min\left(|y_i|^p, (\log n)^{\frac{p-2}{p}}y_i^2\right),$$

where the first equality holds for any function and the last inequality follows as $x \in B$. We claim that $\min\left(|y_i|^p, (\log n)^{\frac{p-2}{p}}y_i^2\right) \geq c(\log n)^{\frac{p-2}{p}}y_i^2$. Indeed, if this minimum is $|y_i|^p$ then by (41) and (40) we have that $|y_i| \leq C$ and therefore $|y_i|^p \geq cy_i^2 \geq c(\log n)^{\frac{p-2}{p}}y_i^2$. Thus, we get the bound

$$f_i(0) + f_i(1) - 2f_i(1/2) \geq c(\log n)^{\frac{p-2}{p}}y_i^2.$$

Substituting this bound into (40) we get that

$$|y|^2 = \sum_{i=1}^n y_i^2 \leq C(\log n)^{\frac{2-p}{p}}.$$

This finishes the proof of the corollary. ◻

In the last two corollaries we saw that the Euclidean spherical shell and the ℓ_p spherical shell can be used to obtain upper bounds on $L(B_p^n, 1/2)$. The main idea of the proof of the following lemma is to consider a set which looks like a Euclidean shell for coordinates close to 0 and like an ℓ_p shell for larger coordinates.

Lemma 3.9. *For all $2 < p < \infty$ we have that $L(B_p^n, 1/2) \leq Cn^{\frac{p-2}{4p+2}}$.*

Proof. Define the convex function

$$h(r) := \begin{cases} \frac{p}{2}n^{\frac{2-p}{2p+1}}r^2 + \left(1 - \frac{p}{2}\right)n^{-\frac{p}{2p+1}} & |r| \leq n^{-\frac{1}{2p+1}} \\ |r|^p & |r| \geq n^{-\frac{1}{2p+1}} \end{cases}$$

Next, let $E := n\mathbb{E}[h(X_i)]$ and consider the set

$$A := \left\{x \in B_p^n : \Big|\sum_{i=1}^n h(x_i) - E\Big| \leq C_0\right\},$$

where C_0 is a sufficiently large constant that will be determined later. We start by proving that Vol$(A) \geq 1/2$. To this end, let $g(r) = h(r) - |r|^p$ and define the sets

$$A_1 := \left\{x \in B_p^n : \big|||x||_p^p - E_1\big| \leq C_0/2\right\}$$

and

$$A_2 := \Big\{x \in B_p^n : \ \Big|\sum_{i=1}^n g(x_i) - E_2\Big| \le C_0/2\Big\},$$

where $E_1 := n\mathbb{E}\big[|X_i|^p\big]$ and $E_2 := n\mathbb{E}\big[g(X_i)\big]$. It suffices to lower bound the volumes of A_1 and A_2 since $A_1 \cap A_2 \subseteq A$. By the same arguments as in the proof of Corollary 3.8 we have that $\mathrm{Vol}(A_1) \ge 3/4$ as long as C_0 is sufficiently large.

In order to lower bound the volume of A_2 we estimate the variance of $\sum g(X_i)$. To this end, let

$$\varphi(t) := \mathbb{1}\{|t| \le 1\} \cdot \big(\frac{p}{2}t^2 + 1 - \frac{p}{2} - |t|^p\big), \quad t \in \mathbb{R}$$

and note that φ is differentiable and φ' is Lipschitz. For all $r \in \mathbb{R}$ we have that

$$g(r) = n^{-\frac{p}{2p+1}}\varphi\big(n^{\frac{1}{2p+1}}r\big)$$

and therefore, by the second part of Claim 3.4 we have that

$$\begin{aligned}\mathrm{Var}\Big(\sum_{i=1}^n g(X_i)\Big) &= n^{-\frac{2p}{2p+1}} \cdot \mathrm{Var}\Big(\sum_{i=1}^n \varphi\big(n^{\frac{1}{2p+1}}X_i\big)\Big)\\ &= n^{-\frac{2p}{2p+1}}\sum_{i=1}^n \mathrm{Var}\Big(\varphi\big(n^{\frac{1}{2p+1}}X_i\big)\Big) + n^{-\frac{2p}{2p+1}}\sum_{i\ne j}\mathrm{Cov}\Big(\varphi\big(n^{\frac{1}{2p+1}}X_i\big), \varphi\big(n^{\frac{1}{2p+1}}X_j\big)\Big) \le C.\end{aligned}$$

Thus, as long as C_0 is sufficiently large, we have by Chebyshev's inequality $\mathrm{Vol}(A_2) \ge 3/4$ and therefore $\mathrm{Vol}(A) \ge 1/2$.

We turn to show that $|\ell \cap A| \le Cn^{\frac{p-2}{4p+2}}$ for any line ℓ. This part of the proof is identical to the corresponding part in the proof of Corollary 3.8 and therefore some of the details are omitted. Let $x, y \in \mathbb{R}^n$ such that the line segment from x to $x + y$ is contained in A. It suffices to show that $|y| \le Cn^{\frac{p-2}{4p+2}}$. We have that $h''(r) \ge cn^{-\frac{p-2}{2p+1}}$ for all r except for two points where h' is not differentiable and therefore the function $f(t) := \sum_{i=1}^n h(x_i + ty_i)$ satisfies

$$C \ge f(0) + f(1) - 2f(1/2) \ge cn^{-\frac{p-2}{2p+1}}\sum_{i=1}^n y_i^2.$$

This finishes the proof of the lemma. □

3.2 Lower bound when $2 \le p < \infty$

The main result in this section is the following proposition.

Proposition 3.10. *For all $2 \le p < \infty$ we have that $L(B_p^n, 1/2) = \Omega_p\big(n^{\frac{p-2}{4p+2}}\big)$.*

The main idea of the proof is to use a perturbation that changes only coordinates close to 0. Recall that X is uniform random variable in B_p^n. Let ψ be a smooth non-negative bump function supported on $[1,2]$. We think of ψ as a fixed function and allow the constants C and c to depend on ψ. Let $R, r > 0$ and let $\varphi(x) := r\psi(Rx)$. Finally, let δ_i be i.i.d. symmetric $\{-1,1\}$ Bernoulli random variables. Define the random vector $Y := (Y_1, \dots, Y_n)$ where

$$Y_i := X_i + \varphi(X_i)\delta_i. \tag{42}$$

The main idea of the proof of Proposition 3.10 is the following proposition that shows that the perturbation of X given in (42) does not change the distribution of X by much.

Proposition 3.11. *For all $2 < p < \infty$ there exists a small constant $\varepsilon > 0$ such that the following holds. Let $n \geq 1$ sufficiently large depending on ε and let $1 \leq R \leq \sqrt{n}$, $r \leq 1$ such that*

$$nR^3r^4 \leq \varepsilon^6, \quad R^{2p+1} \geq n. \tag{43}$$

Finally, let Y be the random variable defined by (42). *Then,*

$$d_{TV}(X, Y) \leq 1/4.$$

Using Proposition 3.11 we can easily prove Proposition 3.10.

Proof of Proposition 3.10. Let $n \geq 1$ sufficiently large and $R := n^{1/(2p+1)}$. Let X be a uniform point in B_p^n and define the random variable $W = (W_1, \dots, W_n)$ where $W_i := \psi(RX_i)\delta_i$.

We start by showing that $|W|^2 = \sum_{i=1}^n \psi(RX_i)^2$ is typically large. Recall the definition of $\tilde{X}_i$ in (38). We clearly have that

$$\mathbb{E}\big[\psi(R\tilde{X}_i)^2\big] \geq c/R, \quad \mathbb{E}\big[\psi(R\tilde{X}_i)^4\big] \leq C/R$$

and therefore, by the first part of Claim 3.4 we have

$$\mathbb{E}\big[\psi(RX_i)^2\big] \geq c/R, \quad \mathbb{E}\big[\psi(RX_i)^4\big] \leq C/R.$$

It follows that $\mathbb{E}[|W|^2] \geq cn/R$ and moreover, using the second part of Claim 3.4 we have

$$\mathrm{Var}\big(|W|^2\big) \leq \sum_{i=1}^n \mathbb{E}\big[\psi(RX_i)^4\big] + \sum_{i \neq j} \mathrm{Cov}\big(\psi(RX_i)^2, \psi(RX_i)^2\big) \leq \frac{Cn}{R}.$$

Thus, by Chebyshev's inequality there exists some $c_1 \geq 0$ such that

$$\mathbb{P}\big(|W| \geq c_1\sqrt{n/R}\big) \geq 0.99, \tag{44}$$

as long as n is sufficiently large. Next, fix $\varepsilon > 0$ such that Proposition 3.11 hold and let $r_0 := \varepsilon^2 R^{-3/4} n^{-1/4}$. Observe that, by Proposition 3.11, for any $r \leq r_0$ we have

$$d_{TV}(X, X + rW) \leq 1/4.$$

It follows that for all $A \subseteq B_p^n$ with $\mathrm{Vol}(A) \geq 1/2$ we have that

$$\mathbb{P}\big(X + rW \in A\big) \geq 1/4. \tag{45}$$

Since (45) holds for all $r \leq r_0$ it also holds when replacing r with a random variable $U \sim U[0, r_0]$ that is independent of X and W. Thus, by (44) there are some realizations $x \in B_p^n$ and $w \in \mathbb{R}^n$ with $|w| \geq c_1\sqrt{n/R}$ such that

$$\mathbb{P}\big(x + Uw \in A\big) \geq 1/5.$$

The last probability is exactly the normalized Lebesgue measure of the intersection of A with the line segment $[x, x + r_0 w]$. Thus, letting ℓ be the line containing x and $x + r_0 w$ we obtain

$$|\ell \cap A| \geq |r_0 w|/5 \geq c r_0 \sqrt{n/R} \geq c_\varepsilon n^{1/4} R^{-5/4} = c_\varepsilon n^{\frac{p-2}{4p+2}}$$

as needed. □

The rest of this section is devoted to the proof of Proposition 3.11. Throughout the proof we assume that ε is sufficiently small and n sufficiently large depending on ε. We start with the following lemma that gives a closed form expression to the density of Y.

Lemma 3.12. *The density of the random vector $Y = (Y_1, \ldots, Y_n)$ defined in* (42) *is given by*

$$f(y) = \mathbb{E}\Big[\mathbb{1}\big\{x(y,\delta) \in B_p^n\big\} \cdot \prod_{i=1}^n \big(1 + \varphi'(x_i)\delta_i\big)^{-1}\Big], \quad y \in \mathbb{R}^n, \tag{46}$$

where $x(y,\delta) := (x_1, \ldots, x_n)$ and $x_i = x_i(y_i, \delta_i)$ is the random variable defined to be the solution of the equation $y_i = x_i + \varphi(x_i)\delta_i$.

Proof. Assuming that $rR < 1/2$ (which follows from (43) and $R \leq \sqrt{n}$), and $z \in \{-1, 1\}$ the map $t \mapsto t + \varphi(t)z$ is a diffeomorphism and it has the Jacobian $1 + \varphi'(t)z > 0$. Thus, by the change of variables formula, the density at y conditioning on $\delta = (\delta_1, \ldots, \delta_n)$ is given by

$$f^{(\delta)}(y) = \mathbb{1}\big\{x(y,\delta) \in B_p^n\big\} \cdot \prod_{i=1}^n \big(1 + \varphi'(x_i)\delta_i\big)^{-1}.$$

It follows that the unconditional density is given by

$$f(y) = \mathbb{E}\Big[\mathbb{1}\{x(y,\delta) \in B_p^n\} \cdot \prod_{i=1}^n \big(1 + \varphi'(x_i)\delta_i\big)^{-1}\Big].$$

This finishes the proof of the lemma. □

In order to estimate the density given in Lemma 3.12 we restrict our attention to a subset of B_p^n of almost full measure. To this end, for $y \in \mathbb{R}^n$ let $I(y) := \{i : 1 \le Ry_i \le 2\}$ and for $t \in \mathbb{R}$ let $g(t) := \varphi''(t)\varphi(t) + \varphi'(t)^2$. Consider the set $A = A_1 \cap A_2 \cap A_3$ where

$$A_1 := \big\{y \in B_p^n : ||y||_p^p \le \kappa_{p,n}^p - \varepsilon\big\}, \quad A_2 := \Big\{y : |I(y)| \le \frac{n}{R\varepsilon}\Big\}$$

and

$$A_3 := \Big\{y : \Big|\sum_{i=1}^n g(y_i)\Big| \le \varepsilon\Big\}.$$

Proposition 3.11 follows immediately from the following two lemmas.

Lemma 3.13. *We have that Vol*$(A) \ge 1 - C\varepsilon$.

Lemma 3.14. *For all* $y \in A$ *we have that* $|f(y) - 1| \le C\varepsilon$.

Proof of Proposition 3.11. By Lemma 3.14 we have $\big|\mathbb{P}\big(Y \in A\big) - \mathbb{P}\big(X \in A\big)\big| \le C\varepsilon$ and by Lemma 3.13 we have $\mathbb{P}(X \notin A) \le C\varepsilon$. It follows that $\mathbb{P}(Y \notin A) \le C\varepsilon$ and therefore, using Lemma 3.14 once again we obtain

$$\begin{aligned} d_{TV}(X,Y) &= \frac{1}{2}\int_{\mathbb{R}^n} \big|f(y) - \mathbb{1}\{y \in B_p^n\}\big| dy \\ &\le \frac{1}{2}\int_A \big|f(y) - 1\big| dy + \mathbb{P}(X \notin A) + \mathbb{P}(Y \notin A) \le C\varepsilon. \end{aligned}$$

This finishes the proof of the proposition as long as ε is sufficiently small. ☐

It remains to prove Lemma 3.13 and Lemma 3.14.

Proof of Lemma 3.13. The first part of this proof is similar to the proof of Corollary 3.8. Let X be the uniform point in B_p^n given by (37). Then,

$$\mathbb{P}(\|X\|_p^p \le \kappa_{p,n}^p - \varepsilon) = \left(1 - \frac{\varepsilon}{\kappa_{p,n}^p}\right)^{n/p} \ge 1 - C\varepsilon$$

and therefore $\mathrm{Vol}(A_1) \ge 1 - C\varepsilon$.

We turn to bound the volume of A_2. To this end we claim that $\mathbb{P}(RX_i \in [1,2]) \le C/R$. Indeed, using the notation of Theorem 3.1, there exist some constant C_0 such that $\mathbb{P}(|X_i| \ge C_0|g_i|) \le Ce^{-cn}$. Thus,

$$\mathbb{P}(RX_i \in [1,2]) \le Ce^{-cn} + \mathbb{P}(|g_i| \le 2C_0/R) \le C/R,$$

where the last inequality follows as the density of g_i is bounded. Thus, by linearity of expectation $\mathbb{E}\big[I(X)\big] \le Cn/R$. Finally, by Markov's inequality we have that $\mathbb{P}\big(I(X) \ge n/(R\varepsilon)\big) \le C\varepsilon$ and therefore $\mathrm{Vol}(A_2) \ge 1 - C\varepsilon$.

Lastly, we bound the volume of A_3. Recall the definition of $\tilde{X}_i$ in (38) and note that the density of $\tilde{X}_i$ is given by $e^{-|t|^p/a_n^p}/\big(2a_n\Gamma(1+1/p)\big)$ where a_n satisfies $1/6 \le a_n \le 1$. Using integration by parts twice and the fact that $g = (\varphi^2/2)''$ we obtain

$$\begin{aligned}\big|\mathbb{E}[g(\tilde{X}_i)]\big| &\le C\Big|\int_0^\infty (\varphi^2)''(x)e^{-cx^p/a_n^p}dx\Big| \le C\Big|\int_0^\infty (\varphi^2)'(x)e^{-x^p/a_n^p}x^{p-1}dx\Big| \\ &\le C\int_0^\infty \varphi^2(x)x^{p-2}dx \le CR^{1-p}r^2 \le C\varepsilon^2/n,\end{aligned}$$

where in the fourth inequality we used that $\varphi(x) \le r$ for all x and that φ supported on $[1/R, 2/R]$ and in the last inequality we used (43). We now let $h := \psi''\psi + (\psi')^2$ and note that $g(x) = R^2r^2h(Rx)$. By the first part of Claim 3.4 with the function h we have

$$\Big|\mathbb{E}\Big[\sum_{i=1}^n g(X_i)\Big]\Big| \le CR^2r^2 + \sum_{i=1}^n \big|\mathbb{E}\big[g(\tilde{X}_i)\big]\big| \le C\varepsilon^2. \tag{47}$$

Next, using the second part of Claim 3.4 we obtain

$$\begin{aligned}\mathrm{Var}\Big(\sum_{i=1}^n g(X_i)\Big) &= R^4r^4\sum_{i=1}^n \mathrm{Var}\big(h(RX_i)\big) + R^4r^4\sum_{i\neq j}\mathrm{Cov}\big(h(RX_i), h(RX_i)\big) \\ &\le CnR^3r^4 \le C\varepsilon^2.\end{aligned} \tag{48}$$

Finally, by (47), (48) and Chebyshev's inequality

$$\mathbb{P}\Big(\Big|\sum_{i=1}^n g(X_i)\Big| \ge \varepsilon\Big) \le C\varepsilon$$

and therefore $\mathrm{vol}(A_3) \ge 1 - C\varepsilon$. □

We turn to prove Lemma 3.14. To this end we need the following claims.

Claim 3.15. *For all $y \in A$ we have that $\mathbb{P}\big(x(y,\delta) \notin B_p^n\big) \le \exp\big(-c\varepsilon^4R^{-2}r^{-2}\big)$.*

Claim 3.16. *We have that*

$$\mathbb{E}\big[\big(1+\varphi'(x_1)\delta_1\big)^{-1}\big] = 1 + g(y_1) + O(R^4r^4) \quad \textit{and} \quad \mathbb{E}\big[\big(1+\varphi'(x_1)\delta_1\big)^{-2}\big] = 1 + O(R^2r^2).$$

Using these claims we can easily prove Lemma 3.14.

Proof of Lemma 3.14. Let $y \in A$ and recall that $I = I(y) := \{i \le n : 1 \le Ry_i \le 2\}$ satisfies $|I| \le n/(R\varepsilon)$. Note that $\varphi'(x_i) = 0$ for any $i \notin I$ and therefore the product in (46) can be written as a product over $i \in I$. Thus, by Lemma 3.12 and Cauchy-Schwarz inequality

$$\begin{aligned}\Big|f(y) - \prod_{i\in I}\mathbb{E}\big[\big(1+\varphi'(x_i)\delta_i\big)^{-1}\big]\Big| &= \mathbb{E}\Big[\mathbb{1}\{x(y,\delta)\notin B_p^n\}\cdot\prod_{i\in I}\big(1+\varphi'(x_i)\delta_i\big)^{-1}\Big] \\ &\le \sqrt{\mathbb{P}\big(x(y,\delta)\notin B_p^n\big)}\cdot\prod_{i\in I}\sqrt{\mathbb{E}\big[\big(1+\varphi'(x_i)\delta_i\big)^{-2}\big]} \\ &\le \exp\big(-c\varepsilon^4R^{-2}r^{-2} + C|I|R^2r^2\big) \le \exp\big(-c\varepsilon^4R^{-2}r^{-2} + C\varepsilon^{-1}nRr^2\big),\end{aligned} \tag{49}$$

where in the second inequality we used Claim 3.15 and the second part of Claim 3.16. The right hand side of (49) is at most $e^{-c\varepsilon} \le 1 - C\varepsilon$ for sufficiently large n by (43).

Moreover, using the first part of Claim 3.16 we have

$$\begin{aligned}\prod_{i\in I} \mathbb{E}\big[\big(1+\varphi'(x_i)\delta_i\big)^{-1}\big] &= \prod_{i\in I}\Big(1+g(y_i)+O(R^4r^4)\Big) \\ &= \exp\Big(\sum_{i=1}^{n} g(y_i)+O(|I|R^4r^4)\Big) = \exp\big(O(\varepsilon+\varepsilon^{-1}nR^3r^4)\big) = 1+O(\varepsilon),\end{aligned} \tag{50}$$

where in the third equality we used that $y \in A_2 \cap A_3$ and in the last equality we used (43). The lemma follows from (49) and (50) as long as n is sufficiently large. □

We turn to prove Claim 3.15 amd Claim 3.16

Proof of Claim 3.15. Let $y \in A$ and recall that $x = x(y,\delta)$ is defined by $x = (x_1,\dots,x_n)$ where x_i is the solution to the equation $y_i = x_i + \varphi(x_i)\delta_i$. Define the random variable

$$S := \sum_{i\in I} y_i^{p-1}\varphi(y_i)\delta_i.$$

Using a second order Taylor expansion we have almost surely

$$\begin{aligned}||x||_p^p &= \sum_{i\notin I}|x_i|^p + \sum_{i\in I}\big(y_i-\varphi(x_i)\delta_i\big)^p \\ &= \sum_{i\notin I}|y_i|^p + \sum_{i\in I}\big[y_i^p - py_i^{p-1}\varphi(x_i)\delta_i + O\big(y_i^{p-2}r^2\big)\big] \\ &= ||y||_p^p - p\sum_{i\in I}\big[y_i^{p-1}\varphi(y_i)\delta_i + O\big(R^{2-p}r^2\big)\big] \\ &= ||y||_p^p + O\big(|S|+|I|R^{2-p}r^2\big) \\ &= ||y||_p^p + O\big(|S|+\varepsilon^{-1}nR^{1-p}r^2\big) = ||y||_p^p + O\big(|S|+\varepsilon^2\big),\end{aligned} \tag{51}$$

where in the third equality we used that $|\varphi(x_i)-\varphi(y_i)| \le CRr^2$, in the fifth inequality we used that $y \in A_2$ and in the last inequality we used (43). We turn to bound the sum S with high probability. The terms in this sum are almost surely bounded by $CR^{1-p}r$ and therefore by Azuma's inequality (see for example [1, Theorem 7.4.2]) we have that

$$\mathbb{P}\big(|S| \ge \varepsilon^{3/2}\big) \le \exp\Big(\frac{-c\varepsilon^3}{|I|R^{2-2p}r^2}\Big) \le \exp\big(-c\varepsilon^4 n^{-1}R^{2p-1}r^{-2}\big) \le \exp\big(-c\varepsilon^4R^{-2}r^{-2}\big).$$

where in the last inequality we used (43). Substituting the last estimate into (51) we get that

$$\mathbb{P}(x\notin B_p^n) = \mathbb{P}\big(||x||_p^p > \kappa_{p,n}^p\big) \le \mathbb{P}\big(||x||_p^p \ge ||y||_p^p + \varepsilon\big) \le \exp\big(-c\varepsilon^4R^{-2}r^{-2}\big).$$

where the last inequality holds for a sufficiently small ε. This finishes the proof of the claim 3.15. □

Proof of Claim 3.16. Recall that $\varphi(y) = r\psi(Ry)$ where ψ is a fixed bump function and therefore $\varphi'(y) = O(Rr), \varphi''(y) = O(R^2r)$ and $\varphi'''(y) = O(R^3r)$. We have that

$$x_1 = y_1 - \varphi(x_1)\delta_1 \tag{52}$$

and therefore $x_1 = y_1 + O(r)$. Substituting this estimate into the right hand side of (52) we get that $x_1 = y_1 - \varphi(y_1)\delta_1 + O(Rr^2)$. Substituting the last estimate once again into the right hand side of (52) we get

$$x_1 = y_1 - \delta_1\varphi(y_1) + \varphi'(y_1)\varphi(y_1) + O(R^2r^3).$$

Using the Taylor expansion of the function φ' around y_1 we obtain

$$\varphi'(x_1) = \varphi'(y_1) - \delta_1\varphi''(y_1)\varphi(y_1) + \varphi''(y_1)\varphi'(y_1)\varphi(y_1) + \frac{1}{2}\varphi'''(y_1)\varphi(y_1)^2 + O(R^4r^4).$$

Thus, using the fourth order Taylor expansion of the function $1/(1+w)$ we obtain

$$\begin{aligned}\mathbb{E}\big[\big(1+\varphi'(x_1)\delta_1\big)^{-1}\big] &= 1 - \mathbb{E}\big[\varphi'(x_1)\delta_1\big] + \mathbb{E}\big[\varphi'(x_1)^2\big] - \mathbb{E}\big[\varphi'(x_1)^3\delta_1\big] + O(R^4r^4)\\ &= \varphi''(y_1)\varphi(y_1) + \varphi'(y_1)^2 + O(R^4r^4) = g(y_1) + O(R^4r^4)\end{aligned}$$

This finishes the proof of the first part of the claim. The second part follows using the same arguments. ◻

3.3 Lower bound when $1 < p < 2$

In this section we prove the following proposition.

Proposition 3.17. *For all $1 < p < 2$ we have that $L(B_p^n, 1/2) = \Omega_p\big((\log n)^{\frac{2-p}{2p}}\big)$.*

The proof is similar to the case $p > 2$ but with one additional ingredient. In this case, perturbing each coordinate independently will push the random point outside of B_p^n with high probability. To overcome this issue we perturb each pair of coordinates independently. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a fixed, non-negative, smooth, two dimensional, bump function supported on $[1,2]^2$. Let $R_1, R_2 \geq 1, 0 < r < 1$ and let $\varphi(x_1, x_2) := r\psi(R_1(x_1 - R_2), R_1(x_2 - R_2))$. Finally, let

$$h(x_1, x_2) := \varphi(x_1, x_2) \cdot \big(x_1^{1-p}, -x_2^{1-p}\big). \tag{53}$$

Consider the random variable $Y = (Y_1 \dots, Y_n)$ given by

$$\big(Y_{2i-1}, Y_{2i}\big) := \big(X_{2i-1}, X_{2i}\big) + \delta_i h\big(X_{2i-1}, X_{2i}\big), \quad i \leq \lfloor n/2 \rfloor, \tag{54}$$

where δ_i are i.i.d. symmetric $\{-1, 1\}$ Bernoulli random variables and if n is odd we let $Y_n = X_n$.

Proposition 3.18. *For any $1 < p < 2$ there exists $\varepsilon > 0$ such that the following holds. Let $n \geq 1$ sufficiently large and let $R_1, R_2, r > 0$ such that*

$$1 \leq R_2 \leq \log^{1/p} n, \quad R_1 = \log n, \quad nR_1^{-2}r^2R_2^{-p}e^{-2R_2^p/a_n^p} \leq \varepsilon^2, \quad r^5n^2 \leq 1, \tag{55}$$

where a_n is given in (38). *Then, the random variable Y given in* (54) *satisfies*

$$d_{TV}(X, Y) \leq 1/4.$$

We turn to prove Proposition 3.17. In the proof and throughout this section we use the notation $\tilde{O}$ to hide a poly-logarithmic factor of the form $\log^C n$ where, as usual, we allow the constant C to depend on p. In order to simplify the arguments we also assume throughout the section that n is even. The proof for odd n is identical.

Proof of Proposition 3.10. Let $n \geq 1$ sufficiently large and even, $R_1 := \log n$ and $R_2 = c_1 \log^{1/p} n$ where $c_1 := 0.01$. Let X be a uniform point in B_p^n and define the random variable $W := (W_1, \ldots, W_n)$ where for any $i \leq \lfloor n/2 \rfloor$ we let

$$(W_{2i-1}, W_{2i}) := \psi\big(R_1(X_{2i-1} - R_2), R_2(X_{2i} - R_2)\big)\big(X_{2i-1}^{1-p}, -X_{2i}^{1-p}\big)\delta_i.$$

We turn to show that $|W|$ is typically large. We have that $|W|^2 = \sum_{i=1}^{n/2} \xi(X_{2i-1}, X_{2i})$ where

$$\xi(x_1, x_2) := \psi\big(R_1(x_1 - R_2), R_2(x_2 - R_2)\big)^2\big(x_{2i-1}^{2-2p} + x_{2i}^{2-2p}\big).$$

Define the random variable $N := \sum_{i=1}^{n/2} \xi(\tilde{X}_{2i-1}, \tilde{X}_{2i})$. The function ξ is supported on $[R_2 + 1/R_1, R_2 + 2/R_1]^2$ where the density of the pair $(\tilde{X}_1, \tilde{X}_2)$ is at least $ce^{-2R_2^p/a_n^p} = cn^{-2c_1^p/a_n^p}$. It follows that $\mathbb{E}\big[\xi(\tilde{X}_1, \tilde{X}_2)\big] \geq cR_1^{-2}R_2^{2-2p}n^{-2c_1^p/a_n^p}$ and therefore $\mathbb{E}[N] \geq cnR_1^{-2}R_2^{2-2p}n^{-2c_1^p/a_n^p}$. Note that $1/6 \leq a_n \leq 1$ and therefore $\mathbb{E}[N] \geq n^{3/4}$. Next, since $\tilde{X}_i$ are independent we have $\mathrm{Var}(N) = \tilde{O}(n)$. Thus, by Chebyshev's inequality, there exists some $c_2 > 0$ such that

$$\mathbb{P}\Big(\sum_{i=1}^{n/2} \xi(\tilde{X}_{2i-1}, \tilde{X}_{2i}) \geq c_2 n R_1^{-2} R_2^{2-2p} n^{-2c_1^p/a_n^p}\Big) \geq 0.99, \tag{56}$$

as long as n is sufficiently large. In order to bound $|W|$ it suffices to replace the random variables $\tilde{X}_i$ with X_i in the last estimate. To this end, note that the function ξ and its partial derivatives are bounded by $\tilde{O}(1)$ and therefore by Claim 3.3 we have $\mathbb{E}\big|\xi(X_1, X_2) - \xi(\tilde{X}_1, \tilde{X}_2)\big| = \tilde{O}(n^{-1/2})$. Thus, by (56) there exists $c_3 > 0$ such that

$$\begin{aligned}\mathbb{P}\Big(|W| \geq c_3\sqrt{n}R_1^{-1}R_2^{1-p}n^{-c_1^p/a_n^p}\Big)& \\ = \mathbb{P}\Big(\sum_{i=1}^{n/2} \xi(X_{2i-1}, X_{2i}) &\geq c_3^2 n R_1^{-2} R_2^{2-2p} n^{-2c_1^p/a_n^p}\Big) \geq 0.99.\end{aligned} \tag{57}$$

The rest of the proof is almost identical to the proof of Proposition 3.10 and some of the details are omitted. We let $r_0 := \varepsilon R_1 R_2^{p/2} n^{c_1^p/a_n^p} n^{-1/2}$ and note that, by the choice of c_1 and the definition of a_n in (38), we have that $r_0^5 n^2 \leq 1$. By Proposition 3.18, for all $A \subseteq B_p^n$ and for all $0 < r < r_0$ we have $\mathbb{P}\big(X + rW \in A\big) \geq 1/4$. Thus, by (57), there exist $x, w \in \mathbb{R}^n$ with $|w| \geq c_3\sqrt{n}R_1^{-1}R_2^{1-p}n^{-c_1^p/a_n^p}$ such that for $U \sim U[0, r_0]$ we have that $\mathbb{P}(x + Uw \in A) \geq 1/5$. It follows that the line ℓ containing x and $x + w$ satisfies

$$|\ell \cap A| \geq |r_0 w|/5 \geq c_\varepsilon R_2^{1-p+p/2} \geq c_\varepsilon (\log n)^{\frac{2-p}{2p}}.$$

This finishes the proof of the proposition. ∎

As in the proof of Proposition 3.11, we start by computing the density of Y. To this end, given $y \in \mathbb{R}^n$, for each $i \le n/2$, let x_{2i-1} and x_{2i} be the random variables defined as the unique solutions to the equation

$$(y_{2i-1}, y_{2i}) = (x_{2i-1}, x_{2i}) + \delta_i h(x_{2i-1}, x_{2i}).$$

When $r \le n^{-2/5}$ it is clear that the differential of the map $(x_{2i-1}, x_{2i}) \mapsto (x_{2i-1}, x_{2i}) + \delta_i h(x_{2i-1}, x_{2i})$ is invertible and that the map is a diffeomorphism. We let $x = x(y, \delta) = (x_1, \dots, x_n)$ and let $J(w_1, w_2, z)$ be the Jacobian determinant of the map $(w_1, w_2) \mapsto (w_1, w_2) + zh(w_1, w_2)$ at the point (w_1, w_2). The following lemma follows from the change of variables formula in the same way as Lemma 3.12.

Lemma 3.19. *The density of the random variable $Y = (Y_1, \dots, Y_n)$ defined in* (42) *is given by*

$$f(y) = \mathbb{E}\Big[\mathbb{1}\{x(y,\delta) \in B_p^n\} \cdot \prod_{i=1}^{n/2} J(x_{2i-1}, x_{2i}, \delta_i)^{-1}\Big], \quad y \in \mathbb{R}^n.$$

As in the proof of Proposition 3.11, in order to estimate the density given in Lemma 3.19, we restrict our attention to a set of almost full measure. To this end, define the function

$$g(y_1, y_2) := \frac{1}{2}\frac{\partial^2 h_1^2}{\partial y_1^2}(y_1, y_2) + \frac{\partial^2 (h_1 h_2)}{\partial y_1 \partial y_2}(y_1, y_2) + \frac{1}{2}\frac{\partial^2 h_2^2}{\partial y_2^2}(y_1, y_2), \quad (y_1, y_2) \in \mathbb{R}^2 \tag{58}$$

and the set $A := A_1 \cap A_2$ where

$$A_1 := \{y \in B_p^n : ||y||_p^p \le \kappa_{p,n}^p - \varepsilon\}, \quad A_2 := \Big\{y : \Big|\sum_{i=1}^{n/2} g(y_{2i-1}, y_{2i})\Big| \le \varepsilon\Big\}.$$

Proposition 3.18 clearly follows from the following two lemmas.

Lemma 3.20. *We have that Vol$(A) \ge 1 - C\varepsilon$.*

Lemma 3.21. *For any $y \in A$ we have that $|f(y) - 1| \le C\varepsilon$.*

We start by proving Lemma 3.20.

Proof of Lemma 3.20. We have that $\mathrm{Vol}(A_1) \ge 1 - C\varepsilon$ by the same arguments as in the proof of Lemma 3.13.

We turn to bound the volume of A_2. To this end, we bound $\mathbb{E}\big[g(\tilde{X}_1, \tilde{X}_2)\big]$. Note that the density of $\tilde{X}_i$ at y_i is given by $A_n \exp(-(|y_i|^p)/a_n^p)$ for some sequence $c \le A_n \le C$ and therefore

$$\mathbb{E}\big[g(\tilde{X}_1, \tilde{X}_2)\big] = A_n^2 \int_{\mathbb{R}^2} g(y_1, y_2) e^{-(y_1^p + y_2^p)/a_n^p} dy_1 dy_2 = A_n^2\big(I_1 + I_2 + I_3\big)$$

where I_1, I_2 and I_3 are the tree integrals corresponding to the first, second and third terms in the right hand side of (58). Using integration by parts twice we obtain

$$I_1 = \frac{1}{2}\int_{\mathbb{R}^2} \frac{\partial^2 h_1^2}{\partial y_1^2}(y_1, y_2) e^{-(y_1^p+y_2^p)/a_n^p} dy_1 dy_2 = \int_{\mathbb{R}^2} \frac{\partial h_1^2}{\partial y_1}(y_1, y_2) \frac{p y_1^{p-1}}{2a_n^p} e^{-(y_1^p+y_2^p)/a_n^p} dy_1 dy_2$$
$$= \int_{\mathbb{R}^2} h_1^2(y_1, y_2)\Big(\frac{p^2 y_1^{2p-2}}{2a_n^{2p}} - \frac{p(p-1)y_1^{p-2}}{2a_n^p}\Big) e^{-(y_1^p+y_2^p)/a_n^p} dy_1 dy_2.$$

Similarly we have that

$$I_3 = \int_{\mathbb{R}^2} h_2^2(y_1, y_2)\Big(\frac{p^2 y_2^{2p-2}}{2a_n^{2p}} - \frac{p(p-1)y_2^{p-2}}{2a_n^p}\Big) e^{-(y_1^p+y_2^p)/a_n^p} dy_1 dy_2$$

and

$$I_2 = \int_{\mathbb{R}^2} (h_1 h_2)(y_1, y_2) \frac{p^2 (y_1 y_2)^{p-1}}{a_n^{2p}} e^{-(y_1^p+y_2^p)/a_n^p} dy_1 dy_2.$$

Adding these contributions we obtain

$$\begin{aligned}\big|\mathbb{E}[g(\tilde{X}_1, \tilde{X}_2)]\big| \le C\big|I_1 + I_2 + I_3\big| \le C\int_{\mathbb{R}^2} \Big(\big(h_1(y_1, y_2)y_1^{p-1} + h_2(y_1, y_2)y_2^{p-1}\big)^2 + \\ + h_1^2(y_1, y_2)y_1^{p-2} + h_2^2(y_1, y_2)y_2^{p-2}\Big) e^{-(y_1^p+y_2^p)/a_n^p} dy_1 dy_2 \qquad (59) \\ = C\int_{\mathbb{R}^2} \varphi^2(y_1, y_2)(y_1^{-p} + y_2^{-p}) e^{-(y_1^p+y_2^p)/a_n^p} dy_1 dy_2 \le C R_1^{-2} r^2 R_2^{-p} e^{-2R_2^p/a_n^p} \le C\varepsilon^2/n,\end{aligned}$$

where in the equality we substituted the definition of h in (53) and in the last inequality we used the assumption in (55). Note the important cancellation of the first term in the integral. This cancellation is related to the fact that the perturbation given in (54) typically does not push the random point outside the ball.

Next, let $N := \sum_{i=1}^{n/2} g(\tilde{X}_{2i-1}, \tilde{X}_{2i})$ and note that by (59) we have that $|\mathbb{E}[N]| \le C\varepsilon^2$. Moreover, using that $\tilde{X}_i$ are independent and (55) we obtain $\mathrm{Var}(N) \le n\mathbb{E}\big[g(\tilde{X}_1, \tilde{X}_2)^2\big] = \tilde{O}(nr^4) = \tilde{O}(n^{-3/5})$. Thus, by Chebyshev's inequality, there exists $C_1 > 0$ such that

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{n/2} g(\tilde{X}_{2i-1}, \tilde{X}_{2i})\Big| \le C_1\varepsilon^2\Big) \ge 1 - \varepsilon,$$

as long as n is sufficiently large. In order to bound the volume of A_2 it suffices to replace $\tilde{X}_i$ with X_i in the last estimate. To this end note that g and its partial derivatives are bounded by $\tilde{O}(r^2)$ and therefore by Claim 3.3, we have $\mathbb{E}\big|g(X_1, X_2) - g(\tilde{X}_1, \tilde{X}_2)\big| = \tilde{O}(r^2 n^{-1/2})$. Thus, there exists $C_2 > 0$ such that

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{n/2} g(X_{2i-1}, X_{2i})\Big| \le C_2\varepsilon^2\Big) \ge 1 - 2\varepsilon.$$

It follows that $\mathrm{Vol}(A_2) \geq 1 - 2\varepsilon$ for a sufficiently small ε and a sufficiently large n depending on ε. ◻

We turn to prove Lemma 3.21. To this end we need the following claim.

Claim 3.22. *We have that*

$$\mathbb{E}\big[J(x_1, x_2, \delta_1)^{-1}\big] = 1 + g(y_1, y_2) + \tilde{O}(r^3).$$

Proof. We have that

$$\begin{aligned} J(x_1, x_2, \delta_1) &= \Big(1 + \delta_1 \frac{\partial h_1}{\partial x_1}(x_1, x_2)\Big)\Big(1 + \delta_1 \frac{\partial h_2}{\partial x_2}(x_1, x_2)\Big) - \frac{\partial h_1}{\partial x_2}(x_1, x_2)\frac{\partial h_2}{\partial x_1}(x_1, x_2) \\ &= 1 + \delta_1\Big(\frac{\partial h_1}{\partial x_1}(x_1, x_2) + \frac{\partial h_2}{\partial x_2}(x_1, x_2)\Big) + \frac{\partial h_1}{\partial x_1}(x_1, x_2)\frac{\partial h_2}{\partial x_2}(x_1, x_2) - \frac{\partial h_1}{\partial x_2}(x_1, x_2)\frac{\partial h_2}{\partial x_1}(x_1, x_2) \end{aligned}$$

Next, we replace the random points x_1, x_2 with the deterministic points y_1, y_2. Note that the terms in the brackets are of order $\tilde{O}(r)$ while the other terms are of order $\tilde{O}(r^2)$. We have that $y_i = x_i + \tilde{O}(r)$ and therefore, in the terms outside the brackets, x_1, x_2 can be replaced by y_1, y_2 without changing the overall expression by more than $\tilde{O}(r^3)$. For the terms inside the brackets we use the expansion

$$(x_1, x_2) = (y_1, y_2) - \delta_1 h(y_1, y_2) + \tilde{O}(r^2).$$

We obtain that

$$\begin{aligned} J(x_1, x_2, \delta_1) &= 1 + \delta_1\Big(\frac{\partial h_1}{\partial x_1}(y_1, y_2) - \delta_1 \frac{\partial^2 h_1}{\partial x_1^2}(y_1, y_2)h_1(y_1, y_2) - \delta_1 \frac{\partial^2 h_1}{\partial x_1 \partial x_2}(y_1, y_2)h_2(y_1, y_2) \\ &\quad + \frac{\partial h_2}{\partial x_2}(y_1, y_2) - \delta_1 \frac{\partial^2 h_2}{\partial x_2 \partial x_1}(y_1, y_2)h_1(y_1, y_2) - \delta_1 \frac{\partial^2 h_2}{\partial x_2^2}(y_1, y_2)h_2(y_1, y_2)\Big) \\ &\quad + \frac{\partial h_1}{\partial x_1}(y_1, y_2)\frac{\partial h_2}{\partial x_2}(y_1, y_2) - \frac{\partial h_1}{\partial x_2}(y_1, y_2)\frac{\partial h_2}{\partial x_1}(y_1, y_2) + \tilde{O}(r^3) \\ &= 1 + \delta_1\Big(\frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2}\Big) - \frac{\partial^2 h_1}{\partial x_1^2}h_1 - \frac{\partial^2 h_1}{\partial x_1 \partial x_2}h_2 - \frac{\partial^2 h_2}{\partial x_2 \partial x_1}h_1 - \frac{\partial^2 h_2}{\partial x_2^2}h_2 \\ &\quad + \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} + \tilde{O}(r^3). \end{aligned}$$

Thus, using a second order Taylor expansion of $1/(1+x)$ we obtain

$$\begin{aligned}\mathbb{E}\big[J(x_1,x_2,\delta_1)^{-1}\big] &= 1 + \Big(\frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2}\Big)^2 + \frac{\partial^2 h_1}{\partial x_1^2} h_1 + \frac{\partial^2 h_1}{\partial x_1 \partial x_2} h_2 + \frac{\partial^2 h_2}{\partial x_2 \partial x_1} h_1 \\ &\quad + \frac{\partial^2 h_2}{\partial x_2^2} h_2 - \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} + \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} + \tilde{O}(r^3) \\ &= 1 + \Big(\frac{\partial h_1}{\partial x_1}\Big)^2 + \Big(\frac{\partial h_2}{\partial x_2}\Big)^2 + \frac{\partial^2 h_1}{\partial x_1^2} h_1 + \frac{\partial^2 h_1}{\partial x_1 \partial x_2} h_2 \\ &\quad + \frac{\partial^2 h_2}{\partial x_2 \partial x_1} h_1 + \frac{\partial^2 h_2}{\partial x_2^2} h_2 + \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} + \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} + \tilde{O}(r^3) \\ &= 1 + \frac{1}{2}\frac{\partial^2 h_1^2}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 h_2^2}{\partial x_2^2} + \frac{\partial^2 h_1 h_2}{\partial x_1 \partial x_2} + \tilde{O}(r^3) = 1 + g(y_1,y_2) + \tilde{O}(r^3).\end{aligned}$$

This finishes the proof of the claim. ☐

Proof of Lemma 3.21 . First, we claim that for all $y \in A$ we have that $x = x(y,\delta) \in B_p^n$ with probability one. Using a second order Taylor expansion we obtain

$$\begin{aligned}||y||_p^p &= \sum_{i=1}^{n} |y_i|^p = \sum_{i=1}^{n/2} \big|x_{2i-1} + \delta_i h_1(x_{2i-1},x_{2i})\big|^p + \big|x_{2i} + \delta_i h_2(x_{2i-1},x_{2i})\big|^p \\ &= \sum_{i=1}^{n/2} \Big(|x_{2i-1}|^p + p x_{2i-1}^{p-1} h_1(x_{2i-1},x_{2i})\delta_i + p(p-1) x_{2i-1}^{p-2} h_1(x_{2i-1},x_{2i})^2 \\ &\qquad + |x_{2i}|^p + p x_{2i}^{p-1} h_2(x_{2i-1},x_{2i})\delta_i + p(p-1) x_{2i}^{p-2} h_2(x_{2i-1},x_{2i})^2 + \tilde{O}(r^3)\Big) \\ &= ||x||_p^p + \tilde{O}(nr^3) + p(p-1)\sum_{i=1}^{n/2} x_{2i-1}^{p-2} h_1(x_{2i-1},x_{2i})^2 + x_{2i}^{p-2} h_2(x_{2i-1},x_{2i})^2.\end{aligned}$$

where in the last equality we used the definition of h in (53). This cancellation of the linear term in the expansion is the reason we perturbed pairs of coordinates instead of individual coordinates. Thus, using that $y \in A_1$ we obtain

$$||x||_p^p \le ||y||_p^p + \tilde{O}(nr^3) \le \kappa_{p,n}^p,$$

where the last inequality holds for sufficiently large n as $r^5 n^2 \le 1$. It follows that $x \in B_p^n$. Thus, for $y \in A$ we have

$$f(y) = \prod_{i=1}^{n/2} \mathbb{E}\big[J(x_{2i-1},x_{2i},\delta_i)^{-1}\big] = \exp\Big(\tilde{O}(nr^3) + \sum_{i=1}^{n/2} g(y_{2i-1},y_{2i})\Big) = 1 + O(\varepsilon),$$

where in the second equality we used Claim 3.22 and that $r^5 n^2 \le 1$. ☐

3.4 Lower bound when $p = 1$

Consider the simplex $\Delta^n := B_1^n \cap \{x \in R^n : \forall i,\ x_i \geq 0\}$ and let μ_n be the uniform measure on Δ^n. It suffices to prove the following lemma

Lemma 3.23. *We have that $L(\mu_n, 1/2) = \Omega(n^{1/4})$.*

It follows from the lemma that $L(B_1^n, 1/2) = \Omega(n^{1/4})$. Indeed, let $A \subseteq B_1^n$ with $\mathrm{Vol}(A) \geq 1/2$. There exist a quadrant $Q_\varepsilon := \{x \in \mathbb{R}^n : \forall i,\ \varepsilon_i x_i \geq 0\}$ of the space such that $\mathrm{Vol}(Q_\varepsilon \cap A) \geq 2^{-n-1}$. Without loss of generality suppose that this is the first quadrant (the quadrant corresponding to all coordinates are positive or $\varepsilon = (1, \ldots, 1)$). We have that $\mu_n(Q_\varepsilon \cap A) \geq 1/2$ and therefore by Lemma 3.23 there exists a line ℓ so that $|\ell \cap A| \geq |\ell \cap Q \cap A| \geq cn^{1/4}$. This shows that $L(B_1^n, 1/2) = \Omega(n^{1/4})$.

For the proof of Lemma 3.23 we need the following claim. To state the claim we let $g := (g_1, \ldots, g_n)$ be an i.i.d. sequence of $\exp(1)$ random variables and let $\delta_1, \ldots, \delta_n$ be and i.i.d sequence of symmetric $\{-1, 1\}$ Bernoulli random variables independent of g. We also let ψ be a smooth, non-negative bump function supported on $[1, 2]$. For $r > 0$ define the random variable $f = f^{(r)} := (f_1, \ldots, f_n)$ where $f_i := g_i + r\psi(g_i)\delta_i$.

Claim 3.24. *There exists $\varepsilon > 0$ depending only on ψ such that for all $r \leq \varepsilon n^{-1/4}$ we have $d_{TV}(f, g) \leq 1/4$.*

The proof of Claim 3.24 is left as an exercise to the reader. The proof is a minor modification of the claims in Section 2.1 and Section 2.2. Note that in these sections assumption (ii) from Theorem 1.3 is not required.

Proof of Lemma 3.23. Let $g_1, \ldots, g_n, Z$ be an i.i.d. sequence of $\exp(1)$ random variables. It follows from Theorem 3.1 that the vector $X := (X_1, \ldots, X_n)$ defined by

$$X_i := \frac{(n!)^{1/n} g_i}{2\big(Z + \sum_{j=1}^n g_j\big)}.$$

is uniformly distributed in Δ^n. Fix $\varepsilon > 0$ such that Claim 3.24 holds and let $r \leq \varepsilon n^{-1/4}$. Let $f^{(r)}$ be the random variable from Claim 3.24 and suppose that the variables δ_i in the definition of $f^{(r)}$ are independent of $g_1, \ldots, g_n$ and Z. Define the random variable $Y = Y^{(r)} = (Y_1, \ldots, Y_n)$ by

$$Y_i := \frac{(n!)^{1/n} f_i}{2\big(Z + \sum_{j=1}^n f_j\big)}.$$

By Claim 3.24 we have that $d_{TV}(X, Y^{(r)}) \leq 1/4$ and therefore for any subset $A \subseteq \Delta^n$ with $\mu_n(A) \geq 1/2$ we have that

$$\mathbb{P}\big(Y^{(r)} \in A\big) \geq 1/4 \tag{60}$$

To simplify the notations we define the random variables

$$P := \frac{2\big(Z + \sum_{j=1}^n g_j\big)}{(n!)^{1/n}}, \quad Q := \frac{2\sum_{j=1}^n \psi(g_j)\delta_j}{(n!)^{1/n}}, \quad W_i := \psi(g_i)\delta_i - \frac{Q}{P} g_i$$

and $W := (W_1, \dots, W_n)$. We have that $X_i = g_i/P$ and a straightforward computation shows that

$$Y_i = \frac{g_i + r\psi(g_i)\delta_i}{P + rQ} = \frac{g_i}{P} + \frac{r}{P + rQ}\Big(\psi(g_i)\delta_i - \frac{Q}{P}g_i\Big) = X_i + \frac{r}{P + rQ}W_i. \tag{61}$$

Next, since (60) holds for any $r \le \varepsilon n^{-1/4}$, it holds when replacing r with $U \sim U[0, \varepsilon n^{-1/4}]$ independent of all other random variables. Thus, rewriting (60) using (61) we obtain

$$\mathbb{P}\Big(X + \frac{U}{P + UQ}W \in A\Big) \ge 1/4.$$

By the central limit theorem, Stirling's formula and the fact that $5 < 2e < 6$ we have that

$$\mathbb{P}\big(5 < P < 6\big) \ge 0.99 \quad \mathbb{P}\big(|Q| \le C_1 n^{-1/2}\big) \ge 0.99 \quad \mathbb{P}\big(|W| \ge c_1\sqrt{n}\big) \ge 0.97$$

for some $C_1, c_1 > 0$ and n sufficiently large. Thus, there are $x, w \in \mathbb{R}^n$ with $|w| \ge c_1\sqrt{n}$ and $p, q \in \mathbb{R}$ with $5 < p < 6$ and $|q| \le C_1 n^{-1/2}$ such that

$$\mathbb{P}\Big(x + \frac{U}{p + Uq}w \in A\Big) \ge 1/5.$$

It is straightforward to check that the ratio between the densities of the random variables $U/(p + Uq)$ and U/p tends to 1 as n tends to infinity and therefore for sufficiently large n we have

$$\mathbb{P}\big(x + (U/p)w \in A\big) \ge 1/6.$$

It follows that the line ℓ containing x and $x + w$ satisfies

$$|\ell \cap A| \ge \varepsilon n^{-1/4}|w|/(6p) \ge c_\varepsilon n^{1/4}$$

as needed. □

A Proof of Claim 3.4

Proof of Claim 3.4. First, note that by a straightforward calculation with the density of g_k we have that

$$\mathbb{E}[|g_k|^p] = 1/p \quad \text{and} \quad \mathrm{Var}(|g_k|^p) = 1/p. \tag{62}$$

Thus, the random variable

$$N := \frac{1}{n}\sum_{k=3}^{n}\Big(|g_k|^p - \frac{1}{p}\Big) \tag{63}$$

is roughly normally distributed with variance of order $1/n$. In particular we have that $\mathbb{E}[N] = 0$ and $\mathbb{E}[N^{2m}] \le C_m/n^m$ for all $m \in \mathbb{N}$. In the definition of N we do not sum over $k = 1, 2$ in order to make it independent of g_1 and g_2. By (62), (63) and Theorem 3.1 we have that X_1 is well approximated by $\tilde{X}_1$. We claim that the following, more accurate approximation of X_1, holds. We have

$$X_1 := \tilde{X}_1 - \tilde{X}_1 N + \tilde{X}_1 H_1, \tag{64}$$

where H_1 is some random variables with $\mathbb{E}[H_1^m] \le C_m/n^m$ for all $m \ge 1$. Intuitively, (64) says that X_1 can be written as $\tilde{X}_1$ plus a random variable with 0 expectation of order $n^{-1/2}$ plus a random variable of order n^{-1}. The approximation in (64) follows from Theorem 3.1 and a second order Taylor expansion of the function $(1+x)^{-1/p}$. Indeed, by (37) we have

$$\begin{aligned} X_1 &= \tilde{X}_1\Big(1 + \frac{p}{n}Z + \frac{1}{n}\sum_{k=1}^{n}(p|g_k|^p - 1)\Big)^{-1/p} \\ &= \tilde{X}_1\Big(1 + pN + \frac{1}{n}\big(p|g_1|^p + p|g_2|^p + pZ - 2\big)\Big)^{-1/p} = \tilde{X}_1 - \tilde{X}_1 N + \tilde{X}_1 H_1 \end{aligned}$$

where we define H_1 in such a way that the last equality holds. The fact that $\mathbb{E}[H_i^m] \le C_m/n^m$ follows from $\mathbb{E}[N^{2m}] \le C_m/n^m$.

Next, let φ be a differentiable function supported on $[-C_0, C_0]$ such that φ' is Lipschitz. For all $x, \tilde{x} \in \mathbb{R}$ we have that

$$\varphi(Rx) = \varphi(R\tilde{x}) + R\varphi'(R\tilde{x})(x - \tilde{x}) + O\big(\mathbb{1}\{\min(|x|, |\tilde{x}|) \le C_0/R\} \cdot R^2(x - \tilde{x})^2\big).$$

Substituting (64) into the last estimate we obtain

$$\varphi(RX_1) = \varphi(R\tilde{X}_1) - R\varphi'(R\tilde{X}_1)\tilde{X}_1 N + O\Big(\mathbb{1}_{\mathcal{A}_1}\big(|\tilde{X}_1 H_1| R + R^2\big(\tilde{X}_1 N + \tilde{X}_1 H_1\big)^2\big)\Big)$$

where $\mathcal{A}_1 := \big\{\min(|X_1|, |\tilde{X}_1|) \le C_0/R\big\}$. Define the event $\mathcal{B}_1 := \big\{|\tilde{X}_1| \le 2C_0/R\big\}$ and note that

$$\mathbb{P}(\mathcal{A}_1 \setminus \mathcal{B}_1) \le \mathbb{P}\big(|\tilde{X}_1| \ge 2|X_1|\big) \le \mathbb{P}\Bigg(pZ + \sum_{k=1}^{n}(p|g_k|^p - 1) \ge n\Bigg) \le Ce^{-cn},$$

where the last inequality is by Bernstein's inequality and the fact that $|g_i|^p$ has exponential tails. We have

$$\begin{aligned} \varphi(RX_1) &= \varphi(R\tilde{X}_1) - R\varphi'(R\tilde{X}_1)\tilde{X}_1 N + O\big(\mathbb{1}_{\mathcal{B}_1}\big(H_1 + (N + H_1)^2\big) + \mathbb{1}_{\mathcal{A}_1\setminus\mathcal{B}_1} M_1\big) \\ &= \varphi(R\tilde{X}_1) - R\varphi'(R\tilde{X}_1)\tilde{X}_1 N + O\big(\mathbb{1}_{\mathcal{B}_1} F_1 + \mathbb{1}_{\mathcal{A}_1\setminus\mathcal{B}_1} M_1\big) \end{aligned} \tag{65}$$

where M_1 and F_1 are some random variables with $\mathbb{E}[M_1^m] \le C_m$ and $\mathbb{E}[F_1^m] \le C_m n^{-m}$ for all $m \in \mathbb{N}$. Thus, using that N is independent of $\tilde{X}_1$, $\mathbb{E}[N] = 0$ and Cauchy-Schwarz inequality we obtain

$$\mathbb{E}\big[\varphi(RX_1)\big] = \mathbb{E}\big[\varphi(R\tilde{X}_1)\big] + O(n^{-1}).$$

This finishes the proof of the first part of the claim.

We turn to prove the second part. We clearly have that $\mathbb{E}[\varphi(R\tilde{X}_1)] \le C/R$ and therefore

$$\mathbb{E}\big[\varphi(RX_1)\big]^2 = \mathbb{E}\big[\varphi(R\tilde{X}_1)\big]^2 + O(n^{-1}R^{-1}). \tag{66}$$

Moreover, by the same arguments as in (65) we can write

$$\varphi(RX_2) = \varphi(R\tilde{X}_2) - R\varphi'(R\tilde{X}_2)\tilde{X}_2 N + O\big(\mathbb{1}_{\mathcal{B}_2} F_2 + \mathbb{1}_{\mathcal{A}_2\setminus\mathcal{B}_2} M_2\big) \tag{67}$$

where M_2 and F_2 are some random variables with $\mathbb{E}[M_2^m] \le C_m$ and $\mathbb{E}[F_2^m] \le C_m/n^m$.

We now expand the 16 terms in the product of the right hand sides of (65) and (67) to obtain

$$\mathbb{E}\big[\varphi(RX_1)\varphi(RX_2)\big] = \mathbb{E}\big[\varphi(R\tilde{X}_1)\big]\mathbb{E}\big[\varphi(R\tilde{X}_2)\big] + O(n^{-1}R^{-1}). \tag{68}$$

Since $\tilde{X}_1$, $\tilde{X}_2$ and N are independent we have

$$\mathbb{E}\big[\varphi(R\tilde{X}_1)\varphi(R\tilde{X}_2)\big] = \mathbb{E}\big[\varphi(R\tilde{X}_1)\big]\mathbb{E}\big[\varphi(R\tilde{X}_2)\big] \quad \text{and} \quad \mathbb{E}\big[R\varphi'(R\tilde{X}_1)\tilde{X}_1 N\varphi(R\tilde{X}_2)\big] = 0.$$

Next, by Cauchy-Schwarz we have

$$\begin{aligned}
&\mathbb{E}\big[\mathbb{1}_{\mathcal{B}_1} F_1\varphi(R\tilde{X}_2)\big] \le C\big(\mathbb{E}[F_1^2]\cdot\mathbb{P}(\mathcal{B}_1\cap\mathcal{B}_2)\big)^{1/2} \le C/(nR)\\
&\mathbb{E}\big[\mathbb{1}_{\mathcal{A}_1\setminus\mathcal{B}_1} M_1\varphi(R\tilde{X}_2)\big] \le C\big(\mathbb{E}[M_1^2]\cdot\mathbb{P}(\mathcal{A}_1\setminus\mathcal{B}_1)\big)^{1/2} \le Ce^{-cn}\\
&\mathbb{E}\big[\mathbb{1}_{\mathcal{B}_1} F_1 R\varphi'(R\tilde{X}_2)\tilde{X}_2 N\big] \le C\big(\mathbb{E}[F_1^2 N^2]\cdot\mathbb{P}(\mathcal{B}_1\cap\mathcal{B}_2)\big)^{1/2} \le C/(nR)\\
&\mathbb{E}\big[\mathbb{1}_{\mathcal{A}_1\setminus\mathcal{B}_1} M_1 R\varphi'(R\tilde{X}_2)\tilde{X}_2 N\big] \le C\big(\mathbb{E}[M_1^2 N^2]\cdot\mathbb{P}(\mathcal{A}_1\setminus\mathcal{B}_1)\big)^{1/2} \le C/(nR).
\end{aligned}$$

The other terms in the product are either clearly small by Cauchy-Schwarz or symmetric to one of the above terms. This finishes the proof of (68). The second part of the claim follows from (66) and (68). Indeed, $\mathrm{Cov}(\varphi(RX_1),\varphi(RX_2)) = \mathbb{E}\big[\varphi(RX_1)\varphi(RX_2)\big] - \mathbb{E}\big[\varphi(RX_1)\big]^2 = O(n^{-1}R^{-1})$. □

The proof of Claim 3.3 follows from (64) above.

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