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Dec 2, 2022

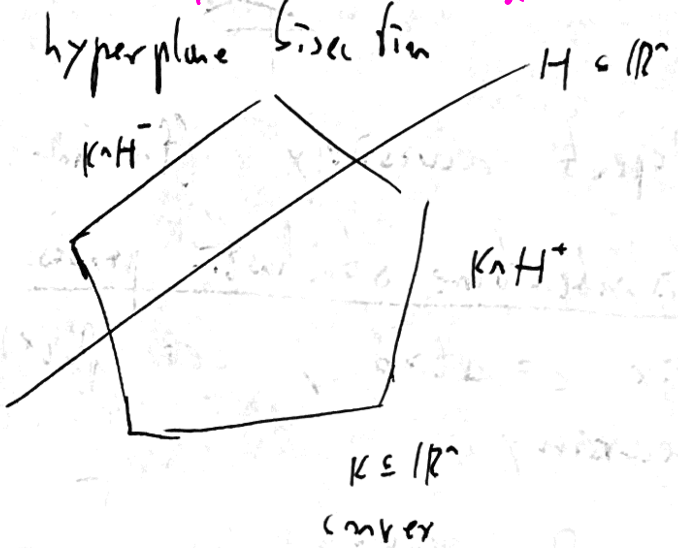
Lecture 5: Eldan's Stochastic Localization

· My original plan was to discuss also heat flow and related applications of the H^{-1} inequality, but now I think that this is too ambitious for 1 hour, and we will continue with our "tradition" of studying just one technique per lecture.

· Today we study stochastic localization, with two applications: Poincaré type inequalities in the style of the KLS conjecture and complex waist inequalities.

Analogous to the app's from Lecture 2

Convex Localization



· Choice of hyperplane cutting rule - lots of freedom

· Used topology to "solve equation(s)".

Modify in 2 ways:

1) <u>Random localization</u>: Hyperplanes are chosen randomly, through barycenter

2) <u>Continuous localization</u>: Rather than bisecting sharply, multiply by an affine functional close to 1.

$\frac{d\mu}{dx} = \rho$

log-concave

$b_\rho = \int x \, d\mu(x)$

$\theta \in \mathbb{R}^n$ direction

$\rho(x) \, 1\!\!1_{\langle x - b_\rho, \pm\theta \rangle \geq 0}$  sharp

$\rho(x) \cdot \left(1 \pm \varepsilon \langle x - b_\rho, \theta \rangle \right)$ ← milder reweighting

- Observation (say μ compactly-supported, $\varepsilon > 0$ small)

We obtain 2 log-concave, prob. measures, whose average is μ.

 log-conc ↓ barycenter.

- Repeat recursively (for now discrete time, in a few minutes - continuous).

- Discrete-time stochastic process

Fix $\varepsilon = \Delta t > 0$, set $\rho_0(x) = \rho(x)$. For $t \in \mathbb{R}^n$ define recursively

$$\rho_{t+\Delta t}(x) = \rho_t(x) \cdot \Big[1 + \langle x - b_{\rho_t}, \underbrace{\sqrt{\Delta t}\, z_t}_{\text{random direction}} \rangle \Big]$$

where $z_0, z_\varepsilon, z_{2\varepsilon}, \ldots$ are ~~i.i.d~~ indep. centered random vectors, maybe uniform in $\sqrt{n} S^{n-1}$, or some other distribution of directions, Gaussian.

- By induction: ρ_t is a random log-concave, prob. measure on $\mathbb{R}^n$ with

$$\forall x \in \mathbb{R}^n, \quad \mathbb{E}\, \rho_t(x) = \mathbb{E}\, \rho_{t-\Delta t}(x) = \cdots = \rho_0(x)$$

In fact, it's a martingale

$$\mathbb{E}\left(\rho_{t+\Delta t}(x) \mid \mathcal{F}_t \right) = \rho_t(x)$$

2)

where $\mathcal{F}_t$ is the σ-algebra generated by $(z_s)_{0 \le s \le t}$.

- For continuous time, let $(W_t)_{t \ge 0}$ be a standard Brownian motion in $\mathbb{R}^n$. Rewrite

which is more convenient

$$P_{t+\Delta t}(x) - P_t(x) = P_t(x) \cdot \langle x - b_{P_t}, \underbrace{\sqrt{\Delta t}\, z_t}_{W_{t+\Delta t} - W_t} \rangle$$

Eldan's Stochastic Localization, Lee-Vempala version

Given initial distribution p_0 solve Stochastic Differential Equation $\forall x \in \mathbb{R}^n$, $t \ge 0$

$$dP_t(x) = P_t(x) \langle x - \underset{b_{P_t}}{b_t}, dW_t \rangle \qquad (*)$$

Can also put a driving matrix C_t (above $C_t = Id$) adapted to the filtration induced by BM

$$dP_t(x) = P_t(x) \langle x - b_t, C_t\, dW_t \rangle$$

Lemma: With prob. one,

$$P_t(x) = p_0(x) \cdot \text{Gaussian factor} \quad \exp(-\text{pos. def. quadratic form})$$

$$\overset{C_t = Id}{=} p_0(x) \cdot \exp\left(\langle \theta_t, x \rangle - \frac{t|x|^2}{2}\right) / Z_t$$

is a prob. measure, more log-concave than a Gaussian $\exp(-t\frac{|x|^2}{2})$ with

$$\mathbb{E}\, P_t(x) = p_0(x)$$

(actually, a martingale).

Remark: Not only we preserve log-concavity as before, but we gain uniform log-concavity.

intuitively

$$(1+\varepsilon\langle x,\theta\rangle)(1-\varepsilon\langle x,\theta\rangle) \approx e^{-\varepsilon^2\langle x,\theta\rangle^2}$$ ← quadratic term

<u>Proof:</u> Apply the Itô formula to $(*)$

$$d\log p_t(x) = \frac{1}{p_t(x)}\left[\langle x-b_t, dw_t\rangle p_t(x)\right] - \frac{1}{2}\cdot\frac{1}{p_t(x)^2}\cdot |x-b_t|^2 p_t(x)^2$$

$$= -\frac{1}{2}|x|^2 dt + \langle x, dw_t\rangle + \langle x, b_t\rangle dt + \text{constant in } x$$

Hence

$$\log p_t(x) = \log p_0(x) - \frac{t}{2}|x|^2 + \langle\theta_t, x\rangle - \log Z_t$$

$$d\theta_t = b_t\,dt + dw_t$$ ← useful for existence/uniqueness strong/weak.

$$Z_t = \int_{\mathbb{R}^n} e^{\langle\theta_t, x\rangle - t|x|^2/2} p_0(x)\,dx$$

Dynamics:

~~Example:~~

p_t 

$b_t = b_{p_t}$ dynamics

$$db_t = d\left[\int x\,p_t(x)\,dx\right]$$

$$= A_t C_t\,dw_t \qquad A_t = \mathrm{Cov}(p_t)$$

$$b_\infty \sim \mu \qquad (\mathbb{E}\,p_t = p_0)$$

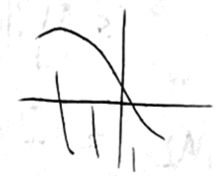
<u>Corollary:</u> For any $t>0$, almost all needles p_t in the decomposition satisfy

$$\lambda_{p_t} \geq t$$ (since more log-concave than $\exp(-t\frac{|x|^2}{2})$)

and by reverse Cheeger,

$$\forall E\subseteq\mathbb{R}^n \qquad p_t^+(\partial E) = \int_{\partial E} p_t \geq c\sqrt{t}\,\min\{p_t(E), 1-p_t(E)\}.$$

3). How can we prove isoperimetric inequalities for the given p_0?
Begin with arbitrary $E \subseteq \mathbb{R}^n$, smooth boundary, $p_0(E) = 1/2$
Then $\forall t > 0$



$$p_0^+(\partial E) = \int_{\partial E} p_0(x)\,dx = \mathbb{E} \int_{\partial E} p_t(x)\,dx$$

$$\geq c\sqrt{t} \cdot \mathbb{E}\ \ p_t(E) \cdot (1 - p_t(E))$$

<u>Proposition</u> (Eldan used by, Lee-Vempala, Chen, ...)
Denote $A_t = \mathrm{Cov}(p_t) \in \mathbb{R}^{n \times n}$. Let $T > 0$ be fixed, and assume

$$\int_0^T \mathbb{E}\, \|A_t\|_{op} \leq \frac{1}{8}$$

Then

$$\lambda_{p_0} \geq c \cdot T$$

log factor is already lost with this Prop.

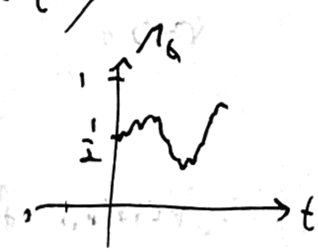
. Thus we only need to control the covariance at A_t, Use of operator norm yields log-factors.

<u>Proof:</u> Denote $M_t = p_t(E) = \int_E p_t(x)\,dx$.

Need $\mathbb{E}\, M_T(1 - M_T) \geq c$.

$$dM_t = \int_E \langle x - b_t, dW_t \rangle\, p_t(x)\,dx = \langle v_t, dW_t \rangle$$

$$v_t = \int_E (x - b_t)\, p_t(x)\,dx.$$



martingale
$M_\infty \in \{0, 1\}$.

$$|v_t| \le \sup_{\theta \in S^{n-1}} \int_E |\langle x - b_t, \theta \rangle| \, p_t(x)\, dx \le \sup_{\theta \in S^{n-1}} \sqrt{\int |\langle x - b_t, \theta\rangle|^2 \, p_t(x)\, dx}$$

$$\le \sqrt{\|A_t\|_{op}}$$

$$dM_t = -|v_t|^2 \, dt + \text{martingale term}$$

$$\frac{d}{dt} \mathbb{E} M_t = -\mathbb{E} |v_t|^2 \ge - \mathbb{E} \|A_t\|_{op}$$

$$\mathbb{E} M_t (1 - M_t) \ge \underbrace{M_0 (1 - M_0)}_{\frac{1}{4}} - \mathbb{E} \int_0^T \|A_t\|_{op} \, dt \ge \frac{1}{8}. \qquad \blacksquare$$

- The game here is to show that A_t doesn't grow too fast (say $A_0 = Id$, isotropic). Differentiate A_t and use 3^{rd} tensors

Eldan: bound using t_n

Chen: Clever regularity for growth of A_t (power-law).
progress towards KLS.

Another application of this method:

<u>Thm</u> (complex waist inequality) Let $f: \mathbb{C}^n \to \mathbb{C}^k$ be holomorphic (say, a polynomial) with $f(0) = 0$. Set $Z = f^{-1}(0)$. Then

$$\forall r > 0 \qquad \gamma_n (Z + rB^n) \ge \gamma_n (\mathbb{C}^{n-k} + rB^n)$$

gaussian gaussian

<u>Remarks:</u>

1) $r \to 0$ Rutishauser, Lelong 1950s:

$$|Z \cap B(0,t)| \ge |\mathbb{C}^{n-k} \cap B(0,t)|$$

versions for minimal surfaces (Alexander-Hoffman-Osserman '74, Brendle - Hung '17)

2) There is a version without $f(0)=0$ (need $Z \neq \emptyset$, replace $\mathbb{C}^{n-k}$ by an affine $(n-k)$-subspace of distance $d(0,Z)$ from the origin).

3) Q1: A proof w/o stochastic localization?

Q2: Version for minimal surfaces? Is there an analogy?

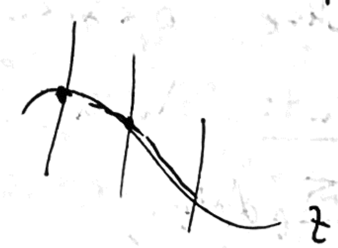
<u>Proof idea:</u> Stochastic Localization with a suitable driving matrix.

<u>Proposition:</u> There exists a decomposition

$$\gamma_n = \mathbb{E}\,\mu_\infty \leftarrow \lim_{t\to\infty} \mu_t$$

where μ_∞ is a random complex Gaussian measure such that $\leftarrow e^{-\frac{(z-a)^* B (z-a)}{2}}$ B Hermitian, pos. def

- It is centered at some point of Z
- Supported in a k-dim. complex subspace
- $\mathrm{Cov}(\mu_\infty) \leq Id$



<u>Proposition $\Rightarrow$ Thm:</u>

$$\gamma_n(Z + rB^n) \geq \mathbb{E}\,\mu_\infty\Big(B\big(\underset{\substack{\uparrow \\ Z}}{\text{center}}, r\big)\Big) \underset{\substack{\uparrow \\ \text{Cov} \leq Id \\ \text{Gaussian} \\ \text{Contraction}}}{\geq} \gamma_k(B(0,r))$$

$$= \gamma_n(\mathbb{C}^{n-k} + rB^n).$$

<u>Proof of Proposition:</u> "Balancing a martingale on a fiber"

Assume 0 regular value of f.

Run stochastic localization with a driving matrix

· $\mu_0 = \gamma_n$, $\mu_\infty = \lim_{t\to\infty} \mu_t$, $\mathrm{Cov}(\mu_t) \leq Id$

· $\gamma_n = \mathbb{E}\, \mu_t \quad \forall t$

$$\frac{d\mu_t}{d\lambda(t)} = \frac{1}{Z_t} e^{-\frac{(z-b_t)^* B_t (z-b_t)}{2}} , \qquad B_0 = Id$$

<u>barycenter dynamics:</u> With $\Sigma_t = B_t^{-1} C_t$ r.~ at our disposal

$$db_t = \Sigma_t \, dW_t$$

$$dB_t = B_t \Sigma_t \Sigma_t^* B_t \, dt \succeq 0$$

Need to choose Σ_t (or equivalently C_t) such that

· The process (B_t) does not blow up in finite time.

· With prob one, $b_t \in Z$

· With prob one, $\lambda_{k+1}(b_t) \xrightarrow{t\to\infty} \infty$ where $0 < \lambda_1 < \dots < \lambda_n$ are the eigenvalues of B_t. k-dim

<u>Useful fact:</u> When f is holomorphic (pluri-harmonic is fine), $db_t = \Sigma_t dW_t$, the process $f(b_t)$ is a martingale for any adapted process (Σ_t), $\Sigma_t \in \mathbb{C}^{n\times n}$.

· No Itô term, and $f(b_t) = 0$ amounts to

$$\mathrm{Image}(\Sigma_t) \subseteq \underbrace{T_{b_t} Z}_{\substack{\shortparallel \\ E_t \,\leq\, \mathbb{C}^n}}$$

$\dim(E_t) = n - k.$

Complex analog of Kakutani, harmonic f-n and B-M.

5) Choose

$$\Sigma_t = B_t^{-1/2} \Pi_{\underbrace{B_t^{1/2}(E_t)}_{= F_t}} , \qquad \Pi \text{ orthogonal proj.}$$

Then

$$\frac{dB_t}{dt} = B_t^{1/2} \Pi_{F_t} B_t^{1/2}$$

does not blow up (only exponential growth).

<u>Lemma:</u> $\lambda_{k+1}(B_t) \xrightarrow{t \to \infty} \infty$

<u>Proof:</u> $$\frac{d}{dt} \sum_{j=1}^{k+1} \lambda_j(B_t) = \operatorname{Tr}\left[\frac{dB_t}{dt} \cdot \Pi_{G_t} \right]$$ span of $k+1$ eigenvectors

$$= \operatorname{Tr}\Big[\underbrace{B_t^{1/2}}_{\geq I\lambda} \underbrace{\Pi_{F_t}}_{n-k} \underbrace{B_t^{1/2}}_{\geq I\lambda} \cdot \underbrace{\Pi_{G_t}}_{k} \Big] \geq 1.$$

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