Convexity in High Dimensions

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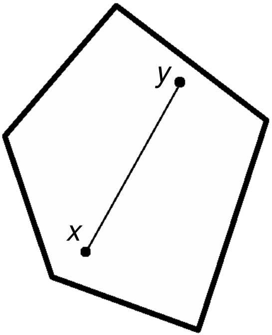
The Poincaré inequality

Theorem (Poincaré, 1890 and 1894)

Let $K \subseteq \mathbb{R}^3$ be convex and open. Let $f : K \to \mathbb{R}$ be C^1-smooth, with $\int_K f = 0$. Then,

$$\lambda_K \int_K f^2 \leq \int_K |\nabla f|^2$$

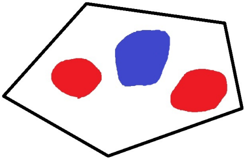
where $\lambda_K \geq (16/9) \cdot Diam^{-2}(K)$.



- In 2D, Poincaré got a better constant, $24/7$.
- Related to Wirtinger's inequality on periodic functions in one dimension (sharp constant, roughly a decade later).
- The largest possible λ_K is the Neumann spectral gap or the inverse **Poincaré constant** of K.
- Proof: Estimate $\int_{K \times K} |f(x) - f(y)|^2 dxdy$ via segments.

Motivation: The heat equation

- Suppose $K \subseteq \mathbb{R}^3$ with ∂K an *'insulator'*, i.e., heat does not escape/enter K.
- Write $u_t(x)$ for the temperature at the point $x \in K$ at time $t \geq 0$.



Heat equation (Neumann boundary conditions)

$$\begin{cases} \dot{u}_t = \Delta u_t & \text{in } K \\ \frac{\partial u_t}{\partial n} = 0 & \text{on } \partial K \end{cases}$$

Fourier's law: Heat flux is proportional to the temp. gradient.

Rate of convergence to equilibrium

$$\frac{1}{|K|}\int_K u_0 = 1 \quad \Longrightarrow \quad \|u_t - 1\|_{L^2(K)} \leq e^{-t\lambda_K}\|u_0 - 1\|_{L^2(K)}$$

Higher dimensions

The Poincaré inequality was generalized to all dimensions:

Theorem (Payne-Weinberger, 1960 – precursor for convex localization)

Let $K \subseteq \mathbb{R}^n$ be convex and open, let μ be the Lebesgue measure on K. If $f : K \to \mathbb{R}$ is C^1-smooth with $\int_K f d\mu = 0$, then,

$$\frac{\pi^2}{Diam^2(K)} \int_K f^2 d\mu \leq \int_K |\nabla f|^2 d\mu.$$

- The constant π^2 is best possible in every dimension n. E.g.,
$$K = (-\pi/2, \pi/2), \quad f(x) = \sin(x).$$
- In contrast, Poincaré's proof would lead to an exponential dependence on the dimension.
- Not only the Lebesgue measure on K, we may consider any log-concave measure.

Higher dimensions

The Poincaré inequality was generalized to all dimensions:

Theorem (Payne-Weinberger, 1960 – precursor for convex localization)

Let $K \subseteq \mathbb{R}^n$ be convex and open, let μ be any **log-concave** measure on K. If $f : K \to \mathbb{R}$ is C^1-smooth with $\int_K f d\mu = 0$, then,

$$\frac{\pi^2}{Diam^2(K)} \int_K f^2 d\mu \leq \int_K |\nabla f|^2 d\mu.$$

- The constant π^2 is best possible in every dimension n. E.g.,

$$K = (-\pi/2, \pi/2), \quad f(x) = \sin(x).$$

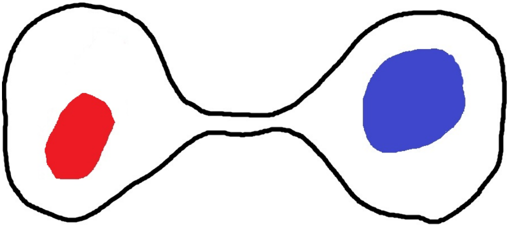
- In contrast, Poincaré's proof would lead to an exponential dependence on the dimension.
- A **log-concave** measure μ on K is a measure with density of the form e^{-H}, where the function H is **convex.**

The role of convexity / log-concavity

- For $\Omega \subseteq \mathbb{R}^n$, the Poincaré coefficient λ_Ω measures the connectivity or conductance of Ω.

Convexity is a strong form of connectedness

Without convexity/log-concavity assumptions:



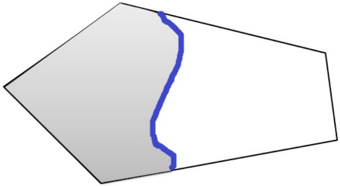
long time to reach equilibrium,
regardless of the diameter

Many other ways to measure connectivity

The isoperimetric constant

For an open set $K \subset \mathbb{R}^n$ define

$$h_K = \inf_{A \subset K} \frac{|\partial A \cap K|}{\min\{|A|, |K \setminus A|\}}$$



- If K is strictly-convex with smooth boundary, the infimum is attained when $|A| = |K|/2$ (Sternberg-Zumbrun, 1999).

Theorem (Cheeger '70, Buser '82, Ledoux '04)

For any open, convex set $K \subseteq \mathbb{R}^n$,

$$\frac{h_K^2}{4} \leq \lambda_K \leq 9h_K^2.$$

- Mixing time of Markov chains, algorithms for estimating volumes of convex bodies (Dyer-Freeze-Kannan '89, ...)

The study of uniform distributions on convex domains

What is this mathematical field about?

(relevant to linear programming, statistics, computer science, slightly to stat. physics)

- The questions are non-trivial in high dimensions, when

$$n \to \infty.$$

The theme

Convexity in high dimension is a source of regularity, comparable to statistical independence. It may compensate for lack of structure or symmetry.

- Playground for various techniques that transcend convex geometry: Convex Localization, Optimal Transport, Bochner identities and curvature, heat flow and Eldan's stochastic localization, geometric measure theory.

The simplest example: the Euclidean ball

- Consider the Euclidean ball $B^n = \{x \in \mathbb{R}^n \, ; \, |x| \leq 1\}$ or the sphere $S^{n-1} = \{x \in \mathbb{R}^n ; |x| = 1\}$.
- Write σ_{n-1} for the uniform probability measure on S^{n-1}.

For a set $A \subseteq S^{n-1}$ and for $\varepsilon > 0$ denote

$$A_\varepsilon = \left\{ x \in S^{n-1} \, ; \, \exists y \in A, \, d(x,y) \leq \varepsilon \right\},$$

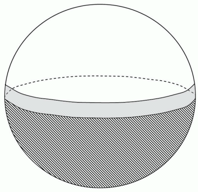
which is the ε-neighborhood of A.

- Consider the hemisphere $H = \{x \in S^{n-1} ; x_1 \leq 0\}$.

A well-known observation:

$$\sigma_{n-1}(H_\varepsilon) = \mathbb{P}(Y_1 \leq \sin \varepsilon) \approx \mathbb{P}\left(\Gamma \leq \varepsilon\sqrt{n}\right)$$

where $Y = (Y_1, \ldots, Y_n)$ is distributed according to σ_{n-1}, and Γ is a standard normal random variable.



The sphere's marginals are approximately Gaussian

Concentration of measure on the sphere

The amount of volume of a distance at least $1/10$ from the equator is at most

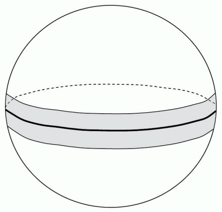
$$C\exp(-cn)$$

of the sphere, for universal constants $c, C > 0$.

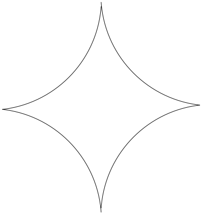
- Most of the mass of the sphere S^{n-1} in high dimensions, is concentrated at a narrow strip near the equator $[x_1 = 0]$
- or any other equator.

“Concentration of Measure”

(à la V. Milman)



$dim \to \infty$



The isoperimetric inequality on the sphere

The isoperimetric inequality (Lévy, Schmidt, '50s).

For any Borel set $A \subset S^{n-1}$ and $t, \varepsilon > 0$,

$$\sigma_{n-1}(A) = t \quad \Longrightarrow \quad \sigma_{n-1}(A_\varepsilon) \geq \sigma_{n-1}(C_\varepsilon),$$

where $C \subseteq S^{n-1}$ is a spherical cap of measure t (when $t = 1/2$ it is a hemisphere).

- For any set $A \subset S^{n-1}$ with $\sigma_{n-1}(A) = 1/2$,

$$\sigma_{n-1}(A_\varepsilon) \geq 1 - 2\exp(-\varepsilon^2 n/2).$$

Therefore, for any subset $A \subset S^{n-1}$ of measure $1/2$, its ε-neighborhood covers almost the entire sphere.

- The isoperimetric inequality is "accompanied" by functional inequalities on the sphere: Sobolev, log-Sobolev, transport-cost, Poincaré and Brunn-Minkowski type.

Concentration of Lipschitz functions on the sphere

Corollary ("Lévy's lemma")

Let $f : S^{n-1} \to \mathbb{R}$ be a 1-Lipschitz function. Denote

$$E = \int_{S^{n-1}} f(x) d\sigma_{n-1}(x).$$

Then, for any $\varepsilon > 0$,

$$\sigma_{n-1}\left(\left\{x \in S^{n-1}; |f(x) - E| \geq \varepsilon\right\}\right) \leq C \exp(-c\varepsilon^2 n),$$

where $c, C > 0$ are universal constants.

- Lipschitz functions on the high-dimensional sphere are "effectively constant".

Concentration phenomena for general convex bodies?

- Does the uniform probability measure on an arbitrary convex body $K \subseteq \mathbb{R}^n$ exhibit similar effects?

A short answer

Pretty much, yes.
(Currently known up to polylogarithmic factors).

- There are rather strong **expansion properties**, and the isoperimetric inequality is nearly saturated by **half-spaces.**

Caveat: exponential tail rather than subgaussian tail

There is no strong small-set-expansion as in the sphere.

The tail distribution of a 1-Lipschitz function is sub-exponential, rather than sub-gaussian.

How would you normalize a convex body?

There is a unit ball B^n and a unit cube $[0,1]^n$.

- How would you choose the correct “units” for a general convex body or a log-concave density?

Definition

A convex body $K \subseteq \mathbb{R}^n$ of volume one is **isotropic** if for the random vector X distributed uniformly on K,

1. $\mathbb{E}X = 0$
2. The covariance matrix $Cov(X) \in \mathbb{R}^{n \times n}$ is a scalar matrix, where $Cov(X)_{ij} = \mathbb{E}X_i X_j - (\mathbb{E}X_i)(\mathbb{E}X_j)$.

- Any convex body can be made isotropic after applying a linear-affine transformation.
- Another common, similar normalization: Rather than requiring $Vol_n(K) = 1$, require $Cov(X) = \mathrm{Id}$.

The Kannan-Lovász-Simonovits (KLS ’95) conjecture

Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body of volume one (or of identity covariance), X distributed uniformly over K.

A conjecture (KLS), which has three equivalent formulations:

1. **Concentration:** For any 1-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathbb{E}f(X) = 0$,
$$\mathbb{P}\left(|f(X)| \geq t\right) \leq Ce^{-ct} \qquad \text{for all } t > 0.$$
2. **Poincaré inequality:** For any smooth f with $\int_K f = 0$,
$$\int_K f^2 \leq C \int_K |\nabla f|^2$$
3. **Isoperimetry:** For any open $A \subseteq \mathbb{R}^n$, smooth boundary,
$$Vol_{n-1}(\partial A \cap K) \geq c \cdot \min\{Vol_n(K \cap A), Vol_n(K \setminus A)\}.$$

Here $c, C > 0$ are universal constants.

The Kannan-Lovász-Simonovits (KLS '95) conjecture

Equivalences in our formulation of the KLS conjecture due to Ball-Nguyen '13, Buser '82, Cheeger '70, Eldan-K. '11, Gromov and V. Milman '83, Ledoux '04, E. Milman '09.

Theorem (K.-Lehec '22, building upon Eldan '13, Lee-Vempala '16 and the breakthrough by Chen '20)

KLS conjecture is true up to polylogarithmic factors.

i.e., replace the universal constants by $C \log^{\alpha} n$.

- Bounds for the α's were improved a few weeks ago by LV+Jambulapati.
- Works for log-concave random vectors too.

Example: Apply the KLS Poincaré inequality with $f(x) = |x| - E$

If X is an isotropic log-concave random vector in $\mathbb{R}^n$, then

$$Var(|X|) \leq C \log^{\alpha} n$$

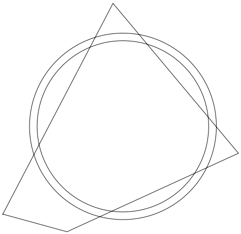
Thin shell phenomenon

- When X is isotropic, $\mathbb{E}|X|^2 \approx n$ and by reverse Hölder inequalities
$$\mathbb{E}|X| \approx \sqrt{n}.$$
However, the variance of $|X|$ is $\leq C \log^{\alpha} n$, much smaller!

Most of the mass of X is contained in a **thin spherical shell.**

- **Sudakov '76,**
Diaconis-Freedman '84:

When most of the mass of the isotropic random vector X is contained in a thin spherical shell, we have **approx. Gaussian marginals.**



Convexity as good as Independence

Current state of the art:

Theorem (Bobkov, Chistyakov, Götze '19 using K.-Lehec '22)

Let X be an isotropic, log-concave random vector in $\mathbb{R}^n$. Then there exists $\Theta \subseteq S^{n-1}$ with $\sigma_{n-1}(\Theta) \geq 9/10$ such that for any $\theta \in \Theta$,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(X \cdot \theta \leq t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2} ds \right| \leq \frac{C \log^{\alpha} n}{n},$$

where $C, \alpha > 0$ are universal constants.

- An optimal result for the Kolmogorov metric, up to the polylogarithmic factor. (No log's for independent random variables with bounded 4^{th} moments, K.-Sodin '11)
- This **Central Limit Theorem for Convex Sets** was originally proven in K. '07, confirming a conjecture by Anttila, Ball and Perissinaki '03 and Brehm-Voigt '00.

No log-factors under symmetry assumptions

- A random vector $X = (X_1, \ldots, X_n)$ is said to be invariant under coordinate reflections if it is equidistributed with $(\pm X_1, \ldots, \pm X_n)$ for any choice of signs.
- For instance, the uniform measure on ℓ_p^n-balls.

Theorem (K. '09)

Let $X = (X_1, \ldots, X_n)$ be a log-concave random vector in $\mathbb{R}^n$, invariant under coordinate refelctions, with $\mathbb{E} X_i^2 = 1$ for all i. Then for any $\theta_1, \ldots, \theta_n \in \mathbb{R}$ with $\sum_i \theta_i^2 = 1$,

$$\sup_{\alpha \leq \beta} \left| \mathbb{P}\left(\alpha \leq \sum_{i=1}^n \theta_i X_i \leq \beta \right) - \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} dt \right| \leq C \sum_{i=1}^n \theta_i^4.$$

- The bound is typically $O(1/n)$, since for a random $\theta \in S^{n-1}$, we have $\sum_i \theta_i^4 \leq C/n$ with prob. at least $1 - C\exp(-c\sqrt{n})$.

Width of thin-shell controls approx. rate in convex CLT

We have the **thin-shell parameter**

$$\sigma_n^2 = \sup_X \mathbb{E}\left(|X| - \sqrt{n}\right)^2 ,$$

where the sup is over isotropic, log-concave random vectors in $\mathbb{R}^n$. A trivial bound is $\sigma_n \leq C\sqrt{n}$. Better estimates:

$\sigma_n \leq Cn^{2/5+o(1)}$ (K. '07)

$\sigma_n \leq Cn^{3/8}$ (Fleury '09)

$\sigma_n \leq Cn^{1/3}$ (Guédon-E. Milman '11)

$\sigma_n \leq Cn^{1/4}$ (Lee-Vempala '16)

$\sigma_n \leq C\exp\left((\log n)^{1/2+o(1)}\right) = n^{o(1)}$ (Chen '20)

$\sigma_n \leq C\log^4 n$ (K.-Lehec '22)

$\sigma_n \leq C\log^{2.23} n$ (Jambulapati-L.-V. '22+)

- Concentration of measure (hinted by Paouris LDP '06).

Isoperimetry and Thin Shell

Definition

Write ψ_n for the minimal number such that for any log-concave random vector X, $Cov(X) = \mathrm{Id}$, and any 1-Lipschitz function φ,

$$Var\varphi(X) \leq \psi_n^2.$$

- Alternatively, if ρ is the density of X, then for any $A \subseteq \mathbb{R}^n$ with smooth boundary,

$$\int_{\partial A} \rho \geq \frac{c}{\psi_n} \cdot \min\{\mathbb{P}(X \in A), \mathbb{P}(X \notin A)\}.$$

- We actually explained that

$$\sigma_n \leq C\psi_n.$$

Theorem (Eldan '13 – breakthrough towards KLS conjecture)

$$\psi_n \leq C \log n \cdot \sigma_n$$

Thin Shell and Slicing

In the 1980s, Bourgain considered the following:

Definition

Write L_n for the minimal number such that for any convex body $K \subseteq \mathbb{R}^n$ of volume one, there exists a hyperplane $H \subseteq \mathbb{R}^n$ such that

$$Vol_{n-1}(K \cap H) \geq \frac{1}{L_n}.$$

- **The hyperplane conjecture**: $L_n \leq C$.
  Any convex body of volume one should have a hyperplane section whose volume is at least c.
- We are still waiting for a short and sweet proof, but it is not coming...

Theorem (Eldan-K. '11)

$$L_n \leq C\sigma_n \leq C'\psi_n$$

The slicing problem

- The relation between Slicing and KLS was promoted by K. Ball, and Ball-Nguyen '13 proved $L_n \leq \exp(C\psi_n^2)$.

Corollary (because "thin shell implies slicing")

$L_n \leq C \log^{\alpha} n$ for a universal constant $C > 0$ and $\alpha \leq 2.23$

- For many years, the best bound in Bourgain's slicing problem was $Cn^{1/4} \log n$ (Bourgain '89) or $n^{1/4}$ (K. '06). With Lee-Vempala '16, we reached three completely different proofs for $1/4$ bound. Yet it was non-optimal.

An equivalent formulation of Bourgain's slicing problem

Suppose that $K \subseteq \mathbb{R}^n$ with $Vol_n(K) = 1$ is isotropic, so

$$Cov(K) = L_K^2 \cdot \mathrm{Id}.$$

It is known that $c < L_K$. Is it true that $c < L_K < C$?
i.e., do the two normalizations essentially coincide?

Using the Brunn-Minkowski inequality

This is related to Hensley's theorem:

Theorem (Volumes of slices – Hensley '80, Fradelizi '99)

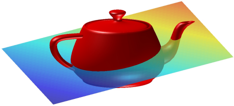
Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body. Then for any hyperplanes $H_1, H_2 \subseteq \mathbb{R}^n$ through the origin,

$$\frac{Vol_{n-1}(K \cap H_2)}{Vol_{n-1}(K \cap H_1)} \leq \sqrt{6}.$$

- In fact, $Vol_{n-1}(K \cap H) \sim 1/L_K$.
- Proven using the **Brunn-Minkowski inequality** (1887): For any Borel sets $A, B \subset \mathbb{R}^n$,

$$Vol\left(\frac{A+B}{2}\right) \geq \sqrt{Vol(A)Vol(B)}$$

Here $(A+B)/2 = \{(a+b)/2\,;\ a \in A, b \in B\}$.



Corrected Busemann-Petty conjecture

In the 1950s, Busemann-Petty conjectured the following:

- Let $K, T \subseteq \mathbb{R}^n$ be centrally-symmetric convex bodies such that

  $$Vol_{n-1}(K \cap \theta^\perp) \leq Vol_{n-1}(T \cap \theta^\perp) \qquad \text{for all } \theta \in S^{n-1}.$$

  Does it follow that $Vol_n(K) \leq Vol_n(T)$?

  True if K is a Euclidean ball or a cross-polytope.
  In general, true if $n \leq 4$ and false if $n \geq 5$.
  (Lutwak '88, Zhang, Gardner-Koldobsky-Schlumprecht '90s)

- Fails already for the cube and Euclidean ball in high dimensions! We lose a factor of $\approx \sqrt{e/2}$ (Ball '86).

Another equivalent formulation of the slicing problem

Does it follow that $Vol_n(K) \leq C \cdot Vol_n(T)$ for some universal constant $C > 0$?

Sharpened Milman ellipsoid?

An equivalent formulation of the slicing problem

Let $K \subset \mathbb{R}^n$ be a convex body. Does there exist an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, with $Vol_n(\mathcal{E}) = Vol_n(K)$, such that

$$Vol_n(K \cap C\mathcal{E}) / Vol_n(K) \geq 1/2.$$

Theorem (V. Milman, '80s)

$$Vol_n(K \cap C\mathcal{E}) / Vol_n(K) \geq c^n,$$

for universal constants $C, c > 0$.

- This suffices for proving the reverse Brunn-Minkowski inequality as well as the Bourgain-Milman inequality

$$Vol_n(K) Vol_n(K^\circ) \geq c^n Vol_n(B^n)^2$$

where K° is the polar body of the convex $K \subseteq \mathbb{R}^n$.

Two more equivalent formulations of the slicing problem

1. **Sylvester problem.** Select $n+2$ independent, random points according to the uniform measure in a convex body K. Let $p(K)$ be the probability that these $n+2$ points are in convex position. Is it true that

$$(1-p(K))^{1/n} \simeq 1/\sqrt{n}?$$

2. **Entropy and covariance.** Is it true that for any log-concave random vector X in $\mathbb{R}^n$,

$$\mathrm{Ent}(X) = \frac{1}{2} \cdot \log\det \mathrm{Cov}(X) + O(n)?$$

Could the cube and simplex be the extreme cases?

- The isotropic constant of a convex body $K \subseteq \mathbb{R}^n$ is an affine-invariant defined via

$$L_K^{2n} = \frac{\det \mathrm{Cov}(K)}{\mathrm{Vol_n}(K)^2}.$$

- We know that $L_K > c$, and is minimized for the Euclidean ball. The conjecture is that $L_K < C$. For example,

$$L_{[0,1]^n} = \frac{1}{\sqrt{12}}, \qquad L_{\Delta^n} = \frac{(n!)^{1/n}}{(n+1)^{(n+1)/(2n)}\sqrt{n+2}} \approx \frac{1}{e}.$$

Relations to classical conjectures

1. If L_K is maximized for the simplex Δ^n, then the Mahler volume-product conjecture follows (the non-symmetric case, proven in 2D by Mahler, 1939). See K. '18.
2. If among centrally-symmetric bodies, L_K is maximized for the cube, then the Minkowski lattice conjecture follows (proven in 2D by Minkowski, 1901). See Magazinov '18.

The end

Thank you!