Abstracts

Isotropic constants and Mahler volumes

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Below is an edited version of informal notes prepared for my lecture at Oberwolfach. Please refer to [10] for a rigorous mathematical discussion and for precise references related to these notes.

The following question is known as Bourgain's slicing problem [3, 4]: Let $K \subseteq \mathbb{R}^n$ be convex with $Vol_n(K) = 1$. Does there always exist a hyperplane $H \subseteq \mathbb{R}^n$ with

$$Vol_{n-1}(K \cap H) \geq \frac{1}{100}?$$

Perhaps with some other universal constant $c > 0$ in place of $1/100$?

This is not merely a curious riddle. In fact it shows up in the study of almost any question pertaining to volume distribution in high dimension under convexity assumptions. We know that it suffices to look at hyperplane sections through the barycenter, according to Makai and Martini [13]. We may furthermore reduce matters to the centrally-symmetric case [6].

Hensley [5] proved the following theorem: Let $K \subseteq \mathbb{R}^n$ be a convex body of volume one. Assume that $K = -K$ (it suffices to require that the barycenter of K lie at the origin). Then for any unit vector $\theta \in \mathbb{R}^n$,

$$c \leq Vol_{n-1}(K \cap \theta^{\perp}) \cdot \sqrt{\int_K \langle x, \theta \rangle^2 dx} \leq C,$$

where $c, C > 0$ are universal constants and $\theta^{\perp} = \{x \in \mathbb{R}^n \,;\, \langle x, \theta \rangle = 0\}$ is the hyperplane orthogonal to θ.

It follows from Hensley's theorem that the slicing problem may be reformulated as a question on the relation between the *covariance matrix* and the *volume* (or entropy) of convex sets. This entropic point of view was emphasized by K. Ball. The covariance matrix $Cov(K) = (Cov_{ij})_{i,j=1,\dots,n}$ is given by

$$Cov_{ij} = \int_K x_i x_j \frac{dx}{|K|} - \int_K x_i \frac{dx}{|K|} \cdot \int_K x_j \frac{dx}{|K|},$$

where $|K| = Vol_n(K)$. In Bourgain's notation, the isotropic constant is defined as

$$L_K = \frac{\det^{\frac{1}{2n}} Cov(K)}{|K|^{1/n}}.$$

The isotropic constant is affinely-invariant. The slicing problem is equivalent to the question of whether $L_K < C$ for some universal constant $C > 0$, for any convex body K in any dimension. It is known that $L_K \geq L_{B^n} \geq c$, where B^n is the Euclidean unit ball centered at the origin in $\mathbb{R}^n$.

Are there any relations between isotropic constants and duality? The polar body to $K \subseteq \mathbb{R}^n$ is

$$K^\circ = \{x \in \mathbb{R}^n\,;\, \forall y \in K,\ \langle x, y\rangle \le 1\}.$$

Note that

$$L_K \cdot L_{K^\circ} = [\det Cov(K) \cdot \det Cov(K^\circ)]^{\frac{1}{2n}} \cdot (|K| \cdot |K^\circ|)^{-1/n}.$$

According to the Bourgain-Milman and Santaló inequalities,

$$c \le n(|K||K^\circ|)^{1/n} \le C$$

whenever the barycenter of K or of K° lies at the origin. We thus learn that Bourgain's slicing problem is equivalent to the question of whether the following inequality holds:

$$\det\left(Cov(K)Cov(K^\circ)\right) \le \left(\frac{C}{n}\right)^{2n}. \tag{1}$$

An idea which appears in the unpublished Ph.D. dissertations by Ball '86 and by Giannopoulos '93 is to consider the trace of the matrix in (1). Perhaps the trace is easier to analyze than the determinant. Given a convex body $K \subseteq \mathbb{R}^n$ with barycenter at zero we set

$$\phi(K) = Tr[Cov(K)Cov(K^\circ)].$$

According to the arithmetic/geometric means inequality,

$$L_K^2 L_{K^\circ}^2 \le Cn\phi(K).$$

The quantity $\phi(K)$ has the following probabilistic interpretation: Let X be a random vector, distributed uniformly in K. Let Y be an independent random vector, distributed uniformly in K°. Then $\phi(K) = \mathbb{E}\langle X, Y\rangle^2$. We thus see that $0 \le \phi(K) \le 1$ when $K = -K$.

In the case where $K \subseteq \mathbb{R}^n$ has the symmetries of the cube (i.e., it is the unit ball of a "1-symmetric norm"), we know quite a lot about the distribution of the random variable $\langle X, Y\rangle$ in high dimensions. In this case, the random variable $\langle X, Y\rangle$ is approximately a Gaussian random variable of mean zero and variance bounded by C/n. This follows from the results of [9].

The central limit theorem for convex sets [7, 8] states that for any convex body $K \subseteq \mathbb{R}^n$, there exists $0 \ne \theta \in \mathbb{R}^n$ such that $\langle X, \theta\rangle$ is approximately a standard Gaussian, in the sense that the total variation distance to the Gaussian distribution does not exceed C/n^α where $C, \alpha > 0$ are universal constants. With X and Y as above, one may wonder whether $\langle X, Y\rangle$ is approximately Gaussian in high dimensions. This would imply that $\phi(K)$ is much smaller than one.

An amusing fact is that $\phi(K)$ attains the same value $n/(n+2)^2$ when K is either a Euclidean ball B^n or a simplex Δ^n, see [1]. Here Δ^n stands for any n-dimensional simplex whose barycenter lies at the origin. It was conjectured by

Kuperberg [11], following earlier unpublished work by Ball and by Giannopoulos, that for any centrally-symmetric, convex body $K \subseteq \mathbb{R}^n$,

$$\phi(K) \leq \frac{C}{n}$$

for a universal constant $C > 0$. In fact, it was conjectured more precisely that

$$\phi(K) \leq n/(n+2)^2.$$

Supporting evidence for this conjecture includes the fact, proven by Kuperberg, that the Euclidean ball is a local maximizer of $\phi(K)$ among C^2-smooth perturbations, and also the result by Alonso–Gutiérrez [1] which verifies the conjecture in the particular case where $K = B_p^n = \{x \in \mathbb{R}^n \,;\, \sum_i |x_i|^p \leq 1\}$ for some $p \geq 1$.

Balls and simplices are extremals for a few well-known functionals in convexity. Nevertheless, we find that there exists a counter-example to Kuperberg's conjecture. Namely, we exhibit a centrally-symmetric convex set $K \subseteq \mathbb{R}^n$ with

$$\phi(K) \geq c$$

where $c > 0$ is a universal constant. In fact, our convex set is unconditional, i.e., for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$(x_1, \ldots, x_n) \in K \qquad \Longleftrightarrow \qquad (|x_1|, \ldots, |x_n|) \in K.$$

Thus there are convex bodies in high dimension for which the random variable $\langle X, Y \rangle$ is far from Gaussian. Our counter-example is essentially a one-dimensional perturbation of the cross-polytope. Its construction exploits the instability of volume under duality in high dimensions. Specifically, we use the fact that for $K_1 = B_1^n \cap \sqrt{3/n} B_2^n$ and $K_2 = B_1^n$, we have $K_1 \subseteq K_2 \subseteq \mathbb{R}^n$ and

(1) $|K_1| \geq \frac{1}{3} \cdot |K_2|$

(2) $|K_1^\circ \cap (1+c) K_2^\circ| \geq \frac{1}{6} |(1+c) K_2^\circ|$ for a universal constant $c > 0$.

Let us now explain the "coincidence" mentioned earlier, that $\phi(K)$ attains the same value $n/(n+2)^2$ when K is a Euclidean ball and when K is a simplex. The reason behind this phenomenon is that both the Euclidean ball and the simplex are hyperplane sections of homogeneous cones.

An open, convex cone $V \subseteq \mathbb{R}^{n+1}$ with apex at 0 is *homogeneous* if for any two points $x, y \in V$ there exists a linear map $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $A(V) = V$ and $Ax = y$. Examples for homogeneous cones include the positive orthant $\mathbb{R}^n_+$, the Lorentz cone and the cone of positive-definite symmetric $n \times n$ matrices.

We shall consider certain canonical constructions in convex cones. Such constructions necessarily respect the symmetries of the cone, when such symmetries exist. Roughly speaking, the quantity $\phi(K) - n/(n+2)^2$ is something like the "Laplacian" of a function $s : V \to \mathbb{R}$ which has the symmetries of V. The function s is constant when the cone is homogeneous, and hence $\phi(K) = n/(n+2)^2$ for all of the hyperplane sections of a homogeneous cone. These canonical constructions are described in detail in [10], and they involve the *Mahler volume* of a convex body $K \subseteq \mathbb{R}^n$, defined as

$$\bar{s}(K) = |K| \cdot \inf_{x \in K} |(K-x)^\circ|.$$

The Mahler conjecture [12] from 1939 suggests that $\bar{s}(K) \geq (n+1)^{n+1}/(n!)^2$ for any convex body $K \subseteq \mathbb{R}^n$, with equality for the simplex Δ^n.

We prove that any local minimizer $K \subset \mathbb{R}^n$ of the functional $K \mapsto \bar{s}(K)$ satisfies

$$L_K \cdot L_{K^\circ} \cdot \bar{s}(K)^{1/n} \geq \frac{1}{n+2}. \tag{2}$$

There is equality in (2) in the case where K is a ball or a simplex. It follows that any global minimizer K of the Mahler volume satisfies $L_K \geq L_{\Delta^n}$ or $L_{K^\circ} \geq L_{\Delta^n}$.

The strong slicing conjecture suggests that $L_K \leq L_{\Delta^n}$ for any convex body $K \subseteq \mathbb{R}^n$. We conclude that the strong slicing conjecture implies Mahler's conjecture. We remark that it was shown by Rademacher [14] that the simplex is the only local maximizer of the isotropic constant L_K in the class of simplicial polytopes.

It is known that the isotropic constant may become bounded after a small perturbation. Our last theorem in this lecture states that the perturbation can be always made projective. That is, for any convex body $K \subseteq \mathbb{R}^n$ with barycenter at the origin and $0 < \varepsilon < 1$, there exists a convex set $T \subseteq \mathbb{R}^n$ with three properties:

(1) $(1-\varepsilon)K \subseteq T \subseteq (1+\varepsilon)K$.
(2) $T^\circ = K^\circ - y$ for some point y in the interior of K°.
(3) $L_T \leq C/\sqrt{\varepsilon}$ where $C > 0$ is a universal constant.

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The Alexandrov-Fenchel inequality via the Bochner method

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(joint work with Ramon van Handel)

One of the deepest theorems in the theory of Convex Bodies is the Alexandrov-Fenchel Inequality [1] which states that the coefficients of the volume polynomial satisfy hyperbolic inequalities. If $K_1, \dots, K_m \subset \mathbb{R}^n$ are convex bodies (m being a positive integer), then it is a result of Minkowski [5] that the function

$$(t_1, \dots, t_m) \mapsto \mathrm{Vol}(t_1K_1 + \cdots + t_mK_m)$$

is a homogeneous polynomial of degree n. The coefficients of this polynomial $V(K_{i_1}, \dots, K_{i_n})$ are called *mixed volumes* and they carry important geometric information about the bodies $K_1, \dots, K_m$ and the relations between them. The Alexandrov-Fenchel inequality reads

$$V(K_1, K_2, K_3, \dots, K_n)^2 \geq V(K_1, K_1, K_3, \dots, K_n)V(K_2, K_2, K_3, \dots, K_n) \tag{1}$$

for any convex bodies $K_1, \dots, K_n \subset \mathbb{R}^n$. In this talk we provide a new proof [6] of (1) which is considerably simpler than all other known proofs of the inequality, and in addition sheds a new light on related inequalities. Our method is spectral in nature (an approach which goes back to Hilbert [4]) and it starts with an integral representation formula for mixed volumes of smooth convex bodies [2], p. 64:

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_1 D(D^2h_2, \dots, D^2h_n) d\mathcal{H}^{n-1}. \tag{2}$$

Here, $h_i : \mathbb{R}^n \to \mathbb{R}$ is the support function of K_i:

$$h_i(u) = \sup_{x \in K_i} \langle u, x \rangle$$

and D^2h_i is the restriction of the Hessian of h_i to the tangent spaces of the unit sphere $\mathbb{S}^{n-1}$. The term $D(D^2h_2, \dots, D^2h_n)$ is called *mixed discriminant* and these quantities arise as the coefficients of the homogeneous polynomial

$$(t_1, \dots, t_m) \mapsto \det(t_1M_1 + \cdots + t_mM_m)$$

for $(n-1) \times (n-1)$ matrices $M_1, \dots, M_m$ (e.g. D^2h_i). It is a result due to Alexandrov [1] (and also a consequence of our method) that under some assumptions on