Approximately Gaussian marginals and the hyperplane conjecture

Bo'az Klartag

Tel-Aviv University

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Lecture based on a joint work with Ronen Eldan.

Many open problems

This talk is concerned with convex bodies in high dimension.

- Despite recent progress, even the simplest questions remain unsolved:

Question [Bourgain, 1980s]

Suppose $K \subset \mathbb{R}^n$ is a convex body of volume one. Does there exist an $(n-1)$-dimensional hyperplane $H \subset \mathbb{R}^n$ such that

$$Vol_{n-1}(K \cap H) > c$$

where $c > 0$ is a universal constant?

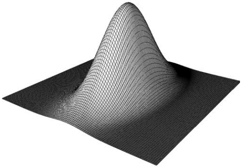
- Known: $Vol_{n-1}(K \cap H) > cn^{-1/4}$ (Bourgain '91, K. '06).
- Affirmative answer for: unconditional convex bodies, zonoids, their duals, random convex bodies, outer finite volume ratio, few vertices/facets, subspaces/quotients of L^p, Schatten class, ...

Logarithmically-Concave densities

As was observed by K. Ball, the hyperplane conjecture is most naturally formulated in the class of **log-concave densities.**

- A probability density on $\mathbb{R}^n$ is log-concave if it takes the form $\exp(-H)$ for a **convex** function $H : \mathbb{R}^n \to [-\infty, \infty)$.

Examples of log-concave densities:
The Gaussian density, the uniform density on a convex body.



1. Product of log-concave densities is (proportional to) a log-concave density.
2. Prékopa-Leindler: If X is a log-concave random vector, so is the random vector $T(X)$ for any linear map T.

Isotropic Constant

For a log-concave density $\rho : \mathbb{R}^n \to [0, \infty)$ set

$$L_\rho = \sup_{x \in \mathbb{R}^n} \rho^{\frac{1}{n}}(x) \det Cov(\rho)^{\frac{1}{2n}}$$

the **isotropic constant** of ρ. The isotropic constant is affinely invariant. What's its meaning?

- Normalization: Suppose X is a random vector in $\mathbb{R}^n$ with density ρ. We say that X (or that ρ) is **isotropic** if

  $$\mathbb{E}X = 0, \qquad Cov(X) = Id$$

  That is, all of the marginals have mean zero and var. one.
- For an isotropic, log-concave density ρ in $\mathbb{R}^n$, we have

  $$L_\rho \sim \rho(0)^{1/n} \sim \int_{\mathbb{R}^n} \rho^{1+\frac{1}{n}} \sim \exp\left(\frac{1}{n} \int_{\mathbb{R}^n} \rho \log \rho \right) > c$$

  where $A \sim B$ means $cA < B < CA$ for universal $c, C > 0$.

An equivalent formulation of the slicing problem

The hyperplane conjecture is *directly* equivalent to the following:

Slicing problem, again:

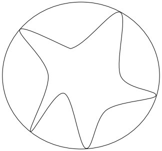
Is it true that for any n and an isotropic, log-concave $\rho : \mathbb{R}^n \to [0, \infty)$,

$$L_\rho < C$$

where $C > 0$ is a universal constant?

(the equivalence follows from works by Ball, Bourgain, Fradelizi, Hensley, Milman, Pajor and others, uses Brunn-Minkowski).

- For a uniform density on $K \subset \mathbb{R}^n$, $L_K = Vol_n(K)^{-1/n}$. Can we have the same covariance as the Euclidean ball, in a substantially smaller convex set?



Remarks

1. It is straightforward to show that $L_\rho > c$, for a universal constant $c > 0$.
2. To summarize, define

$$L_n = \sup_{\rho:\mathbb{R}^n \to [0,\infty)} L_\rho.$$

It is currently known that

$$L_n \leq Cn^{1/4}.$$

3. It is enough to consider the uniform measure on centrally-symmetric convex bodies (Ball '88, K. '05):

$$L_n \leq C \sup_{K \subset \mathbb{R}^n} L_K$$

where $K \subset \mathbb{R}^n$ is convex with $K = -K$.

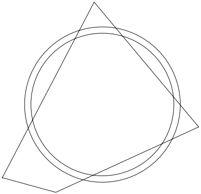
Further open problems

Theorem ("Central Limit Theorem for Convex Bodies", K. '07)

Most of the volume of a log-concave density in high dimensions, with the isotropic normalization, is concentrated near a sphere of radius $\sqrt{n}$.

Define

$$\sigma_n^2 = \sup_X Var(|X|) \sim \sup_X \mathbb{E}\left(|X| - \sqrt{n}\right)^2,$$



where the supremum runs over all log-concave, isotropic random vectors X in $\mathbb{R}^n$.

- The theorem states that

$$\sigma_n \ll \sqrt{n}$$

Approximately Gaussian marginals

The importance of σ_n stems from:

Theorem (Sudakov '78, Diaconis-Freedman '84,...)

Suppose X is an isotropic random vector in $\mathbb{R}^n$, $\varepsilon > 0$. If

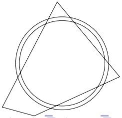
$$\mathbb{P}\left(\left| \frac{|X|}{\sqrt{n}} - 1 \right| \geq \varepsilon \right) \leq \varepsilon.$$

Then for most $\theta \in S^{n-1}$,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(X \cdot \theta \leq t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2} ds \right| \leq C \left(\varepsilon + \frac{1}{n^{1/5}} \right).$$

Many marginals are approx. standard Gaussians.

- Therefore, most of the 1D marginals of a high-dimensional normalized convex body are approx. standard Gaussians.



How thin is the shell?

- Current best bound, due to Fleury '10:
$$\sigma_n \le Cn^{3/8}$$
(improving on a previous bound of $\sigma_n \le Cn^{0.401}$, K. '07).
- Typical marginals of an isotropic log-concave random vector in $\mathbb{R}^n$, are $C\sigma_n/\sqrt{n}$-close to Gaussian.

Conjecture [Antilla-Ball-Perissinaki '03]

Perhaps
$$\sigma_n \le C$$
for a universal constant $C > 0$?

1. Corresponds to a philosophy that "Convexity is as good as independent random variables", in view of Berry-Esseen.
2. True in some cases, including unconditional convex bodies (K. '09) and random convex bodies (Fleury '10).

Thin shell vs. Slicing problem

Theorem (Eldan, K. '10)

There is a universal constant C such that

$$L_n \leq C\sigma_n.$$

Remarks:

1. Pushing σ_n much below $n^{1/4}$ might be *hard* (at the moment only $n^{3/8}$ is known).
2. If L_n is not bounded, then CLT for convex bodies is *weaker* than the classical CLT.
3. Strengthens a result announced by K. Ball '06 (the larger spectral-gap instead of σ_n, exponential dependence).

Proof ideas

Suppose $K \subset \mathbb{R}^n$ a convex body, barycenter at the origin, X is uniformly distributed in K.

The **logarithmic Laplace transform** is the convex function

$$\Lambda(\xi) = \log \mathbb{E} \exp(X \cdot \xi) \qquad (\xi \in \mathbb{R}^n).$$

- The logarithmic Laplace transform helps relate the covariance matrix and the volume of K.

Differentiating the logarithmic Laplace transform

Recall that $\Lambda(\xi) = \log \mathbb{E} \exp(X \cdot \xi)$

- For $\xi \in \mathbb{R}^n$, denote by X_ξ the "tilted" log-concave random vector in $\mathbb{R}^n$ whose density is proportional to

$$x \mapsto 1_K(x) \exp(\xi \cdot x).$$

(any idea for a good coupling when $n \geq 2$?)

Then,

1. $\nabla \Lambda(\xi) = \mathbb{E} X_\xi \in K$.
2. The hessian $\nabla^2 \Lambda(\xi) = Cov(X_\xi)$.
3. Third derivatives? A bit complicated. With $b_\xi = \mathbb{E} X_\xi$,

$$\partial^i \log \det \nabla^2 \Lambda(\xi) = Tr \left[Cov(X_\xi)^{-1} \mathbb{E}(X_\xi^i - b_\xi^i)(X_\xi - b_\xi) \otimes (X_\xi - b_\xi) \right].$$

Transportation of measure

The function $\Lambda(\xi)$ is strictly convex, so $\nabla\Lambda$ is one-to-one.
Recall that $\nabla\Lambda(\xi) \in K$ for all ξ.
From the change of variables formula,

$$Vol_n(K) \geq Vol_n(\nabla\Lambda(\mathbb{R}^n)) = \int_{\mathbb{R}^n} \det \nabla^2\Lambda(\xi) d\xi \geq \int_{nK^\circ} \det \nabla^2\Lambda$$

- In particular, there exists $\xi \in nK^\circ$ with

$$\det \nabla^2\Lambda(\xi) = \det Cov(X_\xi) \leq \frac{Vol_n(K)}{Vol_n(nK^\circ)}.$$

Since $e^{-n} \leq \exp(\xi \cdot x) \leq e^n$ for $x \in K$, then for such $\xi \in nK^\circ$,

$$L_{X_\xi} \leq \frac{C}{Vol_n(K)^{1/n}} \left(\frac{Vol_n(K)}{Vol_n(nK^\circ)} \right)^{1/(2n)} \sim \left(\frac{1}{Vol_n(K) Vol_n(nK^\circ)} \right)^{1/(2n)} .$$

Log-concave densities and convex bodies

Theorem (Bourgain-Milman '87)

$$Vol_n(K) Vol_n(nK^\circ) \geq c^n$$

where $c > 0$ is a universal constant.

- Therefore $L_{X_\xi} < Const$ for **most** $\xi \in nK^\circ$**.**

There is a correspondence between centered log-concave densities and convex bodies due to K. Ball:

- Suppose $f : \mathbb{R}^n \to [0, \infty)$ is a log-concave. Denote

$$K(f) = \left\{ x \in \mathbb{R}^n ; (n+1) \int_0^\infty f(rx) r^n dr \geq 1 \right\},$$

the convex body associated with f.

Isomorphic version of the slicing problem

- When f is log-concave, the body $K(f)$ is convex – closely related to Busemann inequality.
- When f has barycenter at the origin, $K(f)$ and f have roughly the same volume and covariance matrix. So $L_{K(f)} \sim L_f$.
- Suppose f is supported on a convex body K. Denote $a = \inf_K f^{1/n}$ and $b = \sup_K f^{1/n}$. Then

$$aK \subseteq K(f) \subseteq bK.$$

Applying this construction to X_ξ, we deduce:

Corollary [K. '06]

For any convex body $K \subset \mathbb{R}^n$ and $0 < \varepsilon < 1$, there exists another convex body $T \subset \mathbb{R}^n$ with

1. $(1 - \varepsilon)K \subseteq T \subseteq (1 + \varepsilon)K$.
2. $L_T \leq C/\sqrt{\varepsilon}$, where $C > 0$ is a universal constant.

Using Paouris large deviations Theorem

Theorem (Paouris '06)

Suppose X is an isotropic, log-concave random vector in $\mathbb{R}^n$. Then for any $t \geq C\sqrt{n}$,

$$\mathbb{P}\left(|X| \geq t\right) \leq C \exp(-ct)$$

where $c, C > 0$ are universal constants.

Stability of the isotropic constant: It follows immediately that when K and T are convex bodies of volume one, such that

$$Vol_n(K \cap T) \geq e^{-\sqrt{n}},$$

then necessarily $L_K \sim L_T$.

- This leads to the bound $L_n \leq Cn^{1/4}$.

What's the connection to thin shell?

Paouris theorem is about large deviations. How can we use the thin shell estimates?

Suppose $K \subset \mathbb{R}^n$ convex body, X uniform in K, isotropic.

- To prove $L_n \leq C\sigma_n$, we need the lower bound:

$$Vol_n(K) = \int_{\mathbb{R}^n} \det \nabla^2 \Lambda(\xi) d\xi \geq \left(\frac{1}{C\sigma_n} \right)^n$$

  Note that $\det \nabla^2 \Lambda(0) = \det Cov(X) = 1$.

- Third derivatives of Λ again, at the origin:

$$\nabla \log \det \nabla^2 \Lambda(\xi) \Big|_{\xi=0} = \mathbb{E} X |X|^2 = \mathbb{E} X (|X|^2 - n).$$

Relation to thin shell

Recalling that $\sigma_n^2 \sim \mathbb{E}\left(|X|^2 - n\right)^2 / n$, from Cauchy-Schwartz,

$$\left| \nabla \log \det \nabla^2 \Lambda(\xi) \Big|_{\xi=0} \right| = \left| \mathbb{E}X(|X|^2 - n) \right| \leq C\sqrt{n}\sigma_n.$$

- In fact, throughout the proof, in place of σ_n we work with the smaller

  $$\underline{\sigma}_n = \frac{1}{\sqrt{n}} \sup_X \left| \mathbb{E}X|X|^2 \right|$$

  where the supremum runs over all isotropic, log-concave random vectors X in $\mathbb{R}^n$.

To proceed, we have to work with third derivatives at non-zero ξ (or take higher order derivatives at zero and use Taylor's theorem – this could be explained in another talk...).

A Riemannian metric

Computing the third derivatives for $\xi \neq 0$ is slightly easier with respect to a suitable Riemannian metric.

Definition

For $\xi \in \mathbb{R}^n$, consider the positive-definite quadratic form

$$g_\xi(u,v) = Cov(X_\xi)u \cdot v \qquad (u, v \in \mathbb{R}^n)$$

- This Riemannian metric lets X_ξ "feel isotropic".
- This metric does not depend on the Euclidean structure:

$$g_\xi(u,v) = \mathbb{E} u(X_\xi - b_\xi) \cdot v(X_\xi - b_\xi) \qquad (u, v \in \mathbb{R}^{n*})$$

  where $b_\xi = \mathbb{E} X_\xi$ and u, v are viewed as linear functionals.
- The absolute values of the sectional curvatures are bounded by a universal constant. They vanish when $X_1, \ldots, X_n$ are independent r.v.'s.

A Riemannian metric

Our only use of this Riemannian structure is to ease manipulations of third derivatives. Also for a non-zero $\xi \in \mathbb{R}^n$, we have

$$|\nabla_g \log \det Cov(X_\xi)|_g \leq C\sqrt{n}\sigma_n$$

- Consequently, for $\xi \in \mathbb{R}^n$ with $d_g(0,\xi) \leq \sqrt{n}/\sigma_n$,

$$\det Cov(X_\xi) \geq e^{-n}.$$

We need a lower bound for an integral of $\det Cov(X_\xi)$.

- How big is the Riemannian ball of radius $\sqrt{n}/\sigma_n$ around the origin?

Back to log-Laplace

Lemma

$$d_g(0,\xi) \leq \sqrt{\Lambda(2\xi)}$$

Proved by inspecting the Riemannian length of the (Euclidean) segment $[0,\xi]$: By convexity,

$$d_g(0,\xi) \leq \int_0^1 \sqrt{\frac{\partial^2}{\partial \xi^2}\Lambda(r\xi)dr} \leq \sqrt{\Lambda(2\xi)}$$

- Therefore,

$$\left(\frac{1}{L_K}\right)^n \geq c^n Vol_n\left(\left[\Lambda \leq n/\sigma_n^2\right]\right)$$

Now we forget about the Riemannian metric. We were not really able to deeply exploit the Riemannian geometry.

Level sets of Laplace transform

Recall that $K \subset \mathbb{R}^n$ is a convex body whose barycenter is at the origin, X uniform in K.

Lemma

There exist universal $c, C > 0$ such that

$$cnK^\circ \subseteq [\Lambda \leq n] \subseteq CnK^\circ$$

Proved by standard log-concave tricks, nothing more than asymptotics of 1D integrals. Bourgain-Milman: When X is isotropic,

$$Vol_n([\Lambda < n]) \geq \frac{c^n}{Vol_n(K)} = (cL_K)^n \geq \tilde{c}^n,$$

- Suppose X is isotropic, and take an integer $1 \leq k \leq n$. Then, for any k-dimensional subspace $E \subset \mathbb{R}^n$,

$$Vol_k([\Lambda < k] \cap E) \geq c^k.$$

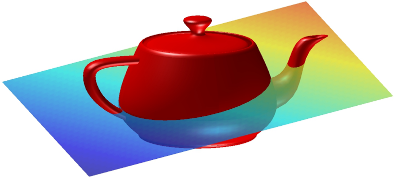
Completing the proof

Recall that

$$\left(\frac{1}{L_K}\right)^n \geq c^n Vol_n\left(\left[\Lambda \leq n/\sigma_n^2\right]\right).$$

Take an integer $k \sim n/\sigma_n^2$. Then, for any k-dimensional subspace $E \subset \mathbb{R}^n$,

$$Vol_k\left([\Lambda < k] \cap E\right) \geq c^k.$$



Without confusion, we deduce

$$Vol_n\left([\Lambda < k]\right) \geq \left(\frac{c\sqrt{k}}{\sqrt{n}}\right)^n \geq \left(\frac{\tilde{c}}{\sigma_n}\right)^n.$$

- This completes the proof of

$$L_n \leq C\sigma_n.$$

Variations of the Riemannian metric

1. If $\Lambda^* : K \to \mathbb{R}$ is the Legendre transform of Λ, then the hessian
$$\nabla^2 \Lambda^*(x) \qquad (x \in K)$$
defined a Riemannian structure on K, isometric to the one described above, linearly invariant.

2. The expression
$$Vol_n(K) = \int \det Cov(X_\xi) d\xi$$
reminds us of the Riemannian volume (a square root is missing!). One may construct a very similar **Kähler metric** on $\mathbb{C}^n / i\mathbb{Z}^n$, whose volume is exactly $Vol_n(K)$. Perhaps it allows a more intrinsic analysis? I have no idea.

Thank you!