T E L   A V I V   U N I V E R S I T Y

The Raymond and Beverley Sackler Faculty of Exact Sciences

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Topics in Asymptotic Convex Geometry

Thesis submitted for the degree “Doctor of Philosophy”

by

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To Ayelet.

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Introduction

1 Asymptotic Convex Geometry

This thesis is concerned with high dimensional geometric phenomena. Based on a low (2 or 3) dimensional intuition, one might expect high dimensional geometry to be rather complicated. A priori, one would think that the diversity and rapid growth of the number of configurations would make it impossible to formulate general, interesting theorems that apply to all high dimensional bodies. In addition, the infinite dimensional experience may strengthen this feeling. In the theory of Banach spaces, pathological examples were found to contradict many reasonable properties. Therefore, one might approach the field of high dimensional geometry with low expectations for a general theory, applicable to all high dimensional bodies.

However, there exists a strong motive in high dimensional geometry that compensates for the enormous diversity - the concentration of measure phenomenon. The fact that high dimensional measures typically have strong concentration estimates, allows for proofs of general theorems relevant to all convex bodies, such as the classical Dvorezky theorem, Milman's quotient of subspace theorem and others. Hence, the high dimensionality or the large number of parameters, when viewed correctly, may sometimes create order and simplicity rather than complication.

Asymptotic Convex Geometry is the study of convex sets in $\mathbb{R}^n$, when the dimension n tends to infinity. While the word "asymptotic" is the essence of this theory - some of the results are even impossible to formulate in a fixed or a low dimension - the term "convex" is less crucial to the theory. Its purpose is just to impose some regularity on the geometric figures we investigate. In many cases, the results also hold under weaker requirements than convexity, such as quasi-convexity. Further, some of the results are correct when formulated appropriately for discrete sets of points, rather than only for convex bodies (e.g. some of the results presented here regarding Minkowski symmetrization).

Historically, Asymptotic Convex Geometry (also known as Asymptotic Geometric Analysis or similar names) emerged in the 1980s from the local theory of Banach spaces. In the local theory, one obtains information regarding an infinite dimensional Banach space from its local structure - the collection of all its finite dimensional subspaces or quotients. In this approach, one considers finite dimensional spaces, which are easier to handle than infinite dimensional spaces (for example, try selecting a random vector in an infinite dimensional sphere). Moreover, one considers families of finite dimensional spaces whose dimensions tend to infinity, rather than a single finite dimensional space. Thus it is important to obtain estimates which are uniform in the dimension, or whose dependence on the dimension is the best possible.

There are scientific disciplines which share common features with Asymptotic Convex Geometry, such as Asymptotic Combinatorics, Statistical Physics, Complexity Theory and also Asymptotic estimates in Probability and Analysis. Some interactions between these fields and Asymptotic Convex Geometry have already appeared, and more are expected to emerge in the future.

2 Topics in this thesis

This thesis is divided into three parts. In the first, which consists of three chapters, we study geometric symmetrizations. One of the symmetrization methods we study is presented in the following definition:

Definition 2.1 *Let $K \subset \mathbb{R}^n$ be a convex body, and let H be a hyperplane. Denote by π_H the reflection operator with respect to H in $\mathbb{R}^n$. The result of a "Minkowski symmetrization of K with respect to H" is defined to be*

$$\tau_H(K) = \frac{K + \pi_H(K)}{2} = \left\{ \frac{x+y}{2}; x \in K,\ y \in \pi_H(K) \right\}.$$

Thus, when applying a Minkowski symmetrization to a body $K \subset \mathbb{R}^n$, we obtain another body $\tau_H(K)$ which shares many of the properties of the original body, yet is symmetric with respect to the hyperplane H. Another symmetrization method is due to Steiner:

Definition 2.2 *Let $K \subset \mathbb{R}^n$ be a convex body, and let H be a hyperplane. "Steiner symmetrization of K with respect to a hyperplane H" yields the unique body $S_H(K)$ such that for any line l perpendicular to H,*

(i) $S_H(K) \cap l$ is a closed segment whose center lies on H.

(ii) $Meas(K \cap l) = Meas(S_H(K) \cap l)$

where $Meas$ is the one dimensional Lebesgue measure on the line l.

When applying a suitable sequence of symmetrizations (either Steiner or Minkowski) to an arbitrary body, one obtains a sequence of bodies that converges to a Euclidean ball. This property renders symmetrizations very useful in proving geometric inequalities in which the equality case is satisfied by the Euclidean ball (see, e.g. [BF]). Here, we consider Minkowski and Steiner symmetrization processes and investigate the rate of the convergence to a Euclidean ball. The following is proved here (we denote by D the unit Euclidean ball in $\mathbb{R}^n$, and $S^{n-1} = \partial D$):

Theorem 2.3 *Let $n \geq 2$ and let $K \subset \mathbb{R}^n$ be a convex body. Then there exist $5n$ Minkowski symmetrizations (or $3n$ Steiner symmetrizations), such that when applied to K, the resulting body $\tilde{K}$ satisfies*

$$\frac{1}{c} rD \subset \tilde{K} \subset crD$$

where $c > 0$ is a numerical constant.

Theorem 2.4 *Let $n \geq 2$, $0 < \varepsilon < \frac{1}{2}$, and let $K \subset \mathbb{R}^n$ be a convex body. Then there exist $cn \log \frac{1}{\varepsilon}$ Minkowski symmetrizations (or $cn^4 \log^2 \frac{1}{\varepsilon}$ Steiner symmetrizations), that transform K into a body $\tilde{K}$ that satisfies*

$$(1-\varepsilon)rD \subset \tilde{K} \subset (1+\varepsilon)rD$$

where $c > 0$ is some numerical constant.

These theorems along with others are proved in Part I. The second part of this thesis, which also consists of three chapters, deals with the slicing problem. The development of ideas from the analysis of symmetrization presented in Part I, has lead to a study of the slicing problem. Any convex body $K \subset \mathbb{R}^n$ whose barycenter is at the origin, has a linear image $\tilde{K}$ with $Vol(\tilde{K}) = 1$ such that

$$\int_{\tilde{K}} \langle x, \theta \rangle^2 dx \tag{1}$$

does not depend on the choice of $\theta \in S^{n-1}$. We say that $\tilde{K}$ is an isotropic linear image of K, or that $\tilde{K}$ is in isotropic position. The isotropic linear image of K is unique up to orthogonal transformations (e.g. [MP1]). The square of the isotropic constant of K, or L_K^2, refers to the quantity in (1) for any $\tilde{K}$ an isotropic linear image of K and for any $\theta \in S^{n-1}$.

A major unsolved problem asks whether there exists a numerical constant C such that $L_K < C$ for every convex body in any finite dimension. This problem is called the slicing problem or the hyperplane conjecture. A positive answer to this question would entail many interesting consequences. One of these is that every convex body of volume one, has at least one hyperplane section whose $n-1$ dimensional volume is greater than some constant $c > 0$. The best current estimate is $L_K < cn^{1/4} \log n$ for an arbitrary convex body $K \subset \mathbb{R}^n$ and is due to Bourgain. For certain classes of convex bodies the question has been affirmatively answered, such as for unconditional bodies, zonoids, duals of zonoids, duals to bodies with finite volume ratio, and more (references are provided in Chapter 4).

In Chapter 4 we present a reduction of the general problem to the boundness of the isotropic constant of a certain class of convex bodies: those which have a finite volume ratio. For $K \subset \mathbb{R}^n$ the volume ratio of K is defined as

$$v.r.(K) = \sup_{\mathcal{E} \subset K} \left(\frac{Vol(K)}{Vol(\mathcal{E})} \right)^{\frac{1}{n}}$$

where the supremum runs over all the ellipsoids that are contained in K. Bodies with a small volume ratio contain a relatively large ellipsoid, and after a linear transformation, most of their proportional sections are close to a Euclidean ball. Using symmetrization techniques, we prove the following conditional proposition:

Proposition 2.5 *There exists $v > 1$ such that the following holds:*

If there exists $c_1 > 0$ such that for any n and for any $K \subset \mathbb{R}^n$, the inequality $v.r.(K) < v$ implies that $L_K < c_1$,

then there exists $c_2 > 0$ such that for any n and for any $K \subset \mathbb{R}^n$ we have $L_K < c_2$.

In Chapter 5 we deal with an isomorphic relaxation of the slicing problem. We show that there exists some $c > 0$ such that the collection of bodies in all dimensions with an isotropic constant smaller than c is "dense" in the space of convex symmetric bodies, in some sense. Formally, we define the Banach-Mazur distance of $K, T \subset \mathbb{R}^n$ as

$$d_{BM}(K,T) = \inf \left\{ ab; \frac{1}{a} K \subset LT \subset bT \ ; a, b > 0, \ L \text{ is a linear operator} \right\}.$$

If $K_n, T_n \subset \mathbb{R}^n$ is a sequence of convex bodies such that $d_{BM}(K_n, T_n) < C$ where C is independent of n, we say that the families $\{K_n\}$ and $\{T_n\}$ are uniformly isomorphic. This terminology originates in Banach space theory: the normed spaces which have K_n and T_n as their unit balls are uniformly isomorphic. We prove that for any convex body we can find another convex body whose isotropic constant is uniformly bounded, and whose distance from the original body is "almost" uniformly bounded (up to a logarithmic factor).

Theorem 2.6 *For any body $K \subset \mathbb{R}^n$ there exists a body $T \subset \mathbb{R}^n$ with $d_{BM}(K,T) < c_1 \log d_{BM}(K, D) \leq \frac{c_1}{2} \log n$ and*

$$L_T < c_2$$

where $c_1, c_2 > 0$ *are numerical constants.*

Furthermore, for bodies that have a non-trivial type, we can eliminate the $\log n$ factor from Theorem 2.6. If $K \subset \mathbb{R}^n$ is such a body, then there exists $T \subset \mathbb{R}^n$ which is uniformly isomorphic to K and such that L_T is uniformly bounded. We would like to emphasize the meaning of uniformity here. When we say that L_T is uniformly bounded we mean that L_T is smaller than some numerical constant that is completely independent. The same applies to the sentence "T is uniformly isomorphic to K". This means that $d_{BM}(K, T)$ is smaller than some numerical constant which is independent of the dimension.

The final chapter in Part II is concerned with a connection between the slicing problem and the problem of rapid Steiner symmetrization. The n-dimensional cube has an interesting property: it requires only $\lfloor \frac{n}{10} \rfloor$ Steiner symmetrizations in order to transform into a body which is uniformly isomorphic to a Euclidean ball. Of course, the number $\frac{1}{10}$ has no special significance and may be replaced by any $0 < \varepsilon < 1$. Note that when applying much fewer than n symmetrizations to a convex body, there remain projections of high dimension that we did not alter. Therefore, for a general convex body we cannot hope for such a fast symmetrization process. As is proved in Chapter 2, the cross-polytope requires roughly at least n symmetrizations in order to be symmetrized into an isomorphic ellipsoid.

However, it is possible that all convex bodies require only a short symmetrization process, after removing a small proportion of their volume. Let us consider the class of bodies $K \subset \mathbb{R}^n$ for which there exists a large part $T \subset K$ with $Vol(T) > \frac{9}{10} Vol(K)$, such that much fewer than n symmetrizations are sufficient in order to symmetrize T into an isomorphic ellipsoid. In Chapter 6 we demonstrate that this class of bodies is the entire collection of convex bodies, if and only if the slicing conjecture is true.

The third part of this thesis is not devoted to a single topic, but rather presents two different results that arose during the author's research. In Chapter 7 we present a new geometric inequality concerning diameters of sections of convex bodies. For a k-dimensional convex body $T \subset \mathbb{R}^n$, define

$$v.rad.(T) = \left(\frac{Vol(T)}{Vol(D)} \right)^{1/k}$$

where Vol is interpreted as the k-dimensional Lebesgue measure in the affine hull of T. Indeed, $v.rad.(T)$ is the radius of a Euclidean ball with the same volume as T. Define also $diam(T) = \sup_{x,y \in T} |x-y|$. We denote the Grassman manifold of all k-dimensional subspaces in $\mathbb{R}^n$ by $G_{n,k}$. This manifold is equipped with a unique rotation invariant probability measure, which we work with in the next Theorem.

Theorem 2.7 *Let $K \subset \mathbb{R}^n$ be a convex body that has the origin in its interior. Let $k = \lambda n$ be a positive integer and let $E \in G_{n,k}$ be a random subspace of dimension k. Then, with probability greater than $1 - e^{-n}$,*

$$diam(K \cap E)^{1-\lambda} v.rad.(K \cap E)^{\lambda} < Cv.rad.(K)$$

where $C > 0$ is a numerical constant.

The proof of Theorem 2.7 is short and elementary, and its main tool is integration in polar coordinates. The theorem has many immediate corollaries. Among these is the known fact that a finite volume ratio body has proportional sections which are isomorphic to a Euclidean ball. One can also deduce a low M^*-estimate from this theorem, but more interestingly, this theorem gives rise to a new inequality, dual in a sense, which we call a "low M-estimate". Denote $\|x\| = \inf\{\lambda; \lambda x \in K\}$, the norm which has K as its unit ball and $M(K) = \int_{S^{n-1}} \|x\| d\sigma(x)$ where σ is the unique rotation invariant probability measure on the sphere. Then for a random $E \in G_{n,\lambda n}$ with probability larger than $1 - e^{-n}$,

$$diam(K \cap E) < (cM(K))^{\frac{\lambda}{1-\lambda}} v.rad.(K)^{\frac{1}{1-\lambda}}$$

where $c > 0$ is a universal constant. Since the low M^*-estimate has important applications in Asymptotic Convex Geometry, we expect to find applications for our low M-estimate as well.

Chapter 8 describes an observation related to the John position. Let $K \subset \mathbb{R}^n$ be a convex, centrally-symmetric body. Denote by $\mathcal{E} \subset K$ the (unique) ellipsoid of maximal volume that is contained in K. This ellipsoid is called the Löwner-John ellipsoid of K. It is known that this ellipsoid may be characterized through its contact points with K. For example, if $D \subset K$ then D is the John ellipsoid of K if and only if there exists a measure ν supported on $S^{n-1} \cap \partial K$ such that the covariance matrix of ν is the identity matrix.

The standard proofs of this fact involve perturbation arguments. We show that a simple linear programming duality relation may replace these arguments. We also prove the following proposition:

Proposition 2.8 *Let $K \subset \mathbb{R}^n$ be a convex, centrally-symmetric body. Then there exists a unique centrally-symmetric ellipsoid $\mathcal{E} \subset K$ such that it is possible to define a measure ν on $\partial\mathcal{E} \cap \partial K$ whose covariance matrix is the identity matrix.*

Hence, given an arbitrary Euclidean structure on $\mathbb{R}^n$, we obtain some unique ellipsoid associated with K. This creates a map between ellipsoids, induced by the body K. We show that this map is continuous, non-trivial (i.e. the map characterizes K) and that the John ellipsoid is its unique fixed point.

The chapters in this thesis are independent of each other. A list of references appears at the end of each chapter and for convenience, we also include a complete bibliography at the end of the thesis.

Part I: Symmetrization

$5n$ Minkowski symmetrizations suffice to arrive at an approximate Euclidean ball[1]

Abstract. Here we prove that for every convex body in $\mathbb{R}^n$ there exist $5n$ Minkowski symmetrizations, which transform the body into an approximate Euclidean ball. This result complements the sharp $cn \log n$ upper estimate by J. Bourgain, J. Lindenstrauss and V.D. Milman, of the number of random Minkowski symmetrizations sufficient for approaching an approximate Euclidean ball.

1 Introduction

Let K be a compact convex set in $\mathbb{R}^n$ and let u be any vector in $S^{n-1} = \{u; |u| = 1\}$ where $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$. Denote by $\pi_u \in O(n)$ the reflection with respect to the hyperplane through the origin orthogonal to u, i.e. $\pi_u x = x - 2\langle x, u\rangle u$.

Minkowski symmetrization (often referred to as Blaschke symmetrization, see [Bla1]) of K with respect to u is defined to be the convex set $\frac{1}{2}(\pi_u K + K)$, where the Minkowski sum of two sets $A, B \subset \mathbb{R}^n$ is defined as $A + B = \{a + b; a \in A, b \in B\}$. Denote by $\|\cdot\|^*$ the dual norm to K (i.e. $\|x\|^* = \sup_{y \in K}\langle x, y\rangle$). Despite the fact that K is not necessarily centrally symmetric and $\|\cdot\|^*$ need not be a norm, this convenient notation will be used for readability. Denote by $M^*(K)$ the half mean width of K, defined as $M^*(K) := \int_{S^{n-1}} \|x\|^* d\sigma(x)$, where σ is the normalized rotation invariant measure on S^{n-1}.

It is easily verified that $M^*(K) = M^*(\frac{1}{2}(\pi_u K + K))$, so the mean width is preserved under Minkowski symmetrizations. Since successive Minkowski symmetrizations make the body more symmetric in some sense, one might expect convergence to a ball of radius $M^*(K)$.

Surprisingly, very few symmetrizations are sufficient for this convergence; In [BLM1] it is proven that $cn \log n$ random symmetrizations suffice to obtain from any convex body, a new body $\tilde{K}$, such that $\frac{1}{2}M^*D \subset \tilde{K} \subset 2M^*D$ with high probability, where $D = \{u; |u| \leq 1\}$ is the standard Euclidean ball in $\mathbb{R}^n$.

The proof in [BLM1] can be slightly refined (see [Kl1]), and rather than an estimate of $cn \log n$ for all bodies, in fact $cn \log \frac{2diam(K)}{M^*(K)}$ random symmetrizations are enough. This quantity is always smaller then $cn \log n$ but in some cases there is a substantial improvement; For example, the n-dimensional cube needs only cn random symmetrizations to be transformed into an almost Euclidean ball.

[1] This chapter corresponds to the paper [Kl2].

In [Kl1] it was proven that the aforementioned estimate is very tight and is actually a formula, as follows: For every convex body K at least $\tilde{c}n \log \frac{diam(K)}{2M^*(K)}$ random symmetrizations are necessary in order for the body to become close to a Euclidean ball. Hence, bodies such as $B(l_1^n)$ - the n-dimensional cross polytope - in fact require at least $cn \log n$ random symmetrizations.

Here we show that there exist symmetrizations which are better than random ones. There is a specific choice of $5n$ symmetrizations that transform any convex body into an approximate Euclidean ball. The basic idea underlying the construction is changing the notion of randomness; Rather than symmetrizing with respect to random vectors, symmetrizations with respect to the vectors of a random orthogonal basis will be performed at each iteration.

Six iterations of this kind suffice (totaling $6n$ symmetrizations[2]), however the role of each iteration is slightly different. Precisely, for the first iteration any orthogonal basis is adequate. The remaining five iterations are required to be with respect to random independent orthogonal bases, and the results hold with large probability that tends to 1 when the dimension n approaches infinity.

There exists a very similar symmetrization process that leads to a slightly better estimate, and consists of $5n$ symmetrizations (only $4n$ symmetrizations, if the body is already unconditional). This process uses symmetrizations with respect to five orthogonal bases, some of which need not be random. An additional basis will be used in this process, and will be referred to here as a Walsh basis. It actually coincides with the regular Walsh basis for dimensions which are powers of two. Let us describe the $5n$ symmetrizations process: The first basis is chosen to be any orthogonal basis, and is used only to create unconditionality. The second basis can be a random basis or a Walsh basis (with respect to the first), and the corresponding symmetrization reduces the diameter of the body to a level of $\log n$ times its mean width. The third basis is a Walsh basis with respect to the previous, and reduces the diameter further, to a level of $\log \log n$ times the mean width. The fourth basis must be, in this proof, a random orthogonal basis and the fifth, either a Walsh basis with respect to the fourth, or a random basis. Once the diameter is small enough, the last two bases together transform the body into an approximate Euclidean ball.

The proof outlined below is mainly concerned with the first process described (which is purely random). Results for the second process are analogous to those of the first, and may be concluded based on remarks throughout the proof.

[2]It seems at first, that six iterations consist of $6n$ symmetrizations; However, after the first iteration, the body becomes centrally symmetric. Following that stage, the last vector in each orthogonal basis is unnecessary, because symmetrizing with respect to that vector would not affect the body.

The symbols $c, C, c', \tilde{c}$ denote numerical constants which are not necessarily identical throughout this text.

2 First step: Initial symmetrizations

Let K be an arbitrary convex body in $\mathbb{R}^n$. For the purpose of normalization, assume $M^*(K) = 1$. Take any orthogonal basis $\{e_1, .., e_n\}$ and symmetrize K with respect to the vectors $e_1, .., e_n$ to obtain the new body $\tilde{K}$. Since orthogonal reflections commute, $\tilde{K}$ is invariant under reflection with respect to e_i, for $1 \le i \le n$. Therefore $\tilde{K}$ is unconditional with respect to the basis $\{e_1, .., e_n\}$. By Lemma 3.2 from [Kl1] there exists a universal constant c such that,

$$\tilde{K} \ \subset \ c\sqrt{n}\ conv\{\pm e_i\}_{i=1}^n = c\sqrt{n}B(l_1^n).$$

A specific body will be referred to in this section: $Q = \sqrt{n}\ conv\{\pm e_i\}_{i=1}^n$. After a certain symmetrization process its diameter decays from $\sqrt{n}$ to $\tilde{c}\log n$ with high probability. Clearly, applying the same set of symmetrizations to $\tilde{K}$ will reduce its diameter to less than $\tilde{c}\log n$.

Proposition 2.1 *Let $\{e_1, .., e_n\}$ be an orthogonal basis in $\mathbb{R}^n$, and let $Q = \sqrt{n}\ conv\{\pm e_i\}_{i=1}^n$. Let μ_n be the unique rotation invariant probability measure on $O(n)$. Suppose that $\{u_1, .., u_n\} \in O(n)$ is chosen randomly, according to μ_n. After symmetrizing Q with respect to $u_1, .., u_{n-1}$ a new body $\tilde{Q}$ is obtained.*
Claim:

$$diam(\tilde{Q}) \le c\log n$$

with probability greater than $1 - \frac{1}{n^{10}}$.

Remark: The number '10' in the expression $1 - \frac{1}{n^{10}}$ is of course arbitrary, and may be replaced by any other constant. Such a replacement will influence the constant 'c' in the concluded inequality "$diam(\tilde{Q}) \le c\log n$".

Corollary 2.2 *For every convex body $K \subset \mathbb{R}^n$ with $M^*(K) = 1$, there exist $2n$ symmetrizations which transform K into $\tilde{K}$, where $diam(\tilde{K}) < c\log n$ and $\tilde{K}$ is unconditional with respect to some orthogonal basis.*

Following is a simple and well-known lemma. For completeness it will be proven at the end of this section.

Lemma 2.3 *Let $\{e_i\}_{i=1}^n$ be any orthogonal basis, and let $\{u_i\}_{i=1}^n$ be a random orthogonal basis. Then for all $1 \le i, j \le n$:*

$$|\langle u_i, e_j\rangle| \le c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$$

with probability greater than $1 - \frac{1}{n^{10}}$.

Proof of Proposition 2.1: Denote by $\|\cdot\|$ the dual norm of Q (i.e. $\|x\| = \sup_{y\in Q}\langle x, y\rangle = \sqrt{n}\max_i |\langle x, e_i\rangle|$). The dual norm of $\tilde{Q}$ is, by definition (recall that the Minkowski sum of bodies is equivalent to the sum of their dual norms):

$$|||x||| = \frac{1}{2^{n-1}} \sum_{D\subset\{1,..,n-1\}} \left\| (\prod_{i\in D} \pi_{u_i}) x \right\| = \frac{1}{2^{n-1}} \sum_{D} \sqrt{n} \max_j \left| \langle \prod_{i\in D} \pi_{u_i} x, e_j\rangle \right|.$$

Substitute $x = \sum_i \langle x, u_i\rangle u_i$. Since reflecting with respect to u_i means switching the sign of the i^{th} coordinate in $\{u_1, .., u_n\}$ basis,

$$|||x||| = \mathbb{E}_\varepsilon \max_j \left| \sqrt{n} \sum_i \varepsilon_i \langle x, u_i\rangle\langle u_i, e_j\rangle \right|$$

where $\varepsilon = (\varepsilon_i)_{i=1}^n$ is uniformly distributed in $\{\pm 1\}^n$. Therefore, $|||x|||$ is the expectation of a maximum of n random variables. Denote:

$$f_x^j(\varepsilon) = \sqrt{n} \left| \sum_i \varepsilon_i \langle x, u_i\rangle\langle u_i, e_j\rangle \right|.$$

Then $|||x||| = \mathbb{E}_\varepsilon[\max_j f_x^j(\varepsilon)]$. For $\alpha > 0$, and for any measurable $f : \Omega \to \mathbb{R}$ define $\|f\|_{\psi_\alpha} = \inf\{\lambda > 0 : \int_\Omega e^{|\frac{f}{\lambda}|^\alpha} \le 2\}$. The following equivalent definitions are frequently used:

$$\|f\|_{\psi_\alpha} < c \;\Leftrightarrow\; (\mathbb{E}|f|^p)^{\frac{1}{p}} < c' p^{\frac{1}{\alpha}} \;\Leftrightarrow\; Prob\{|f| > t\} < e^{-c'' t^\alpha}.$$

Khinchine inequality (e.g. [MS] page 38) shows that the ψ_2 norm of f_x^j is bounded, as follows:

$$\|f_x^j\|_p = \left(\mathbb{E}_\varepsilon \left| \sqrt{n} \sum_i \varepsilon_i \langle x, u_i\rangle\langle u_i, e_j\rangle \right|^p \right)^{\frac{1}{p}} \le c\sqrt{p}\sqrt{n} \sqrt{\sum_i (\langle x, u_i\rangle\langle u_i, e_j\rangle)^2}.$$

By Lemma 2.3 with large probability, $\forall i, j \; |\langle u_i, e_j\rangle| \le c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$.. Hence, with high probability,

$$\|f_x^j\|_p \le c\sqrt{p}\sqrt{\log n}|x| \quad \Rightarrow \quad \|f_x^j\|_{\psi_2} \le c' \sqrt{\log n}|x|$$

Since $|||x||| = \mathbb{E}_\varepsilon[\max_j f_x^j(\varepsilon)]$, the well-known estimate for the expectation of a maximum of ψ_2 variables can be used (e.g. [LT] page 79, or the remark after lemma 3.4 here):

$$\forall x \in \mathbb{R}^n \; |||x||| \le c\sqrt{\log n}(c' \sqrt{\log n}|x|) = c \log n |x|$$

Thus the proposition is proven. $\square$

Remark: For every dimension, there exists an orthogonal basis $\{u_i\}_{i=1}^n$ such that $\forall i, j$:

$$|\langle u_i, e_j\rangle| \le \frac{2}{\sqrt{n}}.$$

Such a basis is called here a "Walsh" basis. Indeed, for dimension $n = 2^k$, the regular Walsh basis is satisfactory, while for other dimensions, an appropriate basis may be constructed using sines and cosines (this basis consists of orthogonal vectors resembling the complex valued characters of the group $\mathbb{Z}/n\mathbb{Z}$). Instead of using Lemma 2.3 in the proof of Proposition 2.1, one can replace the random basis with a Walsh basis, obtaining yet a slightly better result, with "$\log n$" replaced by "$\sqrt{\log n}$" in the conclusion of Proposition 2.1.

Proof of Lemma 2.3: Since for every i the vector u_i distributes uniformly over the sphere, by the standard concentration inequality on the sphere (e.g. first pages of [MS]):

$$Prob\{|\langle u_i, e_j\rangle| > \varepsilon\} \le \sqrt{\frac{\pi}{8}} e^{-\frac{\varepsilon^2 n}{2}}. \tag{1}$$

Select c_1 (i.e. $c_1 = 5$) such that for $\varepsilon = c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$, the probability in (1) is less than $\frac{1}{n^{12}}$. Therefore, the probability that $|\langle u_i, e_j\rangle| < c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$ holds for all $1 \le i, j \le n$ is greater than $1 - \frac{1}{n^{10}}$. $\square$

3 Second step: Logarithmic decay of the diameter

In the second step, symmetrizations will be performed with respect to two random orthogonal bases; This section proves that this step reduces the diameter of the body logarithmically: from $c \log n$ to $C \log\log n$, with probability close to 1. Therefore, after the second step (and a total of $4n$ symmetrizations) the diameter is less than $C \log\log n$. This proof extends that of the former section.

Let K be the convex body obtained from the first step of symmetrizations. According to Corollary 2.2, $M^*(K) = 1$, $diam(K) < c \log n$, and K is unconditional with respect to some orthogonal basis (re-denote this basis as $\{e_1, .., e_n\}$). Once again, by Lemma 3.2 from [Kl1],

$$K \ \subset \ c\sqrt{n}\ conv\{\pm e_i\}_{i=1}^n \ = \ c\sqrt{n} B(l_1^n).$$

Set $t = diam(K) < c \log n$. Clearly, $K \subset \sqrt{n} B(l_1^n) \bigcap tB(l_2^n)$. As in the first step, rather than working directly with the body K, symmetrize $K_t = \sqrt{n} B(l_1^n) \bigcap tB(l_2^n)$.

Proposition 3.1 *Let* $K_t = \sqrt{n}B(l_1^n)\bigcap tB(l_2^n)$. *Assume that* $\{u_1,..,u_n\} \in O(n)$ *and* $\{v_1,..,v_n\} \in O(n)$ *are chosen uniformly and independently. After symmetrizations with respect to* $u_1,..,u_{n-1}$ *and* $v_1,..,v_{n-1}$, *a new body* $\tilde{K}_t$ *is obtained such that:*

$$\tilde{K}_t \subset C\log tB(l_2^n)$$

with probability greater than $1-e^{-c\sqrt{n}}$ *of choosing the orthogonal bases.*

Corollary 3.2 *For every convex body* $K \subset \mathbb{R}^n$ *with* $M^*(K) = 1$, *there exist* $4n$ *symmetrizations which transform* K *into* $\tilde{K}$, *where diam*$(\tilde{K}) < c\log\log n$.

Begin by describing the body K_t through its dual norm. Denote by $\|\cdot\|_t'$ the norm:

$$\|x\|_t' = \inf\{\|x'\|_2 + t\|x''\|_\infty : x = x' + x''\}$$

The dual norm of K_t is exactly $t\|\cdot\|'_{\frac{\sqrt{n}}{t}}$, as can be verified. Put $(a_i^*)_{i=1}^n$ for the non-increasing rearrangement of the absolute values of $(a_i)_{i=1}^n$. The following two lemmas are well-known. The first lemma essentially appears in [BL], but for lack of concise references, attached here are the short elementary proofs.

Lemma 3.3

$$\forall x \in \mathbb{R}^n \quad \|x\|_k' \approx \sqrt{\sum_{i=1}^{k^2}(x_i^*)^2}$$

(and the equivalence constant is not more than $\sqrt{2}$*).*

Proof: For i where $|x_i| \geq x_{k^2}^*$ set $x_i' = (x_i - sgn(x_i)x_{k^2}^*)$. For other i's, set $x_i' = 0$. Let $x'' = x - x'$. Then:

$$\begin{aligned}
\|x\|_k' &\leq \|x'\|_2 + k\|x''\|_\infty \\
&= \sqrt{\sum_{i=1}^{k^2}(x_i^* - x_{k^2}^*)^2} + kx_{k^2}^* \\
&\leq \sqrt{2}\sqrt{\sum_{i=1}^{k^2}\left[(x_i^* - x_{k^2}^*)^2 + (x_{k^2}^*)^2\right]} \\
&\leq \sqrt{2}\sqrt{\sum_{i=1}^{k^2}(x_i^*)^2}.
\end{aligned}$$

On the other hand, assume $x = x' + x''$. Surely $x_i^* \leq x_i'^* + x_1''^*$, so

$$\sqrt{\sum_{i=1}^{k^2}(x_i^*)^2} \leq \sqrt{\sum_{i=1}^{k^2}(x_i'^*)^2} + \sqrt{\sum_{i=1}^{k^2}(x_1''^*)^2} \leq \|x'\|_2 + k\|x''\|_\infty.$$

Lemma 3.4 *Let $(X_i)_{i=1}^n$ be ψ_1 random variables (i.e. random variables that satisfy: $\mathbb{E}e^{|X_i|} \leq C$), and let $(X_i^*)_{i=1}^n$ be the non-increasing rearrangement of the X_i's. Then:*

$$\mathbb{E}\sqrt{\frac{1}{k}\sum_{i=1}^{k}(X_i^*)^2} \leq c_2 \log\frac{2n}{k}.$$

Proof: Since the X_i's are ψ_1 variables,

$$\mathbb{E}\frac{1}{k}\sum_{i=1}^{k} e^{X_i^*} \leq \mathbb{E}\frac{1}{k}\sum_{i=1}^{n} e^{|X_i|} \leq C\frac{n}{k}. \tag{2}$$

Let $(a_i)_{i=1}^k$ be any real numbers such that $\forall i\ a_i \geq 1$. Since the function $e^{\sqrt{x}}$ is convex on $[1,\infty)$, by Jensen inequality:

$$e^{\sqrt{\frac{1}{k}\sum_{i=1}^k (a_i)^2}} \leq \frac{1}{k}\sum_{i=1}^{k} e^{a_i}. \tag{3}$$

Replace X_i^* by $max(X_i^*, 1)$, and combine inequalities (2) and (3):

$$\mathbb{E}e^{\sqrt{\frac{1}{k}\sum_{i=1}^k (X_i^*)^2}} \leq \mathbb{E}\frac{1}{k}\sum_{i=1}^{k} e^{X_i^*+1} \leq C'\frac{n}{k}$$

Another application of Jensen inequality ($\mathbb{E}\log X \leq \log \mathbb{E}X$) yields:

$$\mathbb{E}\sqrt{\frac{1}{k}\sum_{i=1}^{k}(X_i^*)^2} \leq \log C'\frac{n}{k}$$

which concludes the proof. $\square$

Remark: If X_i are ψ_2 variables, then it can be simply verified that:

$$\mathbb{E}\sqrt{\frac{1}{k}\sum_{i=1}^{k}(X_i^*)^2} \leq c\sqrt{\log\frac{2n}{k}}.$$

Proof of Proposition 3.1: Let $\|x\| = t\|x\|'_{\frac{\sqrt{n}}{t}}$, the dual norm of K_t. Take two random bases $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$. The symmetrized norm $|||\cdot|||$ is:

$$|||x||| = \mathbb{E}_{\varepsilon,\varepsilon'}\left\|\sum_{j,k}\varepsilon_j\varepsilon'_k\langle x, v_j\rangle\langle v_j, u_k\rangle u_k\right\|$$

where $\varepsilon, \varepsilon'$ are independent and uniformly distributed in $\{\pm 1\}^n$. For $1 \leq i \leq n$ and $\varepsilon, \varepsilon' \in \{\pm 1\}^n$ define:

$$\phi_x^i(\varepsilon, \varepsilon') = \left|\sum_{j,k}\varepsilon_j\varepsilon'_k\langle x, v_j\rangle\langle v_j, u_k\rangle\langle u_k, e_i\rangle\right|.$$

Then:

$$|||x||| = t\mathbb{E}_{\varepsilon,\varepsilon'}\left[\|\phi_x^1(\varepsilon,\varepsilon'),..,\phi_x^n(\varepsilon,\varepsilon')\|'_{\frac{\sqrt{n}}{t}}\right].$$

By Lemma 3.3,

$$|||x||| \leq \sqrt{2}t\mathbb{E}_{\varepsilon,\varepsilon'}\sqrt{\sum_{i=1}^{\lfloor\frac{n}{t^2}\rfloor+1}\phi_x^{i*}(\varepsilon,\varepsilon')^2}.$$

The following lemma, estimating the ψ_1 norm of those variables, will be proved later.

Lemma 3.5

$$\|\phi_x^i\|_{\psi_1} < \frac{c_3}{\sqrt{n}}|x|$$

with probability greater than $1 - e^{-c\sqrt{n}}$ *of choosing the orthogonal bases.*

Lemma 3.4 may now be used (for $k = \lfloor\frac{n}{t^2}\rfloor + 1$). It shows that:

$$|||x||| \leq \sqrt{2}t\left(\frac{c_3}{\sqrt{n}}|x|\right)\cdot c_2\left(\frac{\sqrt{n}}{t}+1\right)\log\frac{2n}{\frac{n}{t^2}} \leq c|x|\log t$$

with probability greater than $1 - ne^{-c\sqrt{n}}$ of choosing the bases. □

Before turning to the proof of lemma 3.5, prove another lemma, which is believed to be known to experts:

Lemma 3.6 *Let* $x = (x_1,..,x_n)$ *and* $y = (y_1,..,y_n)$ *be two random independent vectors in* S^{n-1}. *Then with probability greater than* $1 - e^{-c\sqrt{n}}$,

$$\sum_i x_i^2y_i^2 \leq \frac{c}{n}.$$

Proof of Lemma 3.6: Let $\{\gamma_i\}_{i=1}^n$ and $\{\eta_i\}_{i=1}^n$ be independent standard Gaussian variables. Since the measure on the sphere is the radial projection of the standard Gaussian measure in $\mathbb{R}^n$, then:

$$Prob\left\{\sum_i x_i^2y_i^2 > t\right\} = Prob\left\{\frac{1}{\sum_j\gamma_j^2\sum_j\eta_j^2}\sum_i\gamma_i^2\eta_i^2 > t\right\}.$$

To prove the lemma, it is sufficient to bound from below $\sum_j\gamma_j^2\sum_j\eta_j^2$ and bound from above $\sum_i\gamma_i^2\eta_i^2$. Begin with the second expression. Note that $\gamma_i^2\eta_i^2$ is a $\psi_{\frac{1}{2}}$ variable:

$$\left(\mathbb{E}\gamma_i^{2p}\eta_i^{2p}\right)^{\frac{1}{p}} = \left(\mathbb{E}\gamma_i^{2p}\right)^{\frac{4}{2p}} \leq (c\sqrt{p})^4 = c^4p^{\frac{1}{\alpha}}$$

for $\alpha = \frac{1}{2}$. Therefore, $\sum_i \gamma_i^2 \eta_i^2$ is a sum of independent copies of a $\psi_{\frac{1}{2}}$ random variable. By a deviation inequality for sums of i.i.d ψ_α random variables (see [Schm]),

$$Prob\left\{\sum_i \gamma_i^2 \eta_i^2 > cn\right\} < \exp(-c'\sqrt{n}).$$

The fact that $Prob\{\sum_j \gamma_j^2 < \frac{n}{2}\} < e^{-cn}$ follows from Large Deviations technique (e.g. Cramér's Theorem, [Var]). To conclude, with probability greater than $1 - e^{-c\sqrt{n}}$,

$$\frac{1}{\sum_j \gamma_j^2 \sum_j \eta_j^2} \sum_i \gamma_i^2 \eta_i^2 < \frac{cn}{\frac{n}{2} \cdot \frac{n}{2}} = \frac{c'}{n}$$

$\square$

Proof of Lemma 3.5: Let $\phi_x^i(\varepsilon, \varepsilon') = |\sum_{j,k} \varepsilon_j \varepsilon'_k \langle x, v_j\rangle \langle v_j, u_k\rangle \langle u_k, e_i\rangle|$. This random variable is a particular case of a Rademacher Chaos variable. It is well known (e.g. see [LT]), that a ψ_1 estimate holds true for such variables:

$$\|\phi_x^i\|_{\psi_1} \leq c\|\phi_x^i\|_2 = c\sqrt{\sum_j \langle x, v_j\rangle^2 \sum_k \langle v_j, u_k\rangle^2 \langle u_k, e_i\rangle^2}.$$

It is sufficient to show that the inequality $\sum_k \langle v_j, u_k\rangle^2 \langle u_k, e_i\rangle^2 \leq \frac{c}{n}$ holds with high probability, since in that case, with the same probability:

$$\sqrt{\sum_j \langle x, v_j\rangle^2 \sum_k \langle v_j, u_k\rangle^2 \langle u_k, e_i\rangle^2} \leq \frac{\sqrt{c}}{\sqrt{n}} \sqrt{\sum_j \langle x, v_j\rangle^2} = \frac{\sqrt{c}}{\sqrt{n}}|x|.$$

The fact that $\sum_k \langle v_j, u_k\rangle^2 \langle u_k, e_i\rangle^2 \leq \frac{c}{n}$ holds with probability greater than $1 - e^{-c\sqrt{n}}$ follows directly from Lemma 3.6: Take $U \in O(n)$ such that $U(u_k) = e_k$. U is distributed uniformly over $O(n)$.

$$\sum_k \langle v_j, u_k\rangle^2 \langle u_k, e_i\rangle^2 = \sum_k \langle Uv_j, e_k\rangle^2 \langle e_k, Ue_i\rangle^2$$

and since Uv_j and Ue_i are independent and distributed uniformly over the sphere - the claim is proven, by Lemma 3.6. $\square$

Remark: Proposition 3.1 may be adapted to suit Walsh-type symmetrizations. If $K_t = \sqrt{n}B(l_1^n) \bigcap tB(l_2^n)$ is symmetrized with respect to Walsh vectors $w_1, .., w_{n-1}$, a slightly better conclusion than that in Proposition 3.1 is obtained; In this setting, it is true that:

$$\tilde{K}_t \subset C\sqrt{\log t}B(l_2^n).$$

The differences between the proofs are minor. Lemma 3.5 becomes much easier as it follows immediately from Khinchine inequality, even with a ψ_2 estimate rather than ψ_1. To take advantage of this improvement, use the remark after Lemma 3.4, to obtain the better conclusion.

Re-iteration of this proposition, where each iteration uses a Walsh basis with respect to the previous, would result in a rapid decay of the body's diameter. After $\log^* n$ iterations, a body whose $\frac{diam(\tilde{K})}{M^*(\tilde{K})}$ ratio is bounded by a universal constant is obtained. Note that this specific choice of symmetrizations decreases the diameter of all possible convex bodies in $\mathbb{R}^n$, to be a constant times their mean width. Of course, once the $\frac{diam(K)}{M^*(K)}$ ratio is bounded, cn random independent Minkowski symmetrizations suffice for transforming the body into an approximate Euclidean ball.

4 Third step: Concentration techniques

Take any convex body K in $\mathbb{R}^n$. According to Corollary 3.2, from the previous steps (which consist of no more than $4n$ symmetrizations) a new body is obtained, with $M^* = 1$ and with diameter less than $c \log\log n$. As before, the third step involves symmetrizing with respect to two random orthogonal bases. A total of $2n$ symmetrizations will make the body very close to Euclidean.

Let $\|\cdot\|$ be the dual norm of the body obtained after the previous steps. Since $M^*(K) = 1$, then $M(\|\cdot\|) \equiv \int_{S^{n-1}} \|x\| d\sigma(x) = 1$, and $b(\|\cdot\|) \equiv \sup_{x \in S^{n-1}} \|x\| \leq c \log\log n$. Let $\{u_i\}_{i=1}^n, \{v_i\}_{i=1}^n$ be random orthogonal bases and consider for $x \in \mathbb{R}^n$ the set:

$$\mathcal{F}(x) = \left\{ \sum_{i,j} \varepsilon_i \varepsilon'_j \langle x, v_i \rangle \langle v_i, u_j \rangle u_j : \varepsilon, \varepsilon' \in \{\pm 1\}^n \right\}.$$

The symmetrized norm $|||\cdot|||$ satisfies $|||x||| = \frac{1}{4^n} \sum_{v \in \mathcal{F}(x)} \|v\|$. This section will prove that for the new norm:

$$\forall x \in \mathbb{R}^n \qquad \frac{1}{2}|x| \leq |||x||| \leq 2|x|$$

with large probability of choosing $\{u_i\}_{i=1}^n, \{v_i\}_{i=1}^n \in O(n)$. In fact, a somewhat stronger theorem is proved, where instead of $\frac{1}{2}$ and 2, better estimates are given.

Useful remark: Let $|||x||| = \frac{1}{4^n} \sum_{v \in \mathcal{F}(x)} \|v\|$ be the norm obtained after symmetrizing with respect to $\{u_i\}$ and $\{v_i\}$. Take $U \in O(n)$, and let $|||\cdot|||_U$ be the norm obtained after symmetrizing with respect to $\{Uu_i\}$ and $\{Uv_i\}$. Then $|||Ux|||_U = \frac{1}{4^n} \sum_{v \in \mathcal{F}(x)} \|Uv\|$. Therefore, due to the rotation invariance of the measure μ_n in $O(n)$, it is possible to fix an orthonormal system $\{u_i\}$, and prove the following:

Theorem 4.1 *With the above definitions,*

$$\forall x \in S^{n-1} \quad (1 - c\frac{(\log\log n)^{\frac{3}{2}}}{\sqrt{\log n}}) \le |||x|||_U \le (1 + c\frac{(\log\log n)^{\frac{3}{2}}}{\sqrt{\log n}})$$

with probability greater than $1 - e^{-Cn}$ of choosing $U \in O(n)$, and probability greater than $1 - \frac{1}{n^{10}}$ of choosing $\{v_i\}$.

The proof shall use three lemmas:

Lemma 4.2 $\forall x, y \in S^{n-1} \quad \frac{1}{4^n}\sum_{v\in\mathcal{F}(x)} |\langle v, y\rangle| \le 2c_1\frac{\sqrt{\log n}}{\sqrt{n}}$

for any $\{u_i\}$, with probability of choosing $\{v_i\}$ greater than $1 - \frac{1}{n^{10}}$.

Proof: According to Lemma 2.3, with probability greater than $1 - \frac{1}{n^{10}}$, for all i, j the inequality $|\langle v_i, u_j\rangle| \le c_1\frac{\sqrt{\log n}}{\sqrt{n}}$ holds. Thus:

$$\frac{1}{4^n}\sum_{v\in\mathcal{F}(x)} |\langle v, y\rangle| = \mathbb{E}_{\varepsilon,\varepsilon'}\left|\sum_{i,j} \varepsilon_i\varepsilon'_j\langle x, v_i\rangle\langle v_i, u_j\rangle\langle u_j, y\rangle\right|$$

$$\le \sqrt{\mathbb{E}_{\varepsilon,\varepsilon'}\left[\sum_{i,j} \varepsilon_i\varepsilon'_j\langle x, v_i\rangle\langle v_i, u_j\rangle\langle u_j, y\rangle\right]^2}$$

$$= \sqrt{\sum_{i,j}\langle x, v_i\rangle^2\langle v_i, u_j\rangle^2\langle u_j, y\rangle^2} \le c_1\frac{\sqrt{\log n}}{\sqrt{n}}$$

since x and y are sphere vectors. $\square$

The next lemma is copied from [BLM1], where it is proven.

Lemma 4.3 *Assume that $\{w_\alpha\}_{\alpha\in A} \subset S^{n-1}$, and for some $\delta > 0$,*

$$\sup_{y\in S^{n-1}} \frac{1}{\#A}\sum_{\alpha\in A} |\langle w_\alpha, y\rangle| \le \delta.$$

Let $0 < \lambda < 1$ and let $k \le n$ be an integer. Then there exist disjoint families $\mathcal{F}_\beta = \{\beta_i\}_{i=1}^k \subset A$, $\beta \in B$, so that $\#(\cup_{\beta\in B}\mathcal{F}_\beta) > (1-\lambda)\#A - k$ and so that for every $\beta \in B$ there is an orthonormal set of vectors $\{v_{\beta_i}\}_{i=1}^k$ satisfying

$$|v_{\beta_i} - w_{\beta_i}| \le \frac{\delta 4^k}{\lambda}.$$

Concentration on the orthogonal group shall be used in the proof of Theorem 4.1, due to [GrM1] (see [MS], page 29):

Lemma 4.4 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ such that $\|x\| \leq b|x|$ $\forall x \in S^{n-1}$. Let $k \leq n$ be a positive integer, and $\{x_i\}_{i=1}^k$ be orthonormal vectors. Denote $M = \int_{S^{n-1}} \|x\| d\sigma_n(x)$. Then:*

$$\mu_n \left\{ U \in O(n) \ ; \ \left| \frac{1}{k} \sum_{i=1}^{k} \|Ux_i\| - M \right| \geq \varepsilon \right\} \leq exp\left(-c_4 \frac{\varepsilon^2 nk}{b^2} \right).$$

Proof of Theorem 4.1: Fix $x \in S^{n-1}$. Let $\varepsilon = c_5 \frac{(\log\log n)^{\frac{3}{2}}}{\sqrt{\log n}}$, $\lambda = \frac{\varepsilon}{b}$ and $k = \frac{\log n}{10}$. According to Lemma 4.2, the collection of vectors $\mathcal{F}(x)$ satisfies the requirement of Lemma 4.3 for $\delta = 2c_1 \frac{\sqrt{\log n}}{\sqrt{n}}$, with large probability of choosing $\{v_i\}$ (and of course independently of U). As a result, $\mathcal{F}(x)$ can be decomposed into disjoint almost orthogonal families $\{\mathcal{F}_\beta\}_{\beta \in B}$, which cover all but a λ fraction of $\mathcal{F}(x)$.
From Lemma 4.3, for each family $\mathcal{F} = \{x_1, .., x_k\} \subset \mathcal{F}(x)$, there exist orthonormal vectors $\{t_1, .., t_k\}$ such that $|t_i - x_i| \leq \frac{\delta 4^k}{\lambda}$. Since $\{t_i\}_{i=1}^k$ are orthonormal, then by Lemma 4.4:

$$\mu_n \left\{ U \in O(n) \ ; \ \left| \frac{1}{k} \sum_{i=1}^{k} \|Ut_i\| - 1 \right| \geq \varepsilon \right\} \leq \exp\left(-c_4 \frac{nk\varepsilon^2}{b^2} \right)$$

where $b = \sup_{x \in S^{n-1}} \|x\|$. Since $\|Ut_i - Ux_i\| \leq b\frac{\delta 4^k}{\lambda}$, then:

$$\left| \frac{1}{k} \sum_{i=1}^{k} \|Ux_i\| - 1 \right| \leq \varepsilon + b\frac{\delta 4^k}{\lambda} \tag{4}$$

with probability (of choosing $U \in O(n)$) of at least $1 - \exp(-c_4 \frac{nk\varepsilon^2}{b^2})$.

This holds for a single family $\mathcal{F}$. The number of families is less than 4^n, so inequality (4) holds for all families $\{\mathcal{F}_\beta\}_{\beta \in B}$ together, with probability greater than $1 - 4^n \exp(-c_4 \frac{nk\varepsilon^2}{b^2}) = 1 - \exp(-c_4 n(\frac{k\varepsilon^2}{b^2} - \log 4))$.
There still remains a λ fraction of the collection $\mathcal{F}(x)$, not covered by the disjoint families $\{\mathcal{F}_\beta\}_{\beta \in B}$. Their contribution to the relevant expression, which is $|\frac{1}{4^n} \sum_{v \in \mathcal{F}(x)} \|Uv\| - 1|$, can be bounded by λb. Hence:

$$| \ |||Ux|||_U - 1 \ | = \left| \frac{1}{4^n} \sum_{v \in \mathcal{F}(x)} \|Uv\| - 1 \right| \leq$$

$$\leq \frac{k}{4^n} \sum_{\beta \in B} \left| \frac{1}{k} \sum_{v \in \mathcal{F}_\beta} (\|Uv\| - 1) \right| + \frac{\lambda 4^n}{4^n} b \leq (1-\lambda)(\varepsilon + b\frac{\delta 4^k}{\lambda}) + \lambda b.$$

In summary: choose $\{v_i\}$ by random. With probability of at least $1 - \frac{1}{n^{10}}$, the following holds: the set of $U \in O(n)$ for which

$$| \ |||Ux|||_U - 1 \ | \leq \varepsilon + b\frac{\delta 4^k}{\lambda} + \lambda b \tag{5}$$

has measure of at least $1-\exp(-c_4 n(\frac{k\varepsilon^2}{b^2}-\log 4))$. From substituting the values of the variables k, ε, λ, it follows that $\lambda b \le \varepsilon$, and also $b\frac{\delta 4^k}{\lambda} < \varepsilon$, for $n > c_6$. Therefore, the quantity discussed in (5) is less than 3ε, for $n > c_6$.
The inequality $\left|\, |||Ux|||_U - 1 \,\right| \le 3\varepsilon$ holds with probability (with respect to U) of at least $1-\exp(-c_4 n(\frac{k\varepsilon^2}{b^2}-\log 4))$. With a suitable universal constant c_5 this probability would be greater than $1-\exp(-10n\log\log n)$.

This analysis considered a fixed $x \in S^{n-1}$. Now, take an ε-net on the sphere denoted by $\mathcal{N}$. There exists such a net with $\#\mathcal{N} \le (\frac{4}{\varepsilon})^n$ (e.g. [MS] page 7). For each $x \in \mathcal{N}$, $\left|\, |||Ux|||_U - 1 \,\right| \le 3\varepsilon$ with probability greater than $1-\exp(-10n\log\log n)$. Since $(\frac{4}{\varepsilon})^n \le \exp(\log\log n)$ for $n > c_6$, then $\left|\, |||Ux|||_U - 1 \,\right| \le 3\varepsilon$ holds for all $x \in \mathcal{N}$, with probability that is more than exponentially close to 1.

For a general $x \in S^{n-1}$, write $x = \sum_{i=0}^{\infty} \theta_i x_i$, where $Ux_i \in \mathcal{N}$, and $\theta_0 = 1, 0 \le \theta_i \le \varepsilon^i$. Then $|||x|||_U \le \sum_{i=0}^{\infty}(1+3\varepsilon)\varepsilon^i = \frac{1+3\varepsilon}{1-\varepsilon} \le 1+5\varepsilon$. Finally, $|||x|||_U \ge |||x_0|||_U - \sum_{i=1}^{\infty} |\theta_i| \cdot |||x_i|||_U \ge 1-5\varepsilon$.

Hence, with slightly better than exponentially close to 1 probability, the new norm $|||\cdot|||$ satisfies

$$\forall x \in \mathbb{R}^n \quad (1-\varepsilon)|x| \le |||x||| \le (1+\varepsilon)|x|$$

where $\varepsilon < c\frac{(\log\log n)^{\frac{3}{2}}}{\sqrt{\log n}}$, and the theorem is proven, for $n > c_6$. $\square$

Remark: Using a Walsh-type symmetrization in the second step, the theorem can be proven with $\varepsilon < c\frac{\log\log n}{\sqrt{\log n}}$, an improvement of a mere $\sqrt{\log\log n}$ factor.

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Isomorphic Steiner symmetrization[3]

Abstract. Here we prove that there exist $3n$ Steiner symmetrizations that transform any convex set $K \subset \mathbb{R}^n$ into an isomorphic Euclidean ball; i.e. if $vol(K) = vol(D_n)$ where D_n is the standard Euclidean unit ball, then K can be transformed into a body $\tilde{K}$ such that $c_1 D_n \subset \tilde{K} \subset c_2 D_n$, where c_1, c_2 are numerical constants. Moreover, for any $c > 2$, cn symmetrizations are also enough.

1 Introduction

Fix a Euclidean structure (i.e. scalar product) in $\mathbb{R}^n$. Let $K \subset \mathbb{R}^n$ be any measurable set, and let $H = \{x \in \mathbb{R}^n; \langle x, h\rangle = 0\}$ be a hyperplane through the origin in $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ there exists a unique decomposition $x = y + th$ where $y \in H, t \in \mathbb{R}$, so we can refer to (y, t) as coordinates in $\mathbb{R}^n$. Define "Steiner symmetrization of K with respect to H" as the body:

$$S_H(K) = \left\{ \ (x,t) \ ; \ K \cap (x + \mathbb{R}h) \neq \emptyset \ , \ |t| \leq \frac{1}{2} Meas\{K \cap (x + \mathbb{R}h)\} \ \right\}$$

where $Meas$ is the one dimensional Lebesgue measure in the line $x + \mathbb{R}h$, normalized such that the measure of a segment is equal to its length. Steiner symmetrization preserves the n dimensional volume of a set and transforms convex sets to convex sets (the so called "Brunn principle" [Br1], [Br2]).

Steiner symmetrizations were invented by Steiner [St] to prove the isoperimetric inequality. Throughout the last 160 years, Steiner symmetrizations have become a major tool for proving various geometric inequalities. Some samples are [Mac], [MeP1], [BZ]. It is clear that consecutive Steiner symmetrizations make the body closer to a Euclidean ball, in some sense. Indeed, in [CS] it is proven that for every K there exists a sequence of symmetrizations of K that converges, in the Hausdorff metric, to a Euclidean ball. In [Man] it is proven that even random symmetrizations are suitable, and convergence occurs with probability one. However, it was believed that many symmetrizations are necessary for that convergence. An estimate of the order of $n^{n/2}$ appears in [H].

Use of concentration phenomenon technique brought a tremendous improvement: a reduction to $cn \log n$ was achieved in [BLM2]. More precisely, here we prove:

[3] This chapter corresponds to the paper [KM1].

Theorem 1.1 *There exist universal constants $c, c_1, c_2 > 0$ such that for every convex body $K \subset \mathbb{R}^n$ with the same volume as the standard Euclidean ball, $D_n = \{x \in \mathbb{R}^n; |x| \leq 1\}$, there exist $cn \log n$ Steiner symmetrizations that transform the body K into $\tilde{K}$ such that:*

$$c_1 D_n \subset \tilde{K} \subset c_2 D_n.$$

In this note we sharpen this estimate. Rather than $O(n \log n)$, we put cn as the upper estimate for the number of required symmetrizations, where the constant "c" can be taken to be arbitrarily close to the value 2. In addition, we prove that for every $\varepsilon > 0$, no more than $\lfloor (1+\varepsilon)n \rfloor$ symmetrizations are necessary in order to transform an arbitrary body into an isomorphic ellipsoid. The latter estimate is asymptotically optimal. The following theorems are proven here:

Theorem 1.2 *For every $\varepsilon > 0$, there exist constants $c_1(\varepsilon), c_2(\varepsilon) > 0$ such that for every convex body $K \subset \mathbb{R}^n$, there exist an ellipsoid $\mathcal{E}$ and $\lfloor (1+\varepsilon)n \rfloor$ Steiner symmetrizations that transform the body K into $\tilde{K}$ such that:*

$$c_1(\varepsilon)\mathcal{E} \subset \tilde{K} \subset c_2(\varepsilon)\mathcal{E}.$$

Theorem 1.3 *For every $\varepsilon > 0$, there exist constants $c_1(\varepsilon), c_2(\varepsilon) > 0$ such that for every convex body $K \subset \mathbb{R}^n$ with the same volume as the standard Euclidean ball, $D_n = \{x \in \mathbb{R}^n; |x| \leq 1\}$, there exist $\lfloor (2+\varepsilon)n \rfloor$ Steiner symmetrizations that transform the body K into $\tilde{K}$ such that:*

$$c_1(\varepsilon) D_n \subset \tilde{K} \subset c_2(\varepsilon) D_n.$$

We would like to point out that in addition to Theorem 1.2, for every convex body K there exists a "position" - meaning an affine image of K - such that regarding this image, $\lfloor (1+\varepsilon)n \rfloor$ symmetrizations are already enough to obtain an isomorphic Euclidean ball, for any $\varepsilon > 0$. Hence, the larger amount of symmetrizations may be necessary only due to a "wrong" position but not because of the geometry of the body itself.

Our results are of "isomorphic" nature, rather than "almost isometric". It is not proven that the symmetrizations make the body arbitrarily close to a Euclidean ball (compare with the situation in [Kl2], Chapter 1 here); We prove that the body becomes close to a Euclidean ball, up to some absolute constant, thus is "uniformly isomorphic" to a Euclidean ball. We are unaware of any good estimates regarding "almost-isometric" symmetrization (less than exponential in the dimension).

The text is organized as follows: first, we prove that the number of symmetrizations required is proportional to the dimension. This proof reduces the general problem to symmetrizing two specific bodies - the cube and the cross polytope. While the cross polytope is one of the most difficult bodies to symmetrize, the cube has a short symmetrization process, consisting only of $\lfloor \varepsilon n \rfloor$ symmetrizations. This method, however, does not lead to the best constants known to us. We then present proofs of the sharper results, using Milman's "quotient of subspace theorem". The reason for presenting the non-optimal proof, is that we think it is more accessible and may be interesting in itself. In addition, the proof that leads to the best constant is simplified when using the conclusion of the other proof as a lemma.

By c, C, c' etc. we denote universal constants, whose value may not be equal at different appearances in the text. A "body" in $\mathbb{R}^n$ is any compact and convex set with a non empty interior. The term "random" is used freely throughout the text, and we seldom bother to define the precise probability measure. The reason is that our standard manifolds (such as $S^n, O(n), G_{n,k}$) obey a canonical rotation invariant (Haar) probability measure, and all of our "random choices" are carried out with respect to these measures.

2 Preliminary facts

In this section we present some specific elementary properties of Steiner symmetrization, together with general known facts to be used throughout the proof. Standard references for fundamental properties of Steiner symmetrization are the books [BZ], [BF]. See also [Schn] for general background on classical convexity theory.

Let $K \subset \mathbb{R}^n$ be a convex body and let $H = \{x \in \mathbb{R}^n; \langle x, h \rangle = 0\}$ be a hyperplane. Steiner symmetrization is denoted here by $S_h(K)$ or $S_H(K)$. When we say "Steiner symmetrization with respect to a vector", we mean with respect to the hyperplane orthogonal to that vector. Clearly,

$$H \cap K \subset Proj_H(K) = S_H(K) \cap H = Proj_H(S_H(K))$$

where for a subspace E, we define $Proj_E$ as the orthogonal projection onto E in $\mathbb{R}^n$. Thus, if $E \subset H$ is a subspace, then $Proj_E(K) = Proj_E(S_h(K))$. Hence, by simple induction we proved:

Lemma 2.1 *Let $H_1, .., H_k$ be hyperplanes, and let $E \subset \bigcap_{i=1}^{k} H_i$ be a subspace. Denote $K' = S_{H_1}(..S_{H_k}(K)..)$. Then:*

$$Proj_E(K) = Proj_E(K'),$$

$$E \cap K \;\subset\; E \cap K'.$$

Useful corollary of Lemma 2.1: If $Proj_E(K) = E \cap K$, and symmetrizations are carried out with respect to vectors in $E^{\perp}$, then both $E \cap K$ and $Proj_E(K)$ remain unchanged.

Another elementary property of Steiner symmetrization is that it works independently in "fibers": The symmetrization acts independently in any affine translation of any subspace that contains h. Formally, as it follows from definitions,

Lemma 2.2 *If $h \in F$ where F is a subspace, then for every $x \in \mathbb{R}^n$,*

$$S_h(K) \cap (x + F) \;=\; S_h(K \cap (x + F)).$$

Therefore, if $Proj_E(K) = E \cap K$, then any symmetrization with respect to any vector in E would not change this. A second conclusion of this lemma is that if K is centrally symmetric (i.e. $K = -K$), then also $S_h(K) = -S_h(K)$. For a vector x define π_x as the reflection with respect to the hyperplane orthogonal to x. We say that K is symmetric with respect to x if $\pi_x(K) = K$. A third corollary of Lemma 2.2 is that if K is symmetric with respect to x, symmetrizations with respect to vectors in $x^{\perp}$ would preserve this symmetry. In particular,

Lemma 2.3 *Let $K \subset \mathbb{R}^n$ be any set, and let $\{e_1, .., e_n\}$ be an orthonormal basis. Then K can be transformed into an unconditional set (i.e. symmetric with respect to the vectors $\{e_i\}$) using n Steiner symmetrizations.*

Generally, when we say that K is an unconditional body, it should be understood that there exists an orthonormal basis, such that K is symmetric with respect to its elements. The next claim is about orthogonal symmetrization. If $H_1, .., H_k$ are orthogonal hyperplanes (i.e. their normals are mutually orthogonal), then the corresponding orthogonal projections $Proj_{H_1}, .., Proj_{H_k}$ commute. Hence:

$$S_{H_1,..,H_k}(K) \supset Proj_{H_1 \cap .. \cap H_k}(K)$$

where $S_{H_1,..,H_k}(K) = S_{H_1}(..S_{H_k}(K)..)$. Combining this with Lemma 2.1, we arrive at the following:

Lemma 2.4 *Let $H_1, .., H_k$ be orthogonal hyperplanes, and denote $E = \bigcap_{i=1}^{k} H_i$ and $K' = S_{H_1,..,H_k}(K)$. Then:*

$$Proj_E(K) = Proj_E(K') = K' \cap E.$$

Roughly speaking, Steiner symmetrizations can transform projections into sections. The next lemma is an addition to Lemma 2.4, and it sometimes represents the opposite idea (obtaining "good" projections, when the body already possesses "good" sections).

Lemma 2.5 *Let $K \subset \mathbb{R}^n$ be a convex body which is centrally symmetric, and let $H_1, .., H_k$ be orthogonal hyperplanes, $E = \bigcap_{i=1}^{k} H_i$ and $K' = S_{H_1,..,H_k}(K)$, as before. Then:*

$$Proj_{E^\perp}(K') = K' \cap E^\perp.$$

Proof: From the discussion following Lemma 2.2, K' is symmetric with respect to the vectors $h_1, .., h_k$ (where $H_i = h_i^\perp$). If $x \in K'$, then also $\pi_{h_1}..\pi_{h_k}(x) \in K'$. By convexity and central symmetry,

$$Proj_{E^\perp}(x) = \frac{x - \pi_{H_1}..\pi_{H_k}(x)}{2} \in K'$$

hence $Proj_{E^\perp}(K') \subset K' \cap E^\perp$. □

Lemma 2.6 *Steiner symmetrization transforms an ellipsoid into an ellipsoid. In addition, for any ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ there exists an orthonormal basis $\{v_1, .., v_n\}$, such that symmetrizations of $\mathcal{E}$ with respect to the vectors $v_1, .., v_n$ (in that order) transform it into a Euclidean ball.*

Proof: The proof is a minor adjustment to the proof of Lemma 1.3 in [BLM2]. According to that lemma, ellipsoids are transformed into ellipsoids under Steiner symmetrization. We prove Lemma 2.6 by induction. Let $\mathcal{E} = \{x \in \mathbb{R}^n; \sum_{i=1}^{n} x_i^2/a_i^2 \leq 1\}$ be an ellipsoid. Let $r = (\prod a_i)^{1/n}$. There exists a direction $u \in S^{n-1}$ such that $\mathcal{E} \cap \mathbb{R}u$ has length $2r$. Let $\mathcal{E}_1$ be the Steiner symmetrization of $\mathcal{E}$ with respect to u. Then $\mathcal{E}_1$ is an ellipsoid, which has a principal axis in the direction of u of length $2r$. Any symmetrizations with respect to vectors inside $u^\perp$ could not change this fact. By the induction hypothesis, the ellipsoid $\mathcal{E}_1 \cap u^\perp$ can be symmetrized into a Euclidean ball of dimension $n-1$ and radius r, using symmetrizations with respect to an orthonormal basis of $u^\perp$. Apply exactly the same symmetrizations to the ellipsoid $\mathcal{E}_1$. The proof follows by Lemma 2.2. □

Before moving on, we would like to define a few relevant notions. For two sets $K, T \subset \mathbb{R}^n$ we denote their "geometric distance" as:

$$d(K,T) = \inf\{ab \ ; \ \exists v \in \mathbb{R}^n \ , \ \frac{1}{b}K \subset T + v \subset aK\}.$$

Another notion of distance is the usual Banach-Mazur distance (see e.g. [MS]) defined as:

$$d_{BM}(K,T) = \inf\{d(uK,T) \ ; \ u \in GL(n)\}$$

where $GL(n)$ is the set of invertible matrices acting on $\mathbb{R}^n$. Also, for $K \subset \mathbb{R}^n$ we denote its volume-radius as

$$v.r.(K) = \left(\frac{vol(K)}{vol(D_n)}\right)^{\frac{1}{n}}.$$

Although the next theorem is not explicitly stated in [BLM2], it can be easily deduced from their proof:

Theorem 2.7 *There exist constants $\bar{c}_1, \bar{c}_2, \bar{c} > 0$ such that for every convex body $K \subset \mathbb{R}^n$, there exist $\bar{c}n \log 2d_{BM}(K, D_n)$ symmetrizations that transform K into a body $\tilde{K}$ with:*

$$\bar{c}_1 \ v.r.(K)D_n \subset \tilde{K} \subset \bar{c}_2 \ v.r.(K)D_n.$$

3 Isotropic position

In this section we use notions and properties related to isotropic positions. For a comprehensive discussion of this topic, see [MP1]. With any body K, $vol(K) = 1$, associate the Binet ellipsoid $\mathcal{B}(K)$, which is the unit ball of the Binet norm:

$$\|x\|^2 = \int_K \langle x, y\rangle^2 dy.$$

If $\mathcal{B}(K)$ is homothetic to D_n, the standard Euclidean ball, we say that K is in "isotropic position". For a centrally symmetric ellipsoid $\mathcal{E}$, denote by $\|\cdot\|_{\mathcal{E}}$ the unique norm whose unit ball coincides with $\mathcal{E}$. Define for any K (with $vol(K) = 1$) a number L_K called "the isotropic constant of K":

$$L_K^2 = \frac{1}{n}\min_{\mathcal{E}} \int_K \|x\|_{\mathcal{E}}^2 dx \tag{1}$$

where the minimum runs over all ellipsoids with the same volume as D_n. If K is in isotropic position, then the minimum is achieved for $\mathcal{E} = D_n$. It is computed in [MP1] that for a centrally symmetric K, or if the origin is the barycenter of K,

$$\left(\frac{vol(\mathcal{B}(K))}{vol(D_n)}\right)^{\frac{1}{n}} = \frac{1}{L_K}. \tag{2}$$

The following lemma is known, and can be traced back to Blaschke (see [Bla2]). We provide its short proof for convenience.

Lemma 3.1 *The parameter L_K decreases under Steiner symmetrization. If K is a centrally symmetric body then $vol(\mathcal{B}(K))$ increases.*

Proof: Let $\mathcal{E}$ be the ellipsoid where the minimum in (1) is achieved, and let H be any hyperplane through the origin in $\mathbb{R}^n$. For any affine line l orthogonal to H, and for every $t > 0$,

$$Meas(l \cap K \cap t\mathcal{E}) \leq Meas(l \cap S_H(K) \cap S_H(t\mathcal{E}))$$

and by the definition of $\mathcal{E}$ and Fubini,

$$nL_K^2 = \int_K \|x\|_{\mathcal{E}}^2 dx \geq \int_{S_H(K)} \|x\|_{S_H(\mathcal{E})}^2 dx \geq nL_{S_H(K)}^2.$$

where the latter inequality follows from (1), since $S_H(\mathcal{E})$ has the same volume as D_n. The other quantity in discussion is monotone in L_K. $\square$

For $\mathcal{E}$ a centrally symmetric ellipsoid and H a hyperplane, denote by $S_H^\circ(\mathcal{E})$ the unique ellipsoid that satisfies three conditions:

(i) $Vol(S_H^\circ(\mathcal{E})) = Vol(\mathcal{E})$.

(ii) $S_H^\circ(\mathcal{E}) \cap H = \mathcal{E} \cap H$.

(iii) $S_H^\circ(\mathcal{E})$ is symmetric with respect to H.

It is easily verified that $(S_H(\mathcal{E}^\circ))^\circ = S_H^\circ(\mathcal{E})$, where the polar (or dual) body of K is defined as:

$$K^\circ = \{x; \forall y \in K \ \langle x, y\rangle \leq 1\}.$$

Lemma 3.2 *Let K be a centrally symmetric body and H a hyperplane. Then:*

$$S_H^\circ(\mathcal{B}(K)) \subset \mathcal{B}(S_H(K)).$$

Proof: Let $x \in H$ be any vector. We claim that:

$$\int_K \langle x, y\rangle^2 dy = \int_{S_H(K)} \langle x, y\rangle^2 dy.$$

Indeed, this follows directly by decomposing $\mathbb{R}^n = H \oplus H^\perp$, and using Fubini theorem. The Binet norms of K and $S_H(K)$ are equal in the hyperplane H, therefore $\mathcal{B}(S_H(K)) \cap H = \mathcal{B}(K) \cap H$. In addition, $S_H(K)$ is symmetric with respect to H. Hence,

$$\int_{S_H(K)} \langle x, y\rangle^2 dy = \int_{S_H(K)} \langle x, \pi_H y\rangle^2 dy = \int_{S_H(K)} \langle \pi_H x, y\rangle^2 dy$$

where π_H is the reflection operator with respect to H. Thus the Binet ellipsoid of $S_H(K)$ is symmetric with respect to H. By Lemma 3.1, $\mathcal{B}(S_H(K))$ has at least the same volume as $\mathcal{B}(K)$. Both ellipsoids have the same intersection with H, the ellipsoid $\mathcal{B}(S_H(K))$ is symmetric with respect to H, and has a greater (or equal) volume. By the definition of the operation S_H°,

$$S_H^\circ(\mathcal{B}(K)) \subset \mathcal{B}(S_H(K)).$$

□

An important fact first observed by Bourgain [MP1], is that unconditional bodies satisfy $L_K \leq C$, with constant $C \leq \frac{1}{\sqrt{2}}$, as calculated in [BN]. We would like to give an ad-hoc definition of "almost isotropicity", to be used in the proof:

Definition 3.3 *A body K of volume one is called "almost isotropic with constant C" if:*

$$CD_n \subset \mathcal{B}(K).$$

Lemma 3.4 *For any body $K \subset \mathbb{R}^n$ there exist $2n$ Steiner symmetrizations that transform K into a new body which is unconditional, and "almost isotropic with constant $\sqrt{2}$". The body will remain "almost isotropic with constant $\sqrt{2}$" after any additional Steiner symmetrizations.*

Proof: The first n symmetrizations will be carried out with respect to any orthonormal basis. By Lemma 2.3 the body obtained $\tilde{K}$ is unconditional. Therefore $L_{\tilde{K}} \leq \frac{1}{\sqrt{2}}$. It is clear that $\tilde{K}$ is centrally symmetric. Combining (2) and the fact that for ellipsoids $vol\mathcal{E} \cdot vol\mathcal{E}^\circ = (vol(D_n))^2$ we get:

$$\left(\frac{vol(\mathcal{B}^\circ(\tilde{K}))}{vol(D_n)}\right)^{\frac{1}{n}} \leq \frac{1}{\sqrt{2}}$$

where $\mathcal{B}^\circ(\tilde{K})$ is the dual body to $\mathcal{B}(\tilde{K})$. By Lemma 2.6, there exist n vectors $v_1, .., v_n$ such that

$$S_{v_1}(S_{v_2}...(\mathcal{B}^\circ(\tilde{K}))...) = tD_n \subset \frac{1}{\sqrt{2}} D_n.$$

Thus, by the definition of the operation S°,

$$\sqrt{2} D_n \subset S_{v_1}^\circ(S_{v_2}^\circ...(\mathcal{B}(\tilde{K}))...) \tag{3}$$

and any additional S° operations preserve (3). By Lemma 3.2, we get that for $K' = S_{v_1}(S_{v_2}...(\tilde{K})..)$, it holds that

$$\mathcal{B}(K') \supset \sqrt{2}D_n \tag{4}$$

and that further Steiner symmetrizations of K' preserve the validity of (4). In addition, the body K' is unconditional, by Lemma 2.6 and Lemma 2.3. □

Remark for Lemma 3.4: Note that from the unproven "hyperplane conjecture" [MP1] it follows that just n symmetrizations are enough for obtaining the conclusion of the lemma, since the first n symmetrizations described in Lemma 3.4 are unnecessary.

In a recent paper by Bobkov and Nazarov ([BN]), the following is proven:

Proposition 3.5 *Let $\{e_i\}$ be an orthonormal basis in $\mathbb{R}^n$. Denote:*

$$B(l_1^n) = \{x \in \mathbb{R}^n; \sum_i |\langle x, e_i\rangle| \le 1\} \quad , \quad B(l_\infty^n) = \{x; \forall i |\langle x, e_i\rangle| \le 1\}$$

the cross polytope and the cube, respectively. Assume that K is a body of volume one, which is unconditional with respect to $\{e_i\}$. Then:

$$\left[\forall i \ \alpha^2 \le \int_K \langle x, e_i\rangle^2 dx \le \beta^2\right] \quad \Rightarrow \quad \frac{\alpha}{2}B(l_\infty^n) \subset K \subset \sqrt{3}\beta n B(l_1^n).$$

Unfortunately, we lack information regarding the lower bound α in Proposition 3.5, even for bodies which are "almost isotropic" and unconditional. However, such bodies clearly satisfy the assumption concerning the upper bound, with $\beta \le \frac{1}{\sqrt{2}}$. Hence, any unconditional "almost isotropic" body is bounded by a cross polytope from above. Note that $Vol(nB(l_1^n))^{1/n} < c$ for some numerical constant c (one can take $c = e$). Thus we should focus on symmetrizing the specific body $B(l_1^n)$ into an isomorphic Euclidean ball. Such a result will allow us to symmetrize any convex body K, transforming it into a body that is bounded from above by a Euclidean ball, whose radius is not more than a constant times $v.r.(K)$.

In the next section we shall present a symmetrization procedure for the cross polytope. The second half of our task, symmetrization from below, will be discussed in Sect. 5.

4 Steiner symmetrizations of $B(l_1^n)$

This section aims at symmetrizing a specific body: $K = \sqrt{n}B(l_1^n)$. Note that $D_n \subset K$. Since Steiner symmetrizations preserve D_n, it suffices to transform K, using Steiner symmetrizations, to a body $\tilde{K}$ such that:

$$\tilde{K} \subset CD_n$$

The general idea underlying the symmetrization process for K is the fact discovered by Kashin [Ka]: the body $B(l_1^n)$ has very large sections which are almost Euclidean. Our symmetrization process consists of two stages. In the first stage, we symmetrize the body to create very large subspaces, such that the projections of the body onto them are almost Euclidean. In the second stage, we bound Steiner symmetrizations by Minkowski symmetrizations, and symmetrize in the small subspaces orthogonal to the large subspaces discussed earlier.

Proposition 4.1 *For any $\varepsilon > 0$, there exist $\lfloor (1+\varepsilon)n \rfloor$ Steiner symmetrizations, that transform $K = \sqrt{n}B(l_1^n)$ into a body $\hat{K}$ such that:*

$$D_n \subset \hat{K} \subset c(\varepsilon)D_n.$$

Some lemmas are used in the proof. The first, which is a quantitative extension of Kashin's result, is due to Gluskin and Garnaev [GG]:

Lemma 4.2 *Let $K = \sqrt{n}B(l_1^n)$, and let E be a random subspace of dimension λn. Then with high probability (greater than $1 - e^{-c\lambda n}$),*

$$K \cap E \subset c_1 \sqrt{\frac{-\log(1-\lambda)}{1-\lambda}} D_n.$$

For a body $K \subset \mathbb{R}^n$, define its supporting functional as $h_K(x) = \sup_{y \in K} \langle x, y \rangle$. The mean width of K is denoted as (see e.g. [MS])

$$w(K) = 2M^*(K) = 2 \int_{S^{n-1}} h_K(x) d\sigma(x)$$

where σ is the unique rotation invariant probability measure on the sphere.

Lemma 4.3 *Let $K \subset \mathbb{R}^n$ be an unconditional body such that $vol(K) = vol(\sqrt{n}B(l_1^n))$, and such that the body $\frac{1}{(vol(K))^{1/n}}K$ is "almost isotropic with constant $\sqrt{2}$". Denote $diam(K) = sup_{x,y \in K}|x - y|$, the diameter of K. Then:*

$$M^*(K) \leq c_2 \sqrt{\log(diam(K))}.$$

Proof: By Bobkov-Nazarov (Proposition 3.5),

$$K \subset c\sqrt{n}B(l_1^n)$$

Since K is clearly symmetric, we have $K \subset \frac{diam(K)}{2}D_n$, therefore $K \subset c\sqrt{n}B(l_1^n) \cap \frac{diam(K)}{2}D_n$. The result follows by a well known calculation of the mean width of the latter body (see e.g. [Kl2], Chapter 1 here). □

The next lemma is proven using a contraction principle for Gaussian variables (see e.g. [GM4], Section 4.2):

Lemma 4.4 *Let K be a body in $\mathbb{R}^n$, E is a subspace of dimension λn. Then:*

$$M_E^*(K \cap E) \leq \frac{c_3}{\sqrt{\lambda}} M_{\mathbb{R}^n}^*(K)$$

For a hyperplane H define $\tau_H(K) = \frac{K+\pi_H(K)}{2}$, the Minkowski symmetrization (π_H is the reflection with respect to H). It is clear that $S_H(K) \subset \tau_H(K)$ (e.g. see [BLM2]). Minkowski symmetrizations preserve $M^*(K)$, and in [Kl2] (Chapter 1 here) it is proven that for every body, there exist $5n$ Minkowski symmetrizations that transform the body into an approximate Euclidean ball, of radius $M^*(K)$. Hence,

Proposition 4.5 *For every body $K \subset \mathbb{R}^n$, there exist $5n$ Steiner symmetrizations such that K is transformed into a body $\tilde{K}$ such that:*

$$\tilde{K} \subset c_4 M^*(K) D_n$$

and $\tilde{K}$ is an unconditional body, since the last n symmetrizations are carried out with respect to an orthonormal basis.

4.1 Initial symmetrizations

Randomly choose an orthonormal basis $\{e_1, .., e_n\} \in O(n)$, and symmetrize K with respect to $e_n, .., e_1$ (in this order) to obtain K'. Then clearly K' is unconditional with respect to $e_1, .., e_n$.

Claim 4.6 *For any $1 \leq k \leq n$, with high probability of choosing the basis $\{e_1, .., e_n\}$:*

$$Proj_{F_k} K' \subset c_1 \sqrt{\frac{n}{k} \log \frac{n}{k}} D_n \tag{5}$$

where $F_k = sp\{e_{k+1}, .., e_n\}$ is the linear span of the vectors $\{e_{k+1}, .., e_n\}$. Furthermore, additional symmetrizations with respect to vectors inside $sp\{e_1, .., e_k\}$ cannot ruin the validity of (5).

Proof: Since the orthonormal basis $\{e_1, .., e_n\}$ is chosen randomly, then the subspaces $F_k = sp\{e_{k+1}, .., e_n\}$ are random. According to Lemma 4.2

$$K \cap F_k \subset c_1 \sqrt{\frac{n}{k} \log \frac{n}{k}} D_n \cap F_k \tag{6}$$

Now, the first $n-k$ symmetrizations cannot affect the validity of (6), since they preserve $D_n \cap F_k$, according to Lemma 2.2. Let K'' be the body obtained after the first $n-k$ symmetrizations. Then by Lemma 2.5 (the body is clearly centrally symmetric),

$$Proj_{F_k}(K'') = K'' \cap F_k \subset c_1 \sqrt{\frac{n}{k} \log \frac{n}{k}} D_n \cap F_k$$

The next k symmetrizations are carried out with respect to vectors orthogonal to F_k. Thus, by the corollary of Lemma 2.1, these k symmetrizations cannot ruin (5). Furthermore, any additional symmetrization with respect to vectors in $sp\{e_1, .., e_k\}$ cannot affect the validity of (5). $\square$

Remark for Claim 4.6: The probability in discussion in Claim 4.6 is greater than $1 - e^{-c(n-k)}$. The probability that (5) holds for all $1 \le k \le \frac{n}{2}$ together is greater than $1 - e^{-c'n}$. We will see that the symmetrization process works well in this case, which happens with high probability.

4.2 Iteration of symmetrizations

So far, we have performed n Steiner symmetrizations to $K = \sqrt{n} B(l_1^n)$, and have obtained a body K' that satisfies Claim 4.6. We plan to perform a short series of iterations. Each iteration reduces both the diameter and the mean width at least logarithmically, and maintains the body unconditional for the next iteration. The i^{th} iteration would consist of $5 \left\lfloor \frac{n}{\log^{(i)} n} \right\rfloor$ Steiner symmetrizations (where $\log^{(0)} n = n$, and $\log^{(i+1)} n = \log \log^{(i)} n$). When we arrive at the first i such that $\log^{(i)} n < c_6(\varepsilon) = \max\{\log 2c_5, 10/\varepsilon\}$ we will stop, where $c_5 = \max\{2c_1, 4c_2c_3c_4\}$ is a numerical constant. Thus, the entire process (including the initial symmetrizations) consists of

$$n + \sum_i 5 \left\lfloor \frac{n}{\log^{(i)} n} \right\rfloor \le \lfloor (1+\varepsilon) n \rfloor$$

symmetrizations. Denote $E_i = sp\{e_1, .., e_{\lfloor \frac{n}{\log^{(i)} n} \rfloor}\}$, and K_i is the body obtained from the i^{th} iteration ($K_0 = K'$). The definition of the i^{th} iteration is simple: Use $K_{i-1} \cap E_i$ as the object for Proposition 4.5. To arrive at K_i, apply the same set of symmetrizations (with respect to the same vectors in E_i) to the entire body K_{i-1}.

Claim 4.7 *For i such that $\log^{(i)} n \geq c_6(\varepsilon)$, the body K_i is unconditional and satisfies:*

$$diam(K_i) < 2c_5 log^{(i)} n$$

Proof: By induction. For $i = 0$ it is clear. Assume that

$$diam(K_{i-1}) < 2c_5 log^{(i-1)} n$$

and K_{i-1} is unconditional. Since Steiner symmetrizations preserve "almost isotropicity", by Lemma 4.3 we have

$$M^*(K_{i-1}) \leq c_2 \sqrt{\log^{(i)} n + \log 2c_5} < 2c_2 \sqrt{\log^{(i)} n}$$

where the latter inequality follows from definition of $c_6(\varepsilon)$. By Lemma 4.4, the mean width of $K_{i-1} \cap E_i$ with respect to its ambient subspace is bounded:

$$M^*_{E_i}(K_{i-1} \cap E_i) \leq c_3 \sqrt{\log^{(i)} n} \left(2c_2 \sqrt{\log^{(i)} n} \right) = 2c_2 c_3 \log^{(i)} n$$

Recall the definition of the iteration: K_i was obtained from K_{i-1} using symmetrizations in the subspace E_i, which was chosen according to Proposition 4.5. Therefore, by Lemma 2.2,

$$diam(K_i \cap E_i) \ < \ 2c_4 \cdot (2c_2 c_3 \log^{(i)} n)$$

and by Lemma 2.5, since K_i is symmetric with respect to an orthonormal basis of E_i,

$$diam(Proj_{E_i}(K_i)) \ = \ diam(K_i \cap E_i) \ < \ 2c_4 \cdot (2c_2 c_3 \log^{(i)} n).$$

This is all regarding the subspace E_i. Take a look at its complement $E_i^{\perp}$. Note that all of the symmetrizations in the previous iterations were carried out inside E_i. By a corollary to Lemma 2.2, K_i is still symmetric with respect to an orthonormal basis in $E_i^{\perp}$, hence in total K_i unconditional. In addition, by Claim 4.6, we still have:

$$diam(Proj_{E_i^{\perp}}(K_i)) < 2c_1 \sqrt{\log^{(i)} n \log^{(i+1)} n} < 2c_1 \log^{(i)} n.$$

Note that

$$diam(K_i) \leq \sqrt{2} \max\{diam(Proj_{E_i}(K_i)), diam(Proj_{E_i^{\perp}}(K_i))\}$$

and therefore $diam(K_i) < \sqrt{2} c_5 \log^{(i)} n$. □

Hence, we have used $\lfloor (1+\varepsilon)n \rfloor$ symmetrizations for the body $K = \sqrt{n} B(l_1^n)$, and have obtained a body with diameter smaller than $2c_5 \log^{(i-1)} n \leq 2c_5 e^{c_6(\varepsilon)}$. Thus, Proposition 4.1 is proven. In addition, it is clear that those $\lfloor (1+\varepsilon)n \rfloor$ symmetrizations, applied to any $K \subset \mathbb{R}^n$, will transform it into an unconditional body. □

Remark for Proposition 4.1: The proven dependence on ε here is exponential. However, the real dependence on ε in the proposition is only $O(\sqrt{\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}})$. This can be proven in an analogous way to the discussion at the end of Sect. 5.2 (and using Santalo inequality, for obtaining the dual form of Lemma 5.5).

5 Symmetrizations of a cube and a box

5.1 Symmetrizing from below

We begin by quoting a well known lemma of Lozanovskii (see [Pi], page 30), which we use in the following geometric form. Our formulation easily follows from the classical formulation, by setting the normalization $Vol(K) = 1$, and using the fact that $Vol(nB(l_1^n))^{1/n} < eVol(B(l_\infty^n))^{1/n}$.

Lemma 5.1 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one, which is unconditional with respect to the orthonormal basis $\{e_1, .., e_n\}$. Then there exist $\lambda_1, .., \lambda_n > 0$ whose product $\prod_{i=1}^n \lambda_i = 1$ such that,*

$$\{x \in \mathbb{R}^n; \forall i \ |\langle x, e_i \rangle| \leq \lambda_i\} \subset cK$$

where c is some absolute constant (not larger than $2e$).

Hence, any unconditional body contains a relatively large rectangular box. As we try to symmetrize the body and fill it from below, the box - and more importantly the cube - will play a special role in the symmetrization process to be described. The next lemma is the "box version" of Lemma 2.6:

Lemma 5.2 *Let $\mathcal{Q} = \{x \in \mathbb{R}^n; \forall i \ |\langle x, e_i \rangle| \leq \lambda_i\} \subset \mathbb{R}^n$ be a box, where $\{e_1, .., e_n\}$ is an orthonormal basis. Denote $2r = (Vol(\mathcal{Q}))^{1/n}$. Then there exist $n-1$ Steiner symmetrizations that transform $\mathcal{Q}$ into a body that contains a cube with edge r (not necessarily with respect to the original axes).*

Before proving Lemma 5.2, we would like to present the entire picture of symmetrizations. Lemma 5.1 and Lemma 5.2 reduce the problem of symmetrizing from below, to the problem of symmetrizing a specific body, the cube. In Sect. 5.2 we shall present a symmetrization process for the cube. For every $0 < \varepsilon < 1$, we shall describe a symmetrization procedure of the cube consisting of only $\lfloor \varepsilon n \rfloor$ symmetrizations. This way we may fulfill our task of transforming any convex body into a body that contains an appropriate Euclidean ball.

To summarize the proposed symmetrization process of an arbitrary convex body $K \subset \mathbb{R}^n$: Fix $0 < \varepsilon < 1$. Apply $2n$ symmetrizations according to Lemma 3.4, creating unconditionality and almost-isotropicity. At this stage, by Proposition 3.5, K is trapped inside an appropriate cross polytope. Apply to K the $\lfloor (1 + \varepsilon)n \rfloor$ symmetrizations according to Proposition 4.1, and obtain a body that is contained inside a Euclidean ball of appropriate radius. The symmetrizations are designed such that the resulting body would be unconditional again and hence will contain an appropriate rectangular box, by Lemma 5.1. Apply n symmetrizations according to Lemma 5.2. The body obtained contains a cube with the same order of volume-radius as the original body. Now, apply the $\lfloor \varepsilon n \rfloor$ symmetrizations that are suitable for the cube. The resulting body $\hat{K}$ satisfies:

$$c_1(\varepsilon)D_n \subset \frac{1}{v.r.(K)}\hat{K} \subset c_2(\varepsilon)D_n$$

where $c_1(\varepsilon), c_2(\varepsilon) > 0$ are numerical constants, depending only on the parameter ε.

The total number of symmetrizations to be carried out is $\lfloor (4+2\varepsilon)n \rfloor$. This will prove that no more than cn symmetrizations are needed in order to transform $K \subset \mathbb{R}^n$ into an isomorphic Euclidean ball (and any $c > 4$ is adequate here).

Proof of Lemma 5.2: Assume by induction that the lemma is correct for dimension $n-1$. Clearly $r^n = \prod_i \lambda_i$. Select two vectors e_i and e_j, such that $\lambda_i \le r \le \lambda_j$. Denote $E = sp\{e_i, e_j\}$. Two cases exist:
(i) $\frac{r}{\sqrt{2}} \le \lambda_i$. In this case, we symmetrize the $n-1$ dimensional box $e_i^\perp \cap \mathcal{Q}$ according to the induction hypothesis. As in the proof of Lemma 2.6, we apply the same symmetrizations to the entire body $\mathcal{Q}$, to obtain $\tilde{\mathcal{Q}}$. Now, $Vol(e_i^\perp \cap \mathcal{Q}) = \frac{(2r)^n}{2\lambda_i} > (2r)^{n-1}$, where Vol, of course, is interpreted as the natural $n-1$ dimensional Lebesgue measure in $e_i^\perp$. Hence the symmetrized body $e_i^\perp \cap \tilde{Q}$ contains an $n-1$ dimensional cube whose edge is r. Note that for any t with $|t| \le \lambda_i$,

$$(\mathcal{Q} \cap e_i^\perp) + te_i = \mathcal{Q} \cap (e_i^\perp + te_i)$$

since $\mathcal{Q}$ is a box. By Lemma 2.2, the same holds for $\tilde{Q}$, and since $\lambda_i \ge \frac{r}{\sqrt{2}} > \frac{r}{2}$, the body $\tilde{\mathcal{Q}}$ contains a cube whose edge is r. Note that in this case, we use only the symmetrizations of an $n-1$ dimensional rectangular box, which are not more than $n-2$ symmetrizations by the induction hypothesis.
(ii) $\lambda_i < \frac{r}{\sqrt{2}}$. We use the following simple observation (appears in [BLM2] or Lemma 2.6): Let $\mathcal{E} \subset \mathbb{R}^2$ be an ellipse with axes $a, b > 0$. Then for any $a < x < b$, there exists a single symmetrization that transforms $\mathcal{E}$ into an ellipse with axes $x, \frac{ab}{x}$.

Consider the ellipse $\mathcal{E} = \left\{ x \in E; \frac{\langle x, e_i \rangle^2}{\lambda_i^2} + \frac{\langle x, e_j \rangle^2}{\lambda_j^2} \le 1 \right\}$, which is obviously contained in $\mathcal{Q}$. According to the aforementioned observation, we can symmetrize $\mathcal{E}$ into an ellipse $\tilde{\mathcal{E}}$, that has an axis of length $2\frac{r}{\sqrt{2}}$ in direction v_1, and an axis of length $2\frac{\sqrt{2}\lambda_i\lambda_j}{r}$ in the orthogonal direction v_2. Note that we used the fact that $\lambda_i < \frac{r}{\sqrt{2}} < \lambda_j$. We symmetrize $\mathcal{Q}$ in the same manner that we symmetrized $\mathcal{E}$, to obtain $\mathcal{Q}'$. The rectangular $\mathcal{R} = \{x \in E; |\langle x, v_1 \rangle| \le \frac{r}{2},\ |\langle x, v_2 \rangle| \le \frac{\lambda_i\lambda_j}{r}\}$ is contained in $\tilde{\mathcal{E}}$, which in turn is contained in $\mathcal{Q}'$, since $\mathcal{E} \subset \mathcal{Q}$. Similarly to case (i), by Lemma 2.2, the box

$$\mathcal{Q}_1 = \{x \in \mathbb{R}^n; Proj_E(x) \in \mathcal{R},\ and\ \forall k \ne i,j\ |\langle x, e_k \rangle| \le \lambda_k\}$$

is contained in $\mathcal{Q}'$. Now, $Vol(v_1^\perp \cap \mathcal{Q}_1) = \frac{2\lambda_i\lambda_j}{r}\prod_{k\neq i,j}(2\lambda_k) = (2r)^{n-1}$. According to the induction hypothesis we can symmetrize the box $v_1^\perp \cap \mathcal{Q}_1$ to contain a cube with edge r. By applying the same set of symmetrizations to $\mathcal{Q}_1$, we obtain a body that contains a cube with edge r (exactly as in case (i)). Since $\mathcal{Q}_1 \subset \mathcal{Q}'$, by symmetrizing Q' appropriately we obtain a new body $\tilde{Q}$ that contains the desired cube. In total, we applied a single symmetrization to $\mathcal{Q}$ plus the necessary symmetrization for an $n-1$ dimensional rectangular box. $\square$

Remark: Lemma 5.2 may be easily generalized to the case of a cross polytope (or any unconditional 1-symmetric body). Combined with Lozanovskii lemma, this could lead to the same conclusion as in Sect. 3, that the problem of symmetrizing from above is reduced to symmetrizing the cross polytope. However, we choose to include Sect. 3, as it leads to a better understanding of the behaviour of the isotropic constant (see [BKM1] or [BKM2]).

5.2 Steiner symmetrizations of the cube

Let $K = \frac{1}{\sqrt{n}}B(l_\infty^n)$. This section aims at proving the following:

Proposition 5.3 *For any $0 < \varepsilon < 1$, there exist $\lfloor \varepsilon n \rfloor$ Steiner symmetrizations, that transform K into a body $\hat{K}$ such that:*

$$c(\varepsilon)D_n \subset \hat{K} \subset D_n$$

where $c(\varepsilon)$ is of the order of $O(\sqrt{\frac{\varepsilon}{-\log\varepsilon}})$, when ε is close to 0.

The proof given in this section resembles the proof in Sect. 4, partially because K is the dual body of $\sqrt{n}B(l_1^n)$ (for information about duality, see [MS] or [GM4]). Instead of bounding Steiner symmetrizations by Minkowski symmetrizations, we use the result of [BLM2], which is based on Dvoretzky type theorems. Note that $K \subset D_n$, and therefore we are only concerned with symmetrizing from below. In a dual manner to $B(l_1^n)$, our body has very large almost Euclidean projections. Recall that the purpose of the first n symmetrizations in the $B(l_1^n)$ process, was to create almost Euclidean projections. This stage is unnecessary for the cube. Let us formulate the dual version of Lemma 4.2:

Lemma 5.4 *Let $K = \frac{1}{\sqrt{n}}B(l_\infty^n)$, and let E be a random subspace of dimension λn. Then with high probability (greater than $1 - e^{-c\lambda n}$),*

$$Proj_E(K) \supset c_1\sqrt{\frac{1-\lambda}{-\log(1-\lambda)}}D_n \cap E.$$

Note that the constants $c_1, c_2, ..$ in this section are not necessarily equal to the constants denoted by the same letters in the previous sections. The next lemma is just a normalization of a result of Vaaler [Vaa], that all sections in all dimensions of the unit cube have volume greater than one, as was first noted in [MeP2].

Lemma 5.5 *Let $K = \frac{1}{\sqrt{n}} B(l_\infty^n)$, and let E be a subspace of dimension k. Then:*

$$v.r.(K \cap E) > c_2 \sqrt{\frac{k}{n}}$$

where $v.r.(K \cap E)$ is calculated with respect to the natural Lebesgue measure in $E \subset \mathbb{R}^n$.

Now we turn to describing the symmetrization process of K, and proving Proposition 5.3. Let $\{v_1, .., v_n\} \in O(n)$ be a random orthonormal basis. Denote $\lambda_i = \frac{1}{(\log^{(i)} n)^2}$, and $E_i = sp\{v_1, .., v_{\lfloor \lambda_i n \rfloor}\}$. An iteration transforms the body K_{i-1} to K_i (where $K_0 = K$), in two steps:

- Symmetrize the body K_{i-1} with respect to any orthogonal basis in E_i, and obtain K_i'.
- Use Theorem 2.7 for the body $K_i' \cap E_i$. Symmetrize the entire body K_i' with respect to the vectors given by this theorem, to arrive at K_i.

Claim 5.6 *For i such that $\lambda_{i-1} < \frac{c_3}{2}$ (where $c_3 = \frac{1}{\sqrt{2}} \min\{c_1, \bar{c}_1 c_2\}$),*

$$c_3 \sqrt{\frac{\lambda_i}{-\log \lambda_i}} D_n \subset K_i,$$

and not more than $(1 + \bar{c}) \lfloor \lambda_i n \rfloor \cdot 2 \log \frac{1}{\lambda_{i-1}}$ symmetrizations were used to construct K_i from K_{i-1}.

Proof: By induction on i. For $i = 0$ this is trivially true. For a general i, note that K_i' was constructed from K using symmetrizations inside E_i. By Lemma 2.1 and Lemma 2.4,

$$Proj_{E_i^\perp}(K) = Proj_{E_i^\perp}(K_i') = K_i' \cap E_i^\perp.$$

Using Lemma 5.4, we find that:

$$c_1 \sqrt{\frac{\lambda_i}{-\log \lambda_i}} D_n \cap E_i^\perp \subset K_i'. \tag{7}$$

Note that $K_{i-1} \subset D_n$. By the induction hypothesis,

$$d_{BM}(K_i', D_n) \leq d(K_{i-1}, D_n) \leq \frac{\sqrt{-\log \lambda_{i-1}}}{c_3 \sqrt{\lambda_{i-1}}}.$$

By Theorem 2.7, not more then $\bar{c}\lfloor \lambda_i n \rfloor \cdot \log 2d_{BM}(K_i', D_n)$ symmetrizations were carried out to construct K_i from K_i'. Since $\lambda_{i-1} < \frac{c_3}{2}$, this number of symmetrizations is less than

$$\bar{c}\lfloor \lambda_i n \rfloor \cdot 2 \log \frac{1}{\lambda_{i-1}}.$$

Because symmetrizations were carried out only inside E_i, then $vol(K \cap E_i) = vol(K_i' \cap E_i)$. By Theorem 2.7 and Lemma 5.5, we can assure that

$$\bar{c}_1 c_2 \sqrt{\lambda_i} D_n \cap E_i \subset K_i. \tag{8}$$

Since K_i was obtained from K_i' using symmetrizations inside E_i, then (7) still holds for K_i. Combining (7), (8) and the definition of c_3,

$$c_3 \sqrt{\frac{\lambda_i}{-\log \lambda_i}} D_n \subset K_i.$$

□

Now, take the minimal t such that $\log^{(t)} n > 8(\bar{c}+1)/\varepsilon$ and also $\lambda_{t-1} < \frac{c_3}{2}$. To obtain K_t, we have used not more than

$$\sum_{i<t} (1+\bar{c})\lfloor \lambda_i n \rfloor \cdot 2 \log \frac{1}{\lambda_{i-1}} < n \sum_{i<t} \frac{4(\bar{c}+1)}{\log^{(i)} n} < \varepsilon n$$

symmetrizations, and obtained a body of distance less than $O(e^{\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}})$ from the Euclidean ball. From the discussion in Sect. 5.1, we have proved:

Theorem 5.7 *Let $\varepsilon > 0$ and let $K \subset \mathbb{R}^n$ be a convex body. Then there exist numbers $c_1(\varepsilon), c_2(\varepsilon) > 0$, such that there exist $\lfloor (4+\varepsilon)n \rfloor$ Steiner symmetrizations that transform K into a body $\tilde{K}$ that satisfies:*

$$c_1(\varepsilon) D_n \subset \frac{1}{v.r.(K)} \tilde{K} \subset c_2(\varepsilon) D_n.$$

However, Proposition 5.3 is not fully proven yet, as the dependence on ε we got is far worse than what was promised in the Proposition. This is easy to fix. Instead of the prescribed iteration process, denote $E = sp\{v_1, .., v_{\lfloor \varepsilon n \rfloor}\}$. According to Theorem 5.7, there exist $5\lfloor \varepsilon n \rfloor$ symmetrizations carried out inside E, that transform the body K into K', such that:

$$c\sqrt{\varepsilon} D_n \cap E \subset c' v.r.(K \cap E) D_n \cap E \subset K'$$

where the first inclusion follows from Lemma 5.5. Since these $5\lfloor \varepsilon n \rfloor$ symmetrizations include symmetrizations with respect to an orthogonal basis of E, then by Lemma 5.4 and Lemma 2.4, we also have

$$c_1 \sqrt{\frac{\varepsilon}{-\log \varepsilon}} D_n \cap E^{\perp} \subset K'.$$

Therefore, using $5\lfloor \varepsilon n \rfloor$ symmetrizations, we have obtained a body that contains a Euclidean ball of radius $O(\sqrt{\frac{\varepsilon}{-\log \varepsilon}})$, which is an optimal dependence on ε, and Proposition 5.3 is proven. $\square$

6 Symmetrizing to an ellipsoid

We would like to investigate the smallest number of symmetrizations that transform an arbitrary convex body $K \subset \mathbb{R}^n$ into an isomorphic Euclidean ball. Unfortunately, our technique provides an exact answer only to a related question, where rather than approaching an isomorphic Euclidean ball, we obtain an isomorphic ellipsoid. Formally, for any convex body $K \subset \mathbb{R}^n$ define:

$$S_T(K) = \inf\{d_{BM}(K_T, D_n); \exists H_1, .., H_n,\ K_T = S_{H_1,..,H_T}(K)\}$$

the minimal distance to an ellipsoid, that can be achieved using T symmetrizations. We are concerned with the following quantities, representing the asymptotic minimal number of symmetrizations:

$$S_n(c) = \min\{T; \forall K \subset \mathbb{R}^n\ S_T(K) \leq c\}$$

$$\overline{S}(c) = \limsup_{n\to\infty} \frac{S_n(c)}{n}\ ,\ \underline{S}(c) = \liminf_{n\to\infty} \frac{S_n(c)}{n}$$

We do not know how to prove that $\underline{S}(c) = \overline{S}(c)$, or even that these expressions are finite for values of c close to 1. However, we can prove the following:

Theorem 6.1

$$\lim_{c\to\infty} \underline{S}(c) = \lim_{c\to\infty} \overline{S}(c) = 1.$$

The lower bound is easy to obtain:

Lemma 6.2 $\forall c > 1\ \ \underline{S}(c) \geq 1$.

Proof: Take $K = B(l_1^n)$. Then for any subspace H of dimension k (cf. a dual of a theorem in [Ba4] page 9),

$$d_{BM}(Proj_H(K), D_k) \geq \sqrt{\frac{k}{2\log n}}.$$

Apply any $T < n$ Steiner symmetrizations to K, with respect to the hyperplanes $H_1, .., H_T$. Denote $H = \cap_i H_i$ and $K_T = S_{H_1,..,H_T}(K)$. Then $dim(H) \geq n - T$ and:

$$d_{BM}(K_T, D_n) \geq d_{BM}(Proj_H(K_T), D_n \cap H) =$$

$$= d_{BM}(Proj_H(K), D_n \cap H) \geq \sqrt{\frac{n-T}{2\log n}}$$

where the equality in the middle follows by Lemma 2.1. Plug in the definition of $S_n(c)$:

$$S_n(c) \geq n - 2c^2 \log n \quad \Rightarrow \quad \underline{S}(c) \geq 1.$$

□

It remains to prove the upper bound. The proof makes an extensive use of the "quotient of subspace theorem", first proven in [M3] for centrally symmetric bodies (see also [Pi], chapter 8). The extension to non-symmetric bodies appears in [MP2]. Let K be a convex body. Its centroid or barycenter is the point $\int_K x dx \in K$. Denote $\overline{K} = conv(K, -K)$ the convex hull of K and $-K$. Let us formulate a variant of the quotient of subspace theorem, which is easy to deduce from the references:

Theorem 6.3 *Let $\varepsilon > 0$ and let $K \subset \mathbb{R}^n$ be a convex body with 0 as a centroid, and with $vol(K) = vol(D_n)$. Then there exist $F \subset E \subset \mathbb{R}^n$ subspaces with $dim(F) = \lceil (1-2\varepsilon)n \rceil$, $dim(E) > (1-\varepsilon)n$, and an ellipsoid $\mathcal{E} \subset F$ such that:*

$$c_1(\varepsilon)\mathcal{E} \;\subset\; F \cap Proj_E(K) \;\subset\; F \cap Proj_E(\overline{K}) \;\subset\; c_2(\varepsilon)\mathcal{E}.$$

This theorem leads to the following lemma:

Lemma 6.4 *Let $\varepsilon > 0$ and let $K \subset \mathbb{R}^n$ be a convex body with 0 as its centroid. Then there exist $\lceil (1-\varepsilon)n \rceil$ Steiner symmetrizations, a positive number λ and a subspace $F \subset \mathbb{R}^n$ with $dim(F) = \lceil (1-2\varepsilon)n \rceil$ such that the symmetrized body, $\tilde{K}$ satisfies:*

$$c_1(\varepsilon)\lambda D_n \cap F \subset \tilde{K} \cap F \subset Proj_F(\tilde{K}) \subset c_2(\varepsilon)\lambda D_n \cap F.$$

Proof: According to Theorem 6.3, given $K \subset \mathbb{R}^n$ there exist some special subspaces $F \subset E \subset \mathbb{R}^n$, and a distinguished ellipsoid $\mathcal{E} \subset F$. Symmetrize K with respect to any orthogonal basis in $E^{\perp}$, to obtain $\hat{K}$. By Lemma 2.4, $\hat{K} \cap E = Proj_E(K)$, and by Theorem 6.3,

$$c_1(\varepsilon)\mathcal{E} \subset \hat{K} \cap F \subset c_2(\varepsilon)\mathcal{E}.$$

According to Lemma 2.6, the ellipsoid $\mathcal{E}$ can be symmetrized to become $\lambda D_{\lceil (1-2\varepsilon)n \rceil}$ using symmetrizations with respect to some orthogonal basis in F. Apply these symmetrizations to the entire body $\hat{K}$, to obtain $\tilde{K}$. A total of $\lceil (1-\varepsilon)n \rceil$ symmetrizations were carried out. At this stage,

$$c_1(\varepsilon)\lambda D_n \cap F \subset \tilde{K} \cap F \subset c_2(\varepsilon)\lambda D_n \cap F.$$

If we were to apply exactly the same symmetrizations to the body $C = \overline{K}$, then by Theorem 6.3, we would obtain a centrally symmetric $\tilde{C}$ such that

$$Proj_F(\tilde{C}) = \tilde{C} \cap F \subset c_2(\varepsilon)\lambda D_n$$

where the equality follows by Lemma 2.5, since $\tilde{C}$ is symmetric with respect to an orthogonal basis in F. Since $\tilde{K} \subset \tilde{C}$, we get that

$$c_1(\varepsilon)\lambda D_n \cap F \subset \tilde{K} \cap F \subset Proj_F(\tilde{K}) \subset c_2(\varepsilon)\lambda D_n \cap F.$$

□

Remark for Lemma 6.4: As can be easily deduced from the proof, a relatively small number of symmetrizations of an arbitrary convex body, may create very large sections which are uniformly isomorphic to some ellipsoid. Specifically, for any convex body $K \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exist a subspace F of dimension $\lceil (1-2\varepsilon)n \rceil$ and $\lfloor \varepsilon n \rfloor$ symmetrizations, such that for the symmetrized body $\hat{K}$,

$$d_{BM}(\hat{K} \cap F, D_n \cap F) < c(\varepsilon).$$

Proof of Theorem 1.2: Let K be a convex body. Assume that 0 is its centroid. Apply Lemma 6.4 to K. After $\lceil (1-\varepsilon)n \rceil$ symmetrizations, we obtain $\hat{K}$ such that there exists a subspace F with $dim(F) = \lceil (1-2\varepsilon)n \rceil$ and a number λ_1 with:

$$c_1(\varepsilon)\lambda_1 D_n \cap F \subset \hat{K} \cap F \subset Proj_F(\hat{K}) \subset c_2(\varepsilon)\lambda_1 D_n \cap F. \tag{9}$$

In addition, the symmetrizations from Lemma 6.4 include symmetrizations with respect to an orthogonal basis of F. By Lemma 2.4,

$$\hat{K} \cap F^{\perp} = Proj_{F^{\perp}} \hat{K} \tag{10}$$

and further symmetrizations inside $F^{\perp}$ cannot hurt the validity of (9) or (10). By Proposition 5.7, there exist $\lceil 9\varepsilon n \rceil$ symmetrizations that transform $\hat{K} \cap F^{\perp}$ into an isomorphic Euclidean ball. Apply these symmetrizations to $\hat{K}$, to obtain $\tilde{K}$. Then:

$$c_1\lambda_2 D \cap F^{\perp} \subset \tilde{K} \cap F^{\perp} = Proj_{F^{\perp}} \tilde{K} \subset c_2\lambda_2 D \cap F^{\perp}.$$

Combining with (9) we conclude that the unique ellipsoid $\mathcal{E}$ such that

$$Proj_F \mathcal{E} = F \cap \mathcal{E} = \lambda_1 D_n \cap F$$

$$Proj_{F^\perp}\mathcal{E} = F^\perp \cap \mathcal{E} = \lambda_2 D_n \cap F^\perp$$

satisfies

$$c_1'(\varepsilon)\mathcal{E} \subset \tilde{K} \subset c_2'(\varepsilon)\mathcal{E}$$

and the proposition is proven, for bodies with 0 as a centroid.

It remains to dispose of the assumption that 0 is the centroid of K. Denote $b(K) = \frac{1}{vol(K)}\int_K xdx$, the centroid of K. We have described a short symmetrization process for the body $K - b(K)$, resulting with the body $\tilde{K}$. We claim that when these symmetrizations are applied to K, the resulting body is again $\tilde{K}$. Indeed, note that:

$$b(S_H(K)) = Proj_H(b(K)).$$

Since we symmetrize with respect to an orthonormal basis of F, and also with respect to an orthonormal basis of $F^\perp$, the resulting body has 0 as a centroid. Therefore, the symmetrizations of $K - b(K)$ and K have the same centroid, and since apriori they differ only by a translation - they must be identical. Thus, Theorem 1.2 is finally proven. □

Plugging in the estimates from "quotient of subspace" theorem, we get that the distance to an ellipsoid in Theorem 1.2 is of the order of $O(\varepsilon^{-1}\log\varepsilon^{-1})$. It is clear that Theorem 6.1 follows from Theorem 1.2 and Lemma 6.2. The deduction of Theorem 1.3 is simple; Given an arbitrary convex body, apply the symmetrizations of Theorem 1.2 to it. Then, symmetrize the resulting "isomorphic ellipsoid" into an isomorphic Euclidean ball, according to Lemma 2.6, using n additional Steiner symmetrizations. To conclude, we obtained an isomorphic Euclidean ball from an arbitrary convex body, using a total of $\lfloor(2+\varepsilon)n\rfloor$ symmetrizations.

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Almost isometric symmetrization processes[4]

Abstract. It is a classical fact, that given an arbitrary convex body $K \subset \mathbb{R}^n$, there exists an appropriate sequence of Minkowski symmetrizations (or Steiner symmetrizations), that converges in Hausdorff metric to a Euclidean ball. Here we provide quantitative estimates regarding this convergence, for both Minkowski and Steiner symmetrizations. Our estimates are polynomial in the dimension and in the logarithm of the desired distance to a Euclidean ball, improving previously known exponential estimates. Inspired by a method of Diaconis [Di], our technique involves spherical harmonics. We also make use of an earlier result by the author regarding "isomorphic Minkowski symmetrization".

1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex body, and denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the usual Euclidean norm and scalar product in $\mathbb{R}^n$. Given a vector $u \in S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$, we denote by $\pi_u(x) = x - 2\langle x, u\rangle u$ the reflection operator with respect to the hyperplane through the origin, which is orthogonal to u in $\mathbb{R}^n$. The result of a Minkowski symmetrization (sometimes called Blaschke symmetrization) of K with respect to u, is the body

$$\tau_u(K) = \frac{K + \pi_u(K)}{2}$$

where the Minkowski sum of two sets $A, B \subset \mathbb{R}^n$ is defined as $A + B = \{a + b; a \in A, b \in B\}$. Let h_K denote the supporting functional of K, i.e. for $u \in \mathbb{R}^n$

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle.$$

Then $h_{\tau_u(K)}(v) = \frac{1}{2}\left[h_K(v) + h_K(\pi_u(v))\right]$. The mean width of K is defined as $w(K) = 2M^*(K) = 2\int_{S^{n-1}} h_K(u) d\sigma(u)$, where σ is the unique rotation invariant probability measure on the sphere. The mean width is preserved under Minkowski symmetrizations.

Steiner symmetrization of K with respect to a hyperplane H yields the unique body $S_H(K)$ such that for any line l perpendicular to H,

(i) $S_H(K) \cap l$ is a closed segment whose center lies on H.

(ii) $Meas(K \cap l) = Meas(S_H(K) \cap l)$.

[4]This section corresponds to the paper [Kl4].

where $Meas$ is the one dimensional Lebesgue measure in the line l. Steiner symmetrization preserves the volume of a set and transforms convex sets to convex sets. See e.g. [BF] for more information about these symmetrizations, and their applications in proving geometric inequalities.

Consecutive Minkowski/Steiner symmetrizations may cause a convex body to resemble a Euclidean ball. Starting with an arbitrary convex body, one may apply a suitable sequence of Minkowski/Steiner symmetrizations, and obtain a sequence of bodies that converges to a Euclidean ball. This Euclidean ball would have the same mean width/volume as had the original body. In this note, we investigate the rate of this convergence. We ask how many symmetrizations are needed, in order to transform an arbitrary convex body $K \subset \mathbb{R}^n$ into a body that is ε-close to a Euclidean ball. Our question is "almost isometric" in its nature, as we try to provide reasonable estimates even for small values of ε. Previous results in the literature are mostly of "isomorphic" nature, in the sense that the symmetrization process is aimed at obtaining a body which is uniformly "isomorphic" to a Euclidean ball (a body is "isomorphic" to a Euclidean ball if its distance to a Euclidean ball is bounded by some fixed, universal constant).

The first quantitative result regarding Minkowski symmetrization appears in [BLM1]. Denote by D the standard Euclidean ball in $\mathbb{R}^n$. Their result reads as follows:

Theorem 1.1 *Let $0 < \varepsilon < 1$, $n > n_0(\varepsilon)$. Given an arbitrary convex body $K \subset \mathbb{R}^n$, there exist $cn \log n + c(\varepsilon)n$ Minkowski symmetrizations that transform K into a body $\tilde{K}$ such that*

$$(1-\varepsilon)M^*(K)D \subset \tilde{K} \subset (1+\varepsilon)M^*(K)D$$

where $c(\varepsilon), n_0(\varepsilon)$ are of the order of $exp(c\varepsilon^{-2}|\log \varepsilon|)$ and $c > 0$ is a numerical constant.

Their proof uses the method of random Minkowski symmetrizations. In [Kl2] (Chapter 1 here), the notion of randomness was altered, and has lead to an improvement of the dependence on the dimension n. The following is proved in [Kl2]:

Theorem 1.2 *Let $n \geq 2$ and let $K \subset \mathbb{R}^n$ be a convex body. Then there exist $5n$ Minkowski symmetrizations, such that when applied to K, the resulting body $\tilde{K}$ satisfies,*

$$\left(1 - c\frac{|\log\log n|}{\sqrt{\log n}}\right) M^*(K)D \subset \tilde{K} \subset \left(1 + c\frac{|\log\log n|}{\sqrt{\log n}}\right) M^*(K)D$$

where $c > 0$ is some numerical constant.

Note that both in [Kl2] and in [BLM1], for any fixed dimension, one cannot even formally conclude that there is convergence to a Euclidean ball. This note fills that gap in the literature, and also provides surprisingly good dependence on ε. The following theorem is proved here:

Theorem 1.3 *Let $n \geq 2$, $0 < \varepsilon < \frac{1}{2}$, and let $K \subset \mathbb{R}^n$ be a convex body. Then there exist $cn \log \frac{1}{\varepsilon}$ Minkowski symmetrizations, that transform K into a body $\tilde{K}$ that satisfies*

$$(1-\varepsilon)M^*(K)D \subset \tilde{K} \subset (1+\varepsilon)M^*(K)D$$

where $c > 0$ is some numerical constant.

Our approach to the problem of Minkowski symmetrization involves a number of novel ideas. First, rather than applying random Minkowski symmetrizations, at each step we apply n symmetrizations with respect to the vectors of some random orthonormal basis. This change of randomness improves the rate of convergence by a factor of $\log n$ (see [Kl1], [Kl2] and also the remark following Corollary 3.3 here). Second, the use of spherical harmonics allows us to obtain good estimates regarding symmetrization of polynomials on the sphere. Finally, we approximate the supporting functional of K with an appropriate polynomial (applying Theorem 1.2 and a Jackson type theorem), and use the estimates obtained for symmetrization of polynomials.

Quantitative estimates regarding Steiner symmetrization are more difficult to obtain, as the problem is non-linear. The earliest estimate in the literature is due to Hadwiger [H]. It gives an estimate of the order of $\left(c\frac{\sqrt{n}}{\varepsilon^2}\right)^n$ for the number of Steiner symmetrizations required in order to transform an arbitrary n-dimensional convex body, to become ε-close to a Euclidean ball. In addition, an isomorphic result appears in [BLM2], which was improved by a logarithmic factor in [KM1] (Chapter 2 here). The following is proved in [KM1]:

Theorem 1.4 *Let $n \geq 2$ and let $K \subset \mathbb{R}^n$ be a convex body, with $Vol(K) = Vol(D)$. Then there exist $3n$ Steiner symmetrizations, such that when applied to K, the resulting body $\tilde{K}$ satisfies,*

$$cD \subset \tilde{K} \subset CD$$

where $c, C > 0$ are some numerical constants.

Some related estimates also appear in [T]. Our result is the first estimate which is polynomial in n and in $\log \frac{1}{\varepsilon}$. This shows that the precise geometric shape of a convex body cannot prevent fast symmetrization of the body into an almost Euclidean ball. In this note we shall prove the following theorem.

Theorem 1.5 *Let $K \subset \mathbb{R}^n$ be a convex body, and let $0 < \varepsilon < \frac{1}{2}$. Let $r > 0$ be such that $Vol(K) = Vol(rD)$. Then there exist $cn^4 \log^2 \frac{1}{\varepsilon}$ Steiner symmetrizations, that transform K into a body $\tilde{K}$ that satisfies*

$$(1-\varepsilon)rD \subset \tilde{K} \subset (1+\varepsilon)rD$$

where $c > 0$ is some numerical constant.

The powers of n and $\log \frac{1}{\varepsilon}$ in Theorem 1.5 seem non optimal. We conjecture that $cn \log \frac{1}{\varepsilon}$ Steiner symmetrizations are sufficient. Regarding Minkowski symmetrizations, our result is tight in the sense that the powers in Theorem 1.3 cannot be improved.

The proof of Theorem 1.5 is an application of Theorem 1.3 and of a geometric result by Bokowski and Heil. Throughout this note, we denote by c, C, c' etc. positive numerical constants whose value is not necessarily equal in different appearances.

2 Spherical harmonics

In this section we summarize a few facts about spherical harmonics, to be used later on. For a comprehensive discussion on the subject, we refer the reader to the concise expositions in [SW], chapter $IV.2$, in [Mu] and in [Gr]. $P_k : \mathbb{R}^n \to \mathbb{R}$ is a homegeneous harmonic of degree k, if P_k is a homogeneous polynomial of degree k in $\mathbb{R}^n$, and P_k is harmonic (i.e. $\triangle P_k \equiv 0$). We denote,

$$\mathcal{S}_k = \{P|_{S^{n-1}} \ ; \ P : \mathbb{R}^n \to \mathbb{R} \ is \ a \ homogenous \ harmonic \ of \ degree \ k\}$$

where $P|_{S^{n-1}}$ is the restriction of the polynomial P to the sphere. $\mathcal{S}_k$ is the space of spherical harmonics of degree k. It is a linear space of dimension $\frac{(2k+n-2)\ (n+k-3)!}{k!\ (n-2)!}$. For $k \neq k'$, the spaces $\mathcal{S}_k$ and $\mathcal{S}_{k'}$ are orthogonal to each other in $L_2(S^{n-1})$. In addition, if P is a polynomial of degree k in $\mathbb{R}^n$, then $P|_{S^{n-1}}$ can be expressed as a sum of spherical harmonics of degrees not larger than k. Therefore, $L_2(S^{n-1}) = \bigoplus_k \mathcal{S}_k$. Spherical harmonics possess many symmetry properties, partly due to their connection with the representations of $O(n)$ (e.g. [Vi], chapter 9). For a fixed dimension n, the Gegenbauer polynomials $\{G_i(t)\}_{i=0}^{\infty}$ are defined by the following three conditions:

(i) $G_i(t)$ is a polynomial of degree i in one variable.

(ii) For any $i \neq j$ we have $\int_{-1}^{1} G_i(t) G_j(t) \left(1-t^2\right)^{\frac{n-3}{2}} dt = 0$.

(iii) $G_i(1) = 1$ for any i.

The Gegenbauer polynomials are closely related to spherical harmonics. Next, we reformulate Lemma 3.5.4 from [Gr], which is credited to Schneider. This useful lemma also follows from Corollary 2.13, chapter IV of [SW], and is true for all $n \geq 2$.

Lemma 2.1 *Let $g \in \mathcal{S}_k$ be such that $\|g\|_2^2 = \int_{S^{n-1}} g^2(x)d\sigma(x) = 1$. Then,*

$$\int_{O(n)} g(U^{-1}x)g(U^{-1}y)d\mu(U) = G_k(\langle x, y\rangle)$$

where μ is the Haar probability measure on $O(n)$.

The following lemma reflects the fact that $\mathcal{S}_k$ is an irreducible representation space of $O(n)$. We denote by $Proj_{\mathcal{S}_k} : L_2(S^{n-1}) \to \mathcal{S}_k$ the orthogonal projection onto $\mathcal{S}_k$.

Lemma 2.2 *Let $f \in L_2(S^{n-1})$, and let $g \in \mathcal{S}_k$ be such that $\|g\|_2 = 1$. Then,*

$$\int_{O(n)} \left(\int_{S^{n-1}} f(Ux)g(x)d\sigma(x) \right)^2 d\mu(U) = \frac{\|Proj_{\mathcal{S}_k}(f)\|_2^2}{dim(\mathcal{S}_k)} \tag{1}$$

where μ is the Haar probability measure on $O(n)$.

Proof: Let $\{g_1, .., g_N\}$ be an orthonormal basis of $\mathcal{S}_k$. Then,

$$\sum_{i=1}^{dim(\mathcal{S}_k)} \int_{O(n)} \left(\int_{S^{n-1}} f(Ux)g_i(x)d\sigma(x) \right)^2 d\mu(U) \tag{2}$$

$$= \int_{O(n)} \|Proj_{\mathcal{S}_k}(f \circ U)\|_2^2 d\mu(U) = \|Proj_{\mathcal{S}_k}(f)\|_2^2$$

because of the rotation invariance of $\mathcal{S}_k$. Therefore, it is sufficient to prove that the integral in (1) does not depend on the choice of $g \in \mathcal{S}_k$, as long as it satisfies $\|g\|_2 = 1$. Indeed, in that case each of the summands in (2) equals $\frac{\|Proj_{\mathcal{S}_k}(f)\|_2^2}{dim(\mathcal{S}_k)}$, for an arbitrary orthonormal basis $\{g_1, .., g_N\}$ of $\mathcal{S}_k$. Let us try to simplify the integral in (1):

$$\int_{O(n)} \int_{S^{n-1}} f(Ux)g(x)d\sigma(x) \int_{S^{n-1}} f(Uy)g(y)d\sigma(y)d\mu(U)$$

$$= \int_{S^{n-1}} \int_{S^{n-1}} f(x)f(y) \int_{O(n)} g(U^{-1}x)g(U^{-1}y)d\mu(U)d\sigma(x)d\sigma(y).$$

By Lemma 2.1, $\int_{O(n)} g(U^{-1}x)g(U^{-1}y)d\mu(U) = G_k(\langle x, y\rangle)$. Hence, the integral in (1) equals

$$\int_{S^{n-1}} \int_{S^{n-1}} f(x)f(y)G_k(\langle x, y\rangle)d\sigma(x)d\sigma(y)$$

which does not depend on g, and the lemma is proved. $\square$

3 Spherical harmonics and Minkowski symmetrization

In this section we apply a series of Minkowski symmetrizations to a convex body $K \subset \mathbb{R}^n$. Each step in the symmetrization process consists of symmetrizing K with respect to the n vectors of an orthonormal basis $\{e_1, .., e_n\}$ in $\mathbb{R}^n$. Such a step is denoted here as an "orthogonal symmetrization" with respect to $\{e_1, .., e_n\}$. Applying an "orthogonal symmetrization" with respect to $\{e_1, .., e_n\}$ to K, yields a body denoted by K'. Let h be the supporting functional of K, and h' be the supporting functional of K'. Then,

$$h'(x) = \mathbb{E}_\varepsilon \; h\left(\sum_{i=1}^n \varepsilon_i \langle x, e_i \rangle e_i\right) \tag{3}$$

where the expectation is over $\varepsilon \in \{\pm 1\}^n$, with respect to the uniform probability measure on the discrete cube. Note that by (3), orthogonal symmetrization may be viewed as an operation on support functions, rather than on convex bodies. Furthermore, we may apply an "orthogonal symmetrization" to any function on the sphere, which is not necessarily a support function of a convex body. Next, we analyze the effect of orthogonal symmetrizations on spherical harmonics.

Let k be a positive integer. A function $g \in L_2(S^{n-1})$ is called "invariant with respect to the orthonormal basis $\{e_1, .., e_n\}$", if for any $\varepsilon \in \{\pm 1\}^n$, we have $g(x) = g\left(\sum_i \varepsilon_i \langle x, e_i \rangle e_i\right)$. For a fixed orthonormal basis $\{e_1, .., e_n\}$ in $\mathbb{R}^n$, we denote by $\mathcal{S}_k^0$ the linear space of all invariant functions in $\mathcal{S}_k$. Let $Proj_{\mathcal{S}_k^0} : \mathcal{S}_k \to \mathcal{S}_k^0$ be the orthogonal projection in $L_2(S^{n-1})$. Then for $g \in \mathcal{S}_k$,

$$g'(x) = \mathbb{E}_\varepsilon \; g\left(\sum_{i=1}^n \varepsilon_i \langle x, e_i \rangle e_i\right) \quad \Longleftrightarrow \quad g' = Proj_{\mathcal{S}_k^0}(g),$$

i.e. the orthogonal symmetrization of g is the projection of g onto $\mathcal{S}_k^0$.

Lemma 3.1 *If k is odd, $dim(\mathcal{S}_k^0) = 0$. Otherwise,*

$$dim(\mathcal{S}_k^0) = \binom{n + \frac{k}{2} - 2}{n-2}.$$

Proof: The odd case is easy, since for $g \in \mathcal{S}_k$ we necessarily have $g(x) = -g(-x)$, and for $g \in \mathcal{S}_K^0$ we have $g(x) = g(-x)$. Hence, only $0 \in \mathcal{S}_k^0$. Next, assume that k is even, and let $g \in \mathcal{S}_k^0$ be an invariant polynomial with respect to the basis $\{e_1, .., e_n\}$. We use the coordinates $x_1, .., x_n$ with respect to this basis. Fixing $x_2, .., x_n$ the polynomial g satisfies $g_{x_2,..,x_n}(x_1) = g_{x_2,..,x_n}(-x_1)$, and hence only even degrees of x_1 occur in $g_{x_2,..,x_n}$. By repeating the argument for the rest of the variables, we get that g is a function of $x_1^2, .., x_n^2$ alone. We can write,

$$g(x_1, .., x_n) = \sum_{j=0}^{k/2} x_n^{2j} A_j(x_1, .., x_{n-1}) \tag{4}$$

where A_j is a homogeneous polynomial of degree $k - 2j$, which depends solely on $x_1^2, .., x_{n-1}^2$. Let us calculate the Laplacian of (4):

$$0 = \sum_{j=1}^{k/2} 2j(2j-1)x_n^{2j-2} A_j(x_1, .., x_{n-1}) + \sum_{j=0}^{k/2-1} x_n^{2j} \triangle A_j(x_1, .., x_{n-1})$$

or equivalently, $g \in \mathcal{S}_k^0$ if and anly if for all $0 \le j \le \frac{k}{2} - 1$,

$$(2j+2)(2j+1)A_{j+1} = -\triangle A_j. \tag{5}$$

Therefore we are free to choose A_0 any way we like, as long as it is a homogeneous polynomial of degree k, which involves only even powers of the $n-1$ variables. When A_0 is fixed, A_1, A_2 etc. are determined by equation (5), and the function g is recovered.

Hence, $dim(\mathcal{S}_k^0)$ equals the dimension of the space of the possible $A_0(x_1^2, .., x_{n-1}^2)$, which is the dimension of the space of all homogeneous polynomials of degree $k/2$ in $n-1$ variables. This number is known to be $\binom{n + \frac{k}{2} - 2}{n-2}$. □

We denote $N_k = dim(\mathcal{S}_k) = \binom{n+k-2}{n-2} \frac{n+2k-2}{n+k-2}$, and for an even k denote $N_k^0 = dim(\mathcal{S}_k^0) = \binom{n+k/2-2}{n-2}$. Clearly, these two quantities depend on n which is absent from the notation, yet the appropriate value of n will be obvious from the context. We are now ready to calculate the L_2 norm of a "random orthogonal symmetrization" of a spherical harmonic - an orthogonal symmetrization with respect to a basis that is chosen uniformly over $O(n)$. Clearly, any "orthogonal symmetrization" of an odd degree spherical harmonic vanishes. The even case is treated in the following proposition.

Proposition 3.2 *Let k be a positive even integer, and let $g \in \mathcal{S}_k$ be a spherical harmonic. We randomly select an orthonormal basis $\{v_1, .., v_n\} \in O(n)$, and symmetrize g with respect to this basis. Then,*

$$\mathbb{E}\|g'_{v_1,..,v_n}\|_2^2 = \frac{N_k^0}{N_k} \|g\|_2^2 < \left(\frac{k}{n-2+k}\right)^{k/2} \|g\|_2^2$$

where the expectation is over the random choice of $\{v_1, .., v_n\} \in O(n)$ (with respect to the Haar probability measure on $O(n)$).

Proof: Fix an orthonormal basis $\{e_1, .., e_n\}$ of $\mathbb{R}^n$, and consider $\mathcal{S}_k^0$ with respect to that basis. Fix also an orthonormal basis $S_1, .., S_{N_k^0}$ of $\mathcal{S}_k^0$. From the discussion before Lemma 3.1,

$$g'_{e_1,..,e_n} = Proj_{\mathcal{S}_k^0}(g)$$

and if the columns of $U \in O(n)$ are $\{v_1, .., v_n\}$, then

$$g'_{v_1,..,v_n} = \Big(Proj_{\mathcal{S}_k^0}(g \circ U)\Big) \circ U^{-1}.$$

Hence,

$$\|g'_{v_1,..,v_n}\|_2^2 = \|Proj_{\mathcal{S}_k^0}(g \circ U)\|_2^2 = \sum_{j=1}^{N_k^0} \left(\int_{S^{n-1}} g(Ux) S_j(x) d\sigma(x) \right)^2$$

and by Lemma 2.2,

$$\mathbb{E}\|g'_{v_1,..,v_n}\|_2^2 = \frac{\sum_{j=1}^{N_k^0} \|g\|_2^2}{N_k} = \frac{N_k^0}{N_k}\|g\|_2^2.$$

Note that

$$\frac{N_k^0}{N_k} = \frac{n+k-2}{n+2k-2} \prod_{i=1}^{k/2} \frac{(n+i-2)(k/2+i)}{(n+2i-3)(n+2i-2)} < \prod_{i=1}^{k/2} \frac{k/2+i}{k/2+i+n-2}$$

which lies between $\left(\frac{k/2}{n-2+k/2}\right)^{k/2}$ and $\left(\frac{k}{n-2+k}\right)^{k/2}$. □

Since $\left(\frac{k}{n-2+k}\right)^{k/2}$ is a decreasing function of k, then $\left(\frac{k}{n-2+k}\right)^{k/2} \leq \frac{2}{n}$ for any $k \geq 2$, and we obtain the following corollary:

Corollary 3.3 *Let $f \in L_2(S^{n-1})$ satisfy $\int_{S^{n-1}} f(x) d\sigma(x) = 0$. We randomly select $\{v_1, .., v_n\} \in O(n)$. Then,*

$$\mathbb{E}\|f'_{v_1,..,v_n}\|_2 < \frac{c}{\sqrt{n}}\|f\|_2$$

where the expectation is taken over the choice of $\{v_1, .., v_n\} \in O(n)$, and $c = \sqrt{2}$.

Proof: Expand f into spherical harmonics: $f = \sum_{k=1}^{\infty} f_k$ where $f_k = Proj_{\mathcal{S}_k}(f)$. Then $f'_{v_1,..,v_n} = \sum_{k=1}^{\infty} (f_k)'_{v_1,..,v_n}$ and

$$\mathbb{E}\|f'_{v_1,..,v_n}\|_2^2 = \sum_{k=2}^{\infty} \mathbb{E}\| (f_k)'_{v_1,..,v_n} \|_2^2 \leq \sum_k \frac{2}{n}\|f_k\|_2^2 \leq \frac{2}{n}\|f\|_2^2.$$

An application of Jensen inequality concludes the proof. □

Remark: Using similar methods, one can prove that if $g \in \mathcal{S}_k$ and $\tau_u(g)(x) = \frac{g(x)+g(\pi_u(x))}{2}$, then

$$\mathbb{E}_u\|\tau_u(g)\|_2^2 = \frac{n-2+k}{n-2+2k}\|g\|_2^2.$$

Note the advantage of symmetrizing with respect to the n vectors of a random orthonormal basis, compared to symmetrization with respect to n random sphere vectors. For instance, if $k = 2$ then

$$\left(\frac{n-2+k}{n-2+2k}\right)^n \approx \frac{1}{e^2}.$$

Hence n random symmetrizations may reduce the expectation of the L_2 norm only by a constant factor.

4 Decay of L_∞ norm

In Proposition 3.2 and Corollary 3.3 we established a sharp estimate for the decay of the L_2 norm under an "orthogonal symmetrization". Now we deal with the more difficult problem of estimating the decay of the L_∞ norm of the function. Our main tool is the following known lemma (see e.g. page 14 of [Mu]):

Lemma 4.1 *Let $g \in \mathcal{S}_k$ be a spherical harmonic of degree k. Then,*

$$\|g\|_\infty \le \sqrt{dim(\mathcal{S}_k)}\|g\|_2 = \sqrt{N_k}\|g\|_2$$

where $\|g\|_\infty = \sup_{x \in S^{n-1}} |g(x)|$.

We make use of the following well-known estimate of binomial coefficients. For any $1 \le k \le n$,

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} < \left(e\frac{n}{k}\right)^k. \tag{6}$$

In the following combinatorial lemmas, "log" is to be understood as the natural logarithm.

Lemma 4.2 *Let $\varepsilon > 0$, $n \ge 3$, and let $k \ge 2$ be an integer. Then,*

$$N_k^{c_1 \frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{k}{n}\right)}} > \frac{n}{\varepsilon^3}$$

where $c_1 > 0$ is some numerical constant.

Proof: Denote $\alpha = k/n$.

Case 1: $\alpha < 2$. In this case, $1 + \log\left(1 + \frac{k}{n}\right) < 3$, and for $c_1 > 9$,

$$N_k^{c_1 \frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{k}{n}\right)}} > N_k^{3+3\log\left(1+\frac{2}{\varepsilon}\right)} > N_k \cdot N_k^{3\log\left(1+\frac{2}{\varepsilon}\right)} > \frac{N_k}{\varepsilon^3} \ge \frac{n}{\varepsilon^3}$$

since for $k \ge 2$ we always have $N_k \ge n \ge 3$.

Case 2: $\alpha \geq 2$. In this case, $1 + \log\left(1 + \frac{k}{n}\right) < 2\log\left(1 + \frac{k}{n-2}\right)$. By (6), $N_k > \left(\frac{n+k-2}{n-2}\right)^{n-2}$. For $c_1 > 6$,

$$N_k^{c_1 \frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{k}{n}\right)}} > \left(\left(1 + \frac{k}{n-2}\right)^{n-2}\right)^{\frac{3+3\log\left(1+\frac{2}{\varepsilon}\right)}{\log\left(1+\frac{k}{n-2}\right)}}$$

$$= e^{3(n-2)}\left(1 + \frac{2}{\varepsilon}\right)^{3(n-2)} > \frac{n}{\varepsilon^3}$$

for any $n \geq 3$. □

Lemma 4.3 *Let $n \geq 3$, and let $k = \alpha n > 0$ be an even number. Then,*

$$\left(\frac{N_k^0}{N_k}\right)^T < \frac{1}{N_k}$$

for $T = c_2\left[1 + \log(1 + \alpha)\right]$, where $c_2 > 0$ is a numerical constant.

Proof: Since $\frac{n+2k-2}{n+k-2} > 1$, it is sufficient to prove that

$$\left(\frac{\binom{n+k/2-2}{k/2}}{\binom{n+k-2}{k}}\right)^T < \frac{1}{\binom{n+k-2}{k}}. \tag{7}$$

Case 1: $\alpha < \frac{1}{2}$. The left hand side of (7) is equal to:

$$\left(\prod_{i=1}^{k/2}\frac{k/2+i}{k/2+i+n-2}\right)^T < \left(\frac{k}{k+n-2}\right)^{\frac{kT}{2}} \leq \left(\frac{k}{n}\right)^{\frac{kT}{2}}.$$

To obtain (7) it is enough to prove that

$$\left(\frac{k}{n}\right)^{\frac{kT}{2}} < \left(\frac{1}{e}\frac{k}{k+n-2}\right)^k$$

according to (6). Now, because $\alpha = \frac{k}{n} < \frac{1}{2}$, for $T = 8$,

$$\left(\frac{k}{n}\right)^{\frac{kT}{2}} < \left(\frac{k}{n}\right)^k\left(\frac{1}{2}\right)^{\frac{k(T-2)}{2}} < \left(\frac{2}{3e}\frac{k}{n}\right)^k < \left(\frac{1}{e}\frac{k}{k+n-2}\right)^k.$$

Case 2: $\alpha \geq \frac{1}{2}$. Since $\binom{m}{l} = \binom{m}{m-l}$, the left hand side of (7) also equals:

$$\left(\prod_{i=1}^{n-2}\frac{k/2+i}{k+i}\right)^T < \left(\frac{n-2+k/2}{n-2+k}\right)^{(n-2)T} < \left(\frac{5}{6}\right)^{(n-2)T}$$

since $n - 2 < 2k$ and because $\frac{x+k/2}{x+k}$ is an increasing function of x. Now, for any $T > \frac{1+\log(1+\frac{k}{n-2})}{\log(6/5)}$,

$$\left(\frac{5}{6}\right)^{(n-2)T} < \left(\frac{1}{e}\frac{n-2}{n+k-2}\right)^{(n-2)} < \frac{1}{\binom{n+k-2}{n-2}}.$$

Since for $n \geq 3$, we have $\frac{1+\log(1+\frac{k}{n-2})}{\log(6/5)} < 10\left[1 + \log(1 + \alpha)\right]$, the lemma is proved. □

5 Proof of the Minkowski symmetrization result

We make use of Jackson's theorem for the sphere, due to Newman and Shapiro [NS]:

Theorem 5.1 *Let $n, k > 0$ be integers, and let $f : S^{n-1} \to \mathbb{R}$ be a λ-Lipschitz function on the sphere (i.e. $|f(x) - f(y)| \le \lambda|x - y|$ for any $x, y \in S^{n-1}$). Then there exists a polynomial P_k of degree k in n variables, such that for any $x \in S^{n-1}$,*

$$|f(x) - P_k(x)| \le c_3 \lambda \frac{n}{k}$$

where $c_3 > 0$ is some numerical constant.

Proof of Theorem 1.3: We assume that $M^*(K) = 1$. Begin with $5n$ symmetrizations, according to Theorem 1.2, to obtain a centrally-symmetric body $\bar{K}$. Denote by h its supporting functional. Then h is a norm and hence its Lipschitz constant equals $\sup_{x \in S^{n-1}} h(x)$. By Theorem 1.2,

$$\sup_{x \in S^{n-1}} h(x) < 1 + c\frac{|\log\log n|}{\sqrt{\log n}} < c_4 \tag{8}$$

for some numerical constant $c_4 > 0$. Hence h is a c_4-Lipschitz function, and by Theorem 5.1, there exists a polynomial $P_\varepsilon(x)$ of degree $k = \lceil \frac{n}{\varepsilon} \rceil$ such that,

$$\sup_{x \in S^{n-1}} |P_\varepsilon(x) - h(x)| < c_4 c_3 \varepsilon. \tag{9}$$

Let $P_\varepsilon(x) = \sum_{i=0}^k P_i(x)$ be the expansion of P_ε into spherical harmonics. Randomly select T orthonormal bases (i.e. the bases are chosen independently and uniformly in $O(n)$). Apply the corresponding T orthogonal symmetrizations to P_ε and $P_1, .., P_k$, to obtain the random polynomials P'_ε and $P'_1, .., P'_k$. Note that still $P'_\varepsilon = \sum_{i=0}^k P'_i$. Successive application of Proposition 3.2 yields that for an even $i > 0$,

$$\mathbb{E}\|P'_i\|_2^2 = \left(\frac{N_i^0}{N_i}\right)^T \|P_i\|_2^2.$$

Combining this with Lemma 4.1 (assume $n \ge 3$),

$$\mathbb{E}\|P'_i\|_\infty^2 \le N_i \left(\frac{N_i^0}{N_i}\right)^T \|P_i\|_2^2.$$

Assume that $T > (c_1 + 1)c_2 \left[1 + \log\left(1 + \frac{2}{\varepsilon}\right)\right]$. According to Lemma 4.3,

$$\mathbb{E}\|P'_i\|_\infty^2 < N_i \left(\frac{1}{N_i}\right)^{(c_1+1)\frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{i}{n}\right)}} \|P_i\|_2^2$$

$$< N_i^{-c_1\frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{i}{n}\right)}} \|P_i\|_2^2 < \frac{\varepsilon^3}{n}\|P_i\|_2^2$$

where the last inequality follows from Lemma 4.2. Denote $I = P_0 = \int_{S^{n-1}} P_\varepsilon(x) d\sigma(x)$. Then,

$$\mathbb{E}\|P'_\varepsilon(x) - I\|_\infty \leq \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \mathbb{E}\|P'_{2i}(x)\|_\infty \leq \sqrt{\frac{k}{2} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \mathbb{E}\|P'_{2i}(x)\|_\infty^2}$$

$$\leq \sqrt{\frac{1}{2} \left\lceil \frac{n}{\varepsilon} \right\rceil \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{\varepsilon^3}{n} \|P_{2i}(x)\|_2^2} < \varepsilon \|P_\varepsilon\|_2 < \varepsilon \|P_\varepsilon\|_\infty < \varepsilon(c_4 + c_4 c_3 \varepsilon)$$

where the last inequality follows from (8) and (9). Apply the same T orthogonal symmetrizations to $\bar{K}$, and obtain K'. Denote by h' the supporting functional of K'. Then,

$$\sup_{x \in S^{n-1}} |h'(x) - P'_\varepsilon(x)| < c_4 c_3 \varepsilon,$$

and since by (9) we have $1 - c_4 c_3 \varepsilon < I < 1 + c_4 c_3 \varepsilon$, then

$$\mathbb{E} \sup_{x \in S^{n-1}} |h'(x) - 1| < c_4 c_3 \varepsilon + c_4 c_3 \varepsilon + \varepsilon(c_4 + c_4 c_3 \varepsilon) < c' \varepsilon.$$

Clearly,

$$\sup_{x \in S^{n-1}} |h'(x) - 1| < c'\varepsilon \quad \Rightarrow \quad (1 - c'\varepsilon) D \subset K' \subset (1 + c'\varepsilon) D.$$

To summarize, we applied $5n + (c_1 + 1)c_2 \left[1 + \log\left(1 + \frac{2}{\varepsilon}\right)\right] n$ Minkowski symmetrizations to an arbitrary convex body, some of which were chosen randomly. As a result of these symmetrizations, we obtained a body such that the expectation of its distance to a Euclidean ball is no more than $c'\varepsilon$. Therefore, there exists some numerical constant $c > 0$, and $cn \log \frac{1}{\varepsilon}$ symmetrizations that bring the body to be ε-close to a Euclidean ball. $\square$

Remarks:

1. The case $n = 2$ should be treated separately. In this case, $dim(\mathcal{S}_k) = 2, dim(\mathcal{S}_k^\circ) = 1$ for any k. It is easy to verify that the proof works in this case as well.

2. Theorem 1.3 is optimal in the sense that one cannot obtain an estimate for the number of minimal symmetrizations, of the form $f(n)g(\varepsilon)$ with $f(n) << n$ or $g(\varepsilon) << \log \frac{1}{\varepsilon}$. Indeed, the dependence on n should be at least linear, as it takes a segment $n - 1$ symmetrizations just to become n-dimensional. Regarding the dependence on ε, if we take a segment and apply any $\lfloor c \log \frac{1}{\varepsilon} \rfloor$ symmetrizations, then the segment is transformed into a zonotope which is a sum of no more than $\frac{1}{\varepsilon^c}$ segments. Even in dimension two, this zonotope cannot be ε-close to a Euclidean ball, for a small enough c.

3. Note that Theorem 1.3 is not tight for all possible values of n and ε. For example, Theorem 1.2 is better than Theorem 1.3 when $\varepsilon = c\frac{\lfloor \log \log n \rfloor}{\sqrt{\log n}}$.

6 Application to Steiner symmetrization

In this section we prove Theorem 1.5. We make use of a result due to Bokowski and Heil. The following theorem is a special case of Theorem 2 in [BH] (the case $(i,j,k) = (0, d-1, d)$ in the notations of that paper).

Theorem 6.1 *Let $K \subset RD$ be a convex body. Then,*

$$n^2 R^{n-1} M^*(K) \leq \frac{Vol(K)}{Vol(D)} + (n^2-1)R^n.$$

An immediate corollary follows:

Corollary 6.2 *Let $\varepsilon > 0$, and let $K \subset (1+\varepsilon)D$ be a convex body in $\mathbb{R}^n$ with $Vol(K) = Vol(D)$. Then,*

$$M^*(K) < 1 + \left(1 - \frac{1}{n^2}\right)\varepsilon.$$

In addition, if $\varepsilon < \frac{1}{n}$ then,

$$M^*(K) < 1 + \left(1 - \frac{1}{2n}\right)\varepsilon.$$

Proof: By Theorem 6.1, since $\frac{Vol(K)}{Vol(D)} = 1$,

$$M^*(K) \leq (1+\varepsilon)\left(1 - \frac{1}{n^2}\right) + \frac{1}{n^2(1+\varepsilon)^{n-1}} < (1+\varepsilon)\left(1 - \frac{1}{n^2}\right) + \frac{1}{n^2}$$

and therefore $M^*(K) < 1 + \left(1 - \frac{1}{n^2}\right)\varepsilon$. Now, assume that $\varepsilon < \frac{1}{n}$. Using the elementary inequality $\frac{1}{(1+\varepsilon)^{n-1}} < 1 - (n-1)\varepsilon + \frac{n(n-1)}{2}\varepsilon^2$, we obtain

$$M^*(K) \leq (1+\varepsilon)\left(1 - \frac{1}{n^2}\right) + \frac{1}{n^2}\left[1 - (n-1)\varepsilon + \frac{n(n-1)}{2}\varepsilon^2\right]$$

$$< 1 + \varepsilon - \frac{\varepsilon}{n} + \frac{\varepsilon^2}{2} < 1 + \varepsilon - \frac{\varepsilon}{2n}.$$

□

Given a convex body $K \subset \mathbb{R}^n$, define $R(K) = \inf\{R > 0; K \subset RD\}$.

Lemma 6.3 *Let $K \subset \mathbb{R}^n$ be a convex body with $Vol(K) = Vol(D)$. Assume that there exists $0 < \varepsilon < C$ such that $R(K) = 1 + \varepsilon$, where $C > 1$. Then there exist $c_5 n(\log \frac{1}{\varepsilon} + \log n)$ Steiner symmetrizations that transform K into $\tilde{K}$ such that*

$$R(\tilde{K}) < 1 + \left(1 - \frac{1}{2n^2}\right)\varepsilon$$

and if $\varepsilon < \frac{1}{n}$,

$$R(\tilde{K}) < 1 + \left(1 - \frac{1}{4n}\right)\varepsilon$$

where $c_5 = c_5(C) > 0$ *depends solely on* C.

Proof: Let $\tilde{K}$ be the body obtained from K after the $cn \log \frac{4Cn^3}{\varepsilon}$ symmetrizations given by Theorem 1.3. Despite the fact that Theorem 1.3 is concerned with Minkowski symmetrizations, we apply the corresponding Steiner symmetrization (with respect to the same hyperplanes). Since Steiner symmetrizations are contained in Minkowski symmetrizations,

$$R(\tilde{K}) < \left(1 + \frac{\varepsilon}{4Cn^3}\right) M^*(K).$$

Apply corollary 6.2 and the fact that $\varepsilon < C$ to get that

$$R(\tilde{K}) < \left(1 + \frac{\varepsilon}{4Cn^3}\right)\left[1 + \left(1 - \frac{1}{n^2}\right)\varepsilon\right] < 1 + \left(1 - \frac{1}{2n^2}\right)\varepsilon$$

and if $\varepsilon < \frac{1}{n}$,

$$R(\tilde{K}) < \left(1 + \frac{\varepsilon}{4Cn^3}\right)\left[1 + \left(1 - \frac{1}{2n}\right)\varepsilon\right] < 1 + \left(1 - \frac{1}{4n}\right)\varepsilon.$$

□

Proposition 6.4 *Let* $n \geq 2$, $0 < \varepsilon < \frac{1}{2}$, *and let* $K \subset \mathbb{R}^n$ *be a convex body with* $Vol(K) = Vol(D)$. *Then there exist* $c_6 \left[n^3 \log^2 n + n^2 \log^2 \frac{1}{\varepsilon}\right]$ *Steiner symmetrizations, that transform* K *into* $\tilde{K}$ *which satisfies*

$$R(\tilde{K}) < 1 + \varepsilon$$

where $c_6 > 0$ *is a numerical constant.*

Proof: First, apply $3n$ Steiner symmetrizations to K, according to Theorem 1.4, to obtain an isomorphic Euclidean ball $\bar{K}$. Then,

$$\bar{K} \subset CD.$$

Let us define a sequence of convex bodies: $K_0 = \bar{K}$, and K_i is obtained from K_{i-1} using $c_5 n(\log \frac{1}{R(K_{i-1})-1} + \log n)$ Steiner symmetrizations, as in Lemma 6.3. Then,

$$R(K_i) - 1 < \left(1 - \frac{1}{2n^2}\right)\left[R(K_{i-1}) - 1\right] < \left(1 - \frac{1}{2n^2}\right)^i \left[R(K_0) - 1\right]. \tag{10}$$

Let T_1 be the minimal integer such that

$$R(K_{T_1}) < 1 + \frac{1}{n}.$$

Since $R(K_0) < C$, then by (10) necessarily $T_1 < cn^2 \log n$. For any $i \le T_1$ we have $R(K_{i-1}) \ge 1 + \frac{1}{n}$ and hence by Lemma 6.3 we used no more than $c'n \log n$ symmetrizations to obtain K_i from K_{i-1}. In total, we used less than $\tilde{c}n^3 \log^2 n$ symmetrizations to obtain K_{T_1}. By Lemma 6.3 for any $i > 0$,

$$R(K_{T_1+i}) - 1 < \left(1 - \frac{1}{4n}\right)[R(K_{T_1+i-1}) - 1] < \left(1 - \frac{1}{4n}\right)^i.$$

Let T_2 be the first integer such that

$$R(K_{T_1+T_2}) < 1 + \varepsilon.$$

Then $T_2 < cn \log \frac{1}{\varepsilon}$. Define $\tilde{K} = K_{T_1+T_2}$. For any $T_1 < i \le T_1 + T_2$ we used no more than $c'n(\log \frac{1}{\varepsilon} + \log n)$ symmetrizations to obtain K_i from K_{i-1}. In total we applied a maximum of $\tilde{c}n^3 \log^2 n + \tilde{c}n^2 \log^2 \frac{1}{\varepsilon}$ Steiner symmetrizations. $\square$

Proposition 6.4 proves the existence of a rather small circumscribing ball for the symmetrized body. In order to symmetrize the body from below, we use the following standard lemma. Its proof is outlined for completeness.

Lemma 6.5 *Let* $0 < \varepsilon < 1$*, and let* $K \subset \mathbb{R}^n$ *be a convex body with* $M^*(K) \ge 1$*. Assume that* $K \subset [1 + (c_7\varepsilon)^n]\, D$*. Then*

$$(1-\varepsilon)D \subset K$$

where $c_7 > 0$ *is some numerical constant.*

Proof: Assume on the contrary that there exists $x_0 \in S^{n-1}$ with $\|x_0\|_* < 1 - \varepsilon$, where $\|\cdot\|_* = h_K(\cdot)$. Then for $x \in S^{n-1}$ with $|x - x_0| < \frac{\varepsilon}{4}$, we have

$$\|x\|_* \le \|x_0\|_* + \|x - x_0\|_* < 1 - \varepsilon + (1 + (c_7\varepsilon)^n)|x - x_0| < 1 - \frac{\varepsilon}{2}.$$

Denote $A = \{x \in S^{n-1}; |x - x_0| \le \frac{\varepsilon}{4}\}$. Then,

$$M^*(K) = \int_{S^{n-1}} \|x\|_* d\sigma(x) < (1 - \sigma(A))\,(1 + (c_7\varepsilon)^n) + \sigma(A)\left(1 - \frac{\varepsilon}{2}\right).$$

The projection of A onto the hyperplane orthogonal to x_0 contains a Euclidean ball of radius larger than $\frac{\varepsilon}{4\sqrt{2}}$. Therefore,

$$\sigma(A) > \frac{Vol(D_{n-1})}{Vol(S^{n-1})}\left(\frac{\varepsilon}{4\sqrt{2}}\right)^{n-1} > \frac{1}{\sqrt{\pi}n}\left(\frac{\varepsilon}{4\sqrt{2}}\right)^{n-1} > \left(\frac{\varepsilon}{30}\right)^{n-1}$$

where D_{n-1} is the $n-1$ dimensional Euclidean unit ball, and Vol is interpreted here as the $n-1$ dimensional volume. Thus,

$$M^*(K) < 1 + (c_7\varepsilon)^n - \sigma(A)\frac{\varepsilon}{2} < 1 + (c_7\varepsilon)^n - \left(\frac{\varepsilon}{30}\right)^n$$

and for $c_7 = \frac{1}{30}$ we obtain a contradiction. □

Proof of Theorem 1.5: It is sufficient to consider the case $Vol(K) = Vol(D)$. Apply Proposition 6.4 with $\varepsilon' = (c_7\varepsilon)^n$. We use

$$c_6 \left[n^3 \log^2 n + n^2 \log^2 \frac{1}{\varepsilon'} \right] < c' n^4 \log^2 \frac{1}{\varepsilon}$$

Steiner symmetrizations, and obtain a body $\tilde{K}$ such that

$$\tilde{K} \subset (1 + (c_7\varepsilon)^n)\, D.$$

Since $Vol(K) = Vol(D)$, by Urysohn $M^*(K) \geq 1$. Using Lemma 6.5,

$$(1-\varepsilon)D \subset \tilde{K} \subset (1 + (c_7\varepsilon)^n)\, D \subset (1+\varepsilon)D$$

and the theorem is proved. □

Remark: Theorem 1.2 is crucial to the proof of Theorem 1.3. Only after obtaining the precise isomorphic statement regarding Minkowski symmetrization, can we prove the sharp almost isometric version. However, in the proof of Theorem 1.5 we may apply weaker estimates than that in Theorem 1.4, and derive the same conclusion. This could be another indication that the powers in Theorem 1.5 are not optimal.

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Part II: The Slicing Problem

A reduction of the slicing problem to finite volume ratio bodies[5]

Abstract. We investigate the effect of a Steiner type symmetrization on the isotropic constant of a convex body. We reduce the problem of bounding the isotropic constant of an arbitrary convex body, to the problem of bounding the isotropic constant of a finite volume ratio body. We also add two observations concerning the slicing problem. The first is the equivalence of the problem to a reverse Brunn-Minkowski inequality in isotropic position. The second is the essential monotonicity in n of $L_n = \sup_{K\subset\mathbb{R}^n} L_K$ where the supremum is taken over all convex bodies in $\mathbb{R}^n$, and L_K is the isotropic constant of K.

1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex body whose barycenter is at the origin (i.e. $b(K) = \int_K \overrightarrow{x}\,dx = 0$). The inertia matrix of K is the matrix M_K whose entries are $M_{i,j} = \int_K x_i x_j dx$. The isotropic constant of K, denoted by L_K, is defined as

$$L_K^2 = \frac{det(M_K)^{\frac{1}{n}}}{Vol(K)^{1+\frac{2}{n}}}.$$

The isotropic constant is invariant under linear transformations of the body. If M_K is a scalar matrix and $Vol(K) = 1$, we say that K is isotropic, or that K is in isotropic position. In this case, for any $\theta \in \mathbb{R}^n$,

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 |\theta|^2$$

where $|\cdot|$ is the standard Euclidean norm in $\mathbb{R}^n$. Any convex body K has a unique affine image of volume one which is in isotropic position. We refer the reader to [MP1] for more information concerning the isotropic position and the isotropic constant.

A major unsolved problem asks whether there exists a numerical constant C such that $L_K < C$ for every convex body in any finite dimension. This problem is called the slicing problem or the hyperplane conjecture. A positive answer to this question has many interesting consequences, see [MP1]. One of these is that every convex body of volume one, has an $n-1$ dimensional section whose $n-1$ dimensional volume is greater than some constant $c > 0$. The current best estimate is $L_K < cn^{1/4}\log n$, for an arbitrary convex body $K \subset \mathbb{R}^n$ (see [Bou2], or the presentation in [Dar]. See [Pa] for the non-symmetric case). For certain classes of convex bodies the question is affirmatively answered, such as for unconditional bodies (as observed by Bourgain, see [MP1]), zonoids, duals of zonoids (see [Ba2], also for

[5]This chapter corresponds to the paper [BKM2].

the connection with the Gordon-Lewis constant), duals to bodies with finite volume ratio (see [MP1]), and more (e.g. [Ju]). Here, we present a reduction of the general problem to the boundness of the isotropic constant of a certain class of convex bodies: those which have a finite volume ratio. For $K \subset \mathbb{R}^n$, the volume ratio of K is defined as,

$$v.r.(K) = \sup_{\mathcal{E} \subset K} \left(\frac{Vol(K)}{Vol(\mathcal{E})} \right)^{\frac{1}{n}}$$

where the supremum is over all ellipsoids contained in K. Here we prove the following conditional proposition:

Proposition 1.1 *There exists $v > 1$ such that the following holds:*

If there exists $c_1 > 0$ such that for any n and for any $K \subset \mathbb{R}^n$, the inequality $v.r.(K) < v$ implies that $L_K < c_1$,

then there exists $c_2 > 0$ such that for any n and for any $K \subset \mathbb{R}^n$ we have $L_K < c_2$.

Next, we shall state a qualitative version of Proposition 1.1. Denote $L_n = \sup_{K \subset \mathbb{R}^n} L_K$ where the supremum is over all convex sets in $\mathbb{R}^n$, and define

$$L_n(a) = \sup\{L_K \ ; \ K \subset \mathbb{R}^n \ , \ v.r.(K) \le a\}.$$

Then we can bound L_n by a function of $L_n(a)$ for a suitable $a > 1$. As a matter of fact, this function is almost linear:

Proposition 1.2 *For any $\delta > 0$, there exist numbers $v(\delta) > 1$, $c(\delta) > 0$ such that for any n,*

$$L_n < c(\delta) \ L_n(v(\delta))^{1+\delta}.$$

A proof of these propositions, using a symmetrization technique, is presented in Section 4. The technique itself is presented in Section 2. We prove the following proposition in Section 3.

Proposition 1.3 *If $m < n$, then $L_m < cL_n$ where c is a numerical constant.*

As observed by K. Ball (see [MP1]), the hyperplane conjecture implies that a reverse Brunn-Minkowski inequality holds in the isotropic position. Answering a question posed by K. Ball to one of the authors, we show that the slicing problem is actually equivalent to a reverse Brunn-Minkowski inequality in the isotropic position. The following conditional statement is proved in Section 5:

Proposition 1.4 *Assume that there exists a constant $C > 0$, such that for any n, and for any two isotropic convex bodies $K, T \subset \mathbb{R}^n$,*

$$Vol(K+T)^{1/n} \leq C\left(Vol(K)^{1/n} + Vol(T)^{1/n}\right). \tag{1}$$

Then it follows that for any convex body $K \subset \mathbb{R}^n$,

$$L_K < C'(C)$$

where $C'(C)$ is a number that depends solely on C.

Actually, Proposition 1.4 is correct even if we restrict T to be a Euclidean ball, as is evident from the proof. Note that as proved in [M4], inequality (1) which is a reverse Brunn-Minkowski inequality, holds when K and T are in a special position called M-position (see definition in Section 3). However, the connection of an M-position with the isotropic position is not yet clear.

Throughout the text we denote by $c, c', \tilde{c}, C$ etc. some positive universal constants whose value is not necessarily the same on different appearances. Whenever we write $A \approx B$, we mean that there exist universal constants $c, c' > 0$ such that $cA < B < c'A$. Also, $Vol(T)$ denotes the volume of a set $T \subset \mathbb{R}^n$, relative to its affine hull.

The paper [BKM1] serves as an extended introduction to this text.

2 Symmetrization

2.1 Definition

Let $K \subset \mathbb{R}^n$ be a convex body, let $E \subset \mathbb{R}^n$ be a subspace of dimension k, and let $T \subset E$ be a k-dimensional convex body, whose barycenter is at the origin. We define the "(T, E)-symmetrization" of K as the unique body K' such that:

(i) for any $x \in E^{\perp}$, $Vol(K \cap (x+E)) = Vol(K' \cap (x+E))$.

(ii) for any $x \in E^{\perp}$ the body $K' \cap (x+E)$ is homothetic to T, and its barycenter lies in $E^{\perp}$.

In other words, we replace any section of K which is parallel to E, with a homothetic copy of T of the appropriate volume. This procedure of symmetrization is known in convexity, see [BF], page 79. For completeness, we shall next prove that this symmetrization preserves convexity, as follows from Brunn-Minkowski inequality.

Lemma 2.1 *K' is a convex body.*

Proof: For any $z \in E^{\perp}$, the section $(z+E) \cap K'$ is convex, as a homothetic copy of T. Let $x, y \in Proj_{E^{\perp}}(K')$ be any points, where $Proj_{E^{\perp}}$ is the orthogonal projection onto $E^{\perp}$ in $\mathbb{R}^n$. We will show that

$$conv((x+E) \cap K', (y+E) \cap K')$$

$$= \bigcup_{0 \le \lambda \le 1} \lambda \left[(x+E) \cap K'\right] + (1-\lambda)\left[(y+E) \cap K'\right] \subset K'.$$

For $z \in E^{\perp}$, denote $v(z) = Vol((z+E) \cap K') = Vol((z+E) \cap K)$. Since K is convex, by Brunn-Minkowski (e.g. [BF]),

$$v(\lambda x + (1-\lambda)y)^{1/k} \ge \lambda v(x)^{1/k} + (1-\lambda) v(y)^{1/k} \tag{2}$$

where $k = dim(E)$. Since $(z+E) \cap K' = z + \left(\frac{v(z)}{Vol(T)}\right)^{1/k} T$ for any point $z \in E^{\perp}$, inequality (2) entails that

$$(\lambda x + (1-\lambda)y + E) \cap K' \supset \lambda \left[(x+E) \cap K'\right] + (1-\lambda)\left[(y+E) \cap K'\right]$$

and the lemma is proved. $\square$

2.2 The effect of a symmetrization on the isotropic constant

Let us determine the eigenvectors of the inertia matrix $M_{K'}$. These eigenvectors are also called axes of inertia of the body K'. If K is an arbitrary body of volume one with its barycenter at zero, and $\{e_1, .., e_n\}$ are its axes of inertia, then since $L_K^2 = det(M_K)^{1/n}$,

$$L_K^2 = \left(\prod_{i=1}^{n} \int_K \langle x, e_i \rangle^2 dx \right)^{\frac{1}{n}}.$$

Lemma 2.2 *Assume that K is isotropic. Let $e_1, .., e_k$ be axes of inertia of the body $T \subset E$, and let $e_{k+1}, .., e_n$ be any orthonormal basis of $E^{\perp}$. Then the orthonormal basis $\{e_1, .., e_n\}$ is a basis of inertia axes of K'.*

Proof: By property (i) from the symmetrization definition, for any $v \in E^{\perp}$

$$\int_{K'} \langle x, v \rangle^2 dx = \int_K \langle x, v \rangle^2 dx = L_K^2 |v|^2 \tag{3}$$

since K is isotropic. By property (ii), for any $v \in E^{\perp}, u \in E$,

$$\int_{K'} \langle x, v \rangle \langle x, u \rangle dx = \int_{Proj_{E^{\perp}}(K')} \langle y, v \rangle \int_{K' \cap [y+E]} \langle z, u \rangle dz dy = 0$$

since the barycenter of T is at zero. Hence, E and $E^{\perp}$ are invariant subspaces of $M_{K'}$. According to (3), the operator $M_{K'}$ restricted to $E^{\perp}$ is simply a multiple of the identity. Therefore any orthogonal basis $e_{k+1},..,e_n$ of $E^{\perp}$ is a basis of eigenvectors of $M_{K'}$. All that remains is to select k axes of inertia in E. Let $e_1,..,e_k$ be axes of inertia of the k-dimensional body T. It is straightforward to verify that for any $u_1,u_2 \in E$,

$$\int_{K'} \langle x,u_1\rangle\langle x,u_2\rangle dx = c(K,E,T)\int_T \langle x,u_1\rangle\langle x,u_2\rangle dx$$

where $c(K,E,T) = \frac{\int_{Proj_{E^{\perp}}(K)} Vol(K\cap(x+E))^{1+2/k}dx}{Vol(T)^{1+2/k}}$ depends only on K,E,T. Therefore $e_1,..,e_k$ are also axes of inertia of K'. □

We postpone the proof of the following lemma to Section 6.

Lemma 2.3 *Let f be a compactly supported non-negative function on $\mathbb{R}^n$, such that $f^{1/k}$ is concave on its support, and $\int_{\mathbb{R}^n} f(x)dx = 1$. Denote $M = \max_{x\in\mathbb{R}^n} f(x)$. Then,*

$$\frac{(k+1)(k+2)}{(n+k+1)(n+k+2)}M^{2/k} \le \int_{\mathbb{R}^n} f(x)^{1+\frac{2}{k}}dx \le M^{2/k}.$$

Now we can estimate L_2 norms of some linear functionals over K'.

Lemma 2.4 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin. Let $E \subset \mathbb{R}^n$ be a subspace with $dim(E) = k$, and let $T \subset E$ be a k-dimensional convex body of volume one with zero as a barycenter. Denote by K' the "(T,E)-symmetrization" of K. Then for any $v \in E$,*

$$\int_{K'} \langle x,v\rangle^2 dx \ge \left(\frac{k+1}{n+1}\right)^2 Vol(K\cap E)^{2/k}\int_T \langle x,v\rangle^2 dx$$

and,

$$\int_{K'} \langle x,v\rangle^2 dx \le \left(\frac{n+1}{k+1}\right)^2 Vol(K\cap E)^{2/k}\int_T \langle x,v\rangle^2 dx.$$

Proof:

$$\int_{K'} \langle x,v\rangle^2 dx = \int_{Proj_{E^{\perp}}(K')}\int_{K'\cap(E+x)} \langle y,v\rangle^2 dydx$$

$$= \int_{Proj_{E^{\perp}}(K')} Vol(K'\cap(E+x))^{1+\frac{2}{k}}dx\int_T \langle y,v\rangle^2 dy.$$

Denote $g(x) = Vol(K'\cap(x+E)) = Vol(K\cap(x+E))$. Then by Brunn-Minkowski inequality, $g^{1/k}$ is concave on its support in $E^{\perp}$ and $\int g = Vol(K) = 1$. By Lemma 2.3,

$$\frac{(k+1)(k+2)}{(n+1)(n+2)}M^{2/k}\int_T \langle y,v\rangle^2 dy \le \int_{K'} \langle x,v\rangle^2 dx \le M^{2/k}\int_T \langle y,v\rangle^2 dy$$

where $M = \max_{x \in E^\perp} g(x)$. Since the barycenter of K is at the origin, by Theorem 1 in [F],

$$g(0) \leq M \leq \left(\frac{n+1}{k+1}\right)^k g(0)$$

and since $g(0) = Vol(K \cap E)$, we get

$$\left(\frac{k+1}{n+1}\right)^2 \leq \frac{(k+1)(k+2)}{(n+1)(n+2)} \leq \frac{\int_{K'} \langle x, v\rangle^2 dx}{Vol(K \cap E)^{\frac{2}{k}} \int_T \langle x, v\rangle^2 dx} \leq \left(\frac{n+1}{k+1}\right)^2.$$

□

The following theorem connects the isotropic constant of the symmetrized body with the isotropic constants of K, T.

Theorem 2.5 *Let K be an isotropic body of volume one, E a subspace of dimension k, T a k-dimensional convex body with its barycenter at the origin, and K' the "(T,E)-symmetrization" of K. Then*

$$L_{K'} \;\approx\; L_K^{1-\frac{k}{n}} \; L_T^{k/n} \; Vol(K \cap E)^{1/n}.$$

In fact, the ratio of these two quantities is always between $\left(\frac{k+1}{n+1}\right)^{k/n}$ and $\left(\frac{n+1}{k+1}\right)^{k/n}$.

Proof: We may assume that $Vol(T) = 1$. Let $\{e_1, .., e_n\}$ be selected according to Lemma 2.2. Then,

$$L_{K'} = \left(\prod_{i=1}^n \sqrt{\int_{K'} \langle x, e_i\rangle^2 dx}\right)^{1/n} = L_K^{1-\frac{k}{n}} \left(\prod_{i=1}^k \sqrt{\int_{K'} \langle x, e_i\rangle^2 dx}\right)^{1/n}$$

where the right-most equality follows from (3). By Lemma 2.4,

$$\begin{aligned} L_{K'} \;&\geq\; L_K^{1-\frac{k}{n}} \left(\prod_{i=1}^k \sqrt{\left(\frac{k+1}{n+1}\right)^2 Vol(K \cap E)^{\frac{2}{k}} \int_T \langle x, e_i\rangle^2 dx}\right)^{1/n} \\ &=\; L_K^{1-\frac{k}{n}} \left(\frac{k+1}{n+1}\right)^{\frac{k}{n}} Vol(K \cap E)^{\frac{1}{n}} L_T^{\frac{k}{n}} \end{aligned}$$

since the vectors $e_1, .., e_k$ are inertia axes of T. Therefore,

$$L_{K'} > c L_K^{1-\frac{k}{n}} L_T^{k/n} Vol(K \cap E)^{1/n}.$$

Regarding the inverse inequality, according to the opposite inequality in Lemma 2.4 we get,

$$\begin{aligned} L_K' &\leq \left(\frac{n+1}{k+1}\right)^{\frac{k}{n}} L_K^{1-\frac{k}{n}} Vol(K \cap E)^{\frac{1}{n}} L_T^{\frac{k}{n}} \\ &< c L_K^{1-\frac{k}{n}} L_T^{k/n} Vol(K \cap E)^{1/n} \end{aligned}$$

for a different constant c. □

3 Use of an M-ellipsoid

We will need to use a special ellipsoid associated with an arbitrary convex body, called an M-ellipsoid. An M-ellipsoid is defined by the following theorem (see [M4], or chapter 7 in the book [Pi]):

Theorem 3.1 *Let $K \subset \mathbb{R}^n$ be a convex body. Then there exists an ellipsoid $\mathcal{E}$ with $Vol(\mathcal{E}) = Vol(K)$ such that*

$$N(K, \mathcal{E}) = \min\{\sharp A; K \subset A + \mathcal{E}\} < e^{cn}$$

where $\sharp A$ is the number of elements in the set A, and c is a numerical constant. We say that $\mathcal{E}$ is an M-ellipsoid of K (with constant c).

An M-ellipsoid may replace K in various volume computations. For example, assume that $\mathcal{E}$ is an M-ellipsoid of K. If $E \subset \mathbb{R}^n$ is a subspace, and $Proj_E$ is the orthogonal projection onto E in $\mathbb{R}^n$, then by Theorem 3.1,

$$Vol(Proj_E(K))^{1/n} \leq (e^{cn} Vol(Proj_E(\mathcal{E})))^{1/n} = c' Vol(Proj_E(\mathcal{E}))^{1/n}.$$

We shall use the following lemma which appears in [Sp].

Lemma 3.2 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin. Let $E \subset \mathbb{R}^n$ be a subspace of any dimension. Then,*

$$Vol(K \cap E)^{1/n} \geq \frac{1}{Vol(Proj_{E^\perp}(K))^{1/n}}.$$

Proof of Proposition 1.3: First assume that $m \geq \frac{n}{2}$. Recall that $L_n = \sup_{C \subset \mathbb{R}^n} L_C$ where the supremum is taken over all isotropic convex bodies in $\mathbb{R}^n$. This supremum is attained by a compactness argument (the collection of all convex sets modulu affine transformations is compact). Define K to be one of the bodies where the supremum is attained; i.e.

$$L_K = L_n$$

and K is isotropic and of volume one. Let $\mathcal{E}$ be an M-ellipsoid of K. Since $\mathcal{E}$ is an ellipsoid of volume one, it has at least one projection onto a subspace $E^\perp$ of dimension $n - m$, such that

$$Vol(Proj_{E^\perp} K)^{1/n} < c Vol(Proj_{E^\perp} \mathcal{E})^{1/n} < C.$$

By Lemma 3.2,

$$Vol(K \cap E)^{1/n} > c'.$$

Let T be an m-dimensional body such that $L_T = L_m$, and T is of volume one and isotropic. Denote by K' the "(T, E)-symmetrization" of K. Then $L_K = L_n \geq L_{K'}$, and by Theorem 2.5,

$$L_K \geq L_{K'} > cL_K^{1-\frac{m}{n}} L_T^{\frac{m}{n}} Vol(K \cap E)^{1/n} > \tilde{c} L_K^{1-\frac{m}{n}} L_T^{\frac{m}{n}}$$

or equivalently,

$$L_n = L_K > \tilde{c}^{\frac{n}{m}} L_T = \tilde{c}^{\frac{n}{m}} L_m.$$

Since we assumed that $\frac{n}{m} \leq 2$, we get $L_m < c' L_n$. Regarding the case in which $m < \frac{n}{2}$: Note that $L_m \leq L_{2m}$, since the $2m$ dimensional body which is the cartesian product of T with itself, has the same isotropic constant as T. If s is the maximal integer such that $2^s m \leq n$, then clearly $2^s m > \frac{n}{2}$, and therefore

$$L_m \leq L_{2^s m} < c' L_n.$$

□

Remark 3.3: In the proof of Proposition 1.3 we showed that for every convex body $K \subset \mathbb{R}^n$ of volume one, and for any $1 \leq k \leq n$, there exists a k-dimensional subspace E such that $Vol(K \cap E)^{1/n} > c$. This fact is a direct consequence of the existence of an M-ellipsoid, but may not be very trivial to obtain directly.

We would like to mention an additional property attributed to a body $K \subset \mathbb{R}^n$, which has the largest possible isotropic constant. For this purpose, we will quote a useful result which appears in [Ba1] and in [MP1]. Our formulation is closer to the one in [MP1] (Lemma 3.10, and Proposition 3.11 there). Although results in that paper are stated only for centrally-symmetric bodies, the symmetry assumption is rarely used. The generalization to non-symmetric bodies is straightforward, and reads as follows:

Lemma 3.4 *Let $K \subset \mathbb{R}^n$ be an isotropic convex body of volume one. Let $1 \leq k \leq n$ and let E be a k-codimensional subspace. Define C as the unit ball of the (non-symmetric) norm defined on $E^\perp$ as*

$$\|\theta\| = |\theta|^{1+\frac{p}{p+1}} \Bigg/ \left(\int_{K \cap E(\theta)} |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p+1}}$$

for $p = k+1$, where $E(\theta) = \{x + t\theta; x \in E, t > 0\}$ is a half of a $k-1$-codimensional subspace. Then indeed C is convex, and

$$\frac{L_C}{L_K} \approx Vol(K \cap E)^{1/k}.$$

Corollary 3.5 *Let $K \subset \mathbb{R}^n$ be a convex isotropic body of volume one, such that $L_K = L_n$. Then for any subspace $E \subset \mathbb{R}^n$ of codimension k,*

$$Vol(K \cap E)^{1/k} < c$$

where c is a numerical constant.

Proof: By Lemma 3.4,

$$Vol(K \cap E)^{\frac{1}{k}} \approx \frac{L_C}{L_K} = \frac{L_C}{L_n} \leq \frac{L_{n-k}}{L_n} < c$$

where the last inequality follows from Proposition 1.3. □

4 Proof of the reduction to bodies with finite volume ratio

In this section, assume that $K \subset \mathbb{R}^n$ is a convex isotropic body of volume one, such that $L_K = L_n$. Apriori, an M-ellipsoid of K may be very different from a Euclidean ball. We shall see that Corollary 3.5 imposes stringent conditions on the axes of an M-ellipsoid.

4.1 Controlling the axes of an M-ellipsoid

Denote by κ_m the volume of a unit Euclidean ball in $\mathbb{R}^m$. It is well known that $\kappa_m^{1/m} \approx \frac{1}{\sqrt{m}}$. Let $\mathcal{E} = \left\{ x \in \mathbb{R}^n; \sum_i \frac{x_i^2}{n\lambda_i^2} \leq 1 \right\}$ be an M-ellipsoid of K, whose existence is guaranteed in Theorem 3.1. The axes of this ellipsoid are of lengths $\sqrt{n}\lambda_1, .., \sqrt{n}\lambda_n$, and $\left(\prod_{i=1}^n \lambda_i\right)^{1/n} \approx 1$, since $1 = Vol(\mathcal{E}) = \kappa_n \prod \sqrt{n}\lambda_i$. Assume that the λ_i's are ordered, i.e. $\lambda_1 \leq ... \leq \lambda_n$. For convenience, and without loss of generality, we assume that n is divisible by four.

Claim 4.1 $\lambda_{n/2} < c$, *for some numerical constant c.*

Proof: Let $E \subset \mathbb{R}^n$ be any subspace of any dimension. By Lemma 3.2 and Corollary 3.5,

$$Vol(Proj_E(K))^{1/n} > \frac{c}{Vol(K \cap E^{\perp})^{1/n}} > c'.$$

Let $E = sp\{e_1, .., e_{n/2}\}$, the linear space spanned by $e_1, .., e_{n/2}$. Then,

$$c < Vol(Proj_E(K))^{1/n} \leq N(K, \mathcal{E})^{1/n} \left(\kappa_{n/2} \prod_{i=1}^{n/2} \sqrt{n}\lambda_i \right)^{1/n}$$

because $Vol(Proj_E(\mathcal{E})) = \kappa_{n/2} \prod_1^{n/2} \sqrt{n}\lambda_i$. Since $(\kappa_{n/2}\sqrt{n}^{n/2})^{1/n} \approx 1$, we get that

$$\left(\prod_{i=1}^{n/2} \lambda_i\right)^{2/n} > c.$$

Hence we obtain,

$$\lambda_{n/2} \leq \left(\prod_{i=\frac{n}{2}+1}^{n} \lambda_i\right)^{2/n} = \left(\prod_{i=1}^{n} \lambda_i\right)^{2/n} \left(\prod_{i=1}^{n/2} \lambda_i\right)^{-2/n} < \tilde{c}. \qquad (4)$$

□

4.2 Finite volume ratio

The following lemma, whose proof involves the notion of an M-ellipsoid, originally appears in [M2]. It can also be deduced from the proof of Corollary 7.9 in [Pi].

Lemma 4.2 *Let $K \subset \mathbb{R}^n$ be a convex body. Let $0 < \lambda < 1$. Then there exists a subspace G of dimension $\lfloor \lambda n \rfloor$ such that if $P : \mathbb{R}^n \to \mathbb{R}^n$ is a projection (i.e. P is linear and $P^2 = P$) such that $ker(P) = G$, then $P(K)$ has a volume ratio smaller than $c(\lambda)$, where $c(\lambda)$ is some function which depends solely on λ.*

The central theme underlying the proof which follows, is the connection between an M-ellipsoid and the isotropy ellipsoid of a body with the largest possible isotropic constant. This connection arises when we project K onto the subspace $E = sp\{e_1, .., e_{n/2}\}$, together with its covering ellipsoid. According to (4) we get that $Proj_E(\mathcal{E}) \subset c\sqrt{n}D$, so in fact the normalized Euclidean ball is an M-ellipsoid for $Proj_E(K)$. In other words, the isotropy ellipsoid and the selected M-ellipsoid of K are equivalent in a large projection. Therefore, we may combine the properties of an M-ellipsoid with the properties of the isotropy ellipsoid, to create a finite volume ratio body.

Apply Lemma 4.2 to the body $Proj_E(K)$. There exists a subspace $F \subset E$ such that $dim(F) = n/4$ and

$$v.r.(Proj_F(K)) = v.r.(Proj_F(Proj_E(K))) < C.$$

Indeed, F is the orthogonal complement in E, to the subspace G from Lemma 4.2. Denote K' as the $(D_{F^\perp}, F^\perp)$-symmetrization of K, where $D_{F^\perp}$ is the standard Euclidean ball in $F^\perp$. Then,

$$K' \cap F = Proj_F(K') = Proj_F(K)$$

is a finite volume ratio body, i.e. there exists an ellipsoid $\mathcal{F} \subset K' \cap F$ such that $\left(\frac{Vol(K' \cap F)}{Vol(\mathcal{F})}\right)^{4/n} < C$. We claim that K' has a bounded volume ratio. Indeed, the ellipsoid

$$\mathcal{E}' = \left\{\lambda x + \mu y; \lambda^2 + \mu^2 \leq 1, x \in \mathcal{F}, y \in K' \cap F^{\perp}\right\}$$

satisfies

$$\frac{1}{\sqrt{2}}\mathcal{E}' \subset conv\{\mathcal{F}, K' \cap F^{\perp}\} \subset K',$$

$$Vol(\mathcal{E}')^{1/n} \geq \frac{1}{\sqrt{2}C} Vol(Proj_F(K'))^{1/n} Vol(K' \cap F^{\perp})^{1/n} \geq \frac{1}{\sqrt{2}C},$$

by Lemma 3.2. Hence $\mathcal{E}'$ is evidence of the finite volume ratio property of K'. Note also that according to Claim 4.1,

$$Vol(Proj_F(K))^{1/n} \leq N(K, \mathcal{E})^{1/n} Vol(Proj_F(\sqrt{n}\lambda_{n/2}D))^{1/n} < c.$$

Hence by Lemma 3.2 and Theorem 2.5

$$L_{K'} \approx L_n^{1/4} Vol(K \cap F^{\perp})^{1/n} > c\frac{L_n^{1/4}}{Vol(Proj_F(K))^{1/n}} > c' L_n^{1/4}$$

and therefore,

$$L_n < c(L_0)^4$$

where $L_0 = L_n(\tilde{c})$ is the largest possible L_K among all convex bodies in $\mathbb{R}^n$, having volume ratio not larger than $\tilde{c}$, and Proposition 1.1 is proved. $\square$

Remark 4.3: Regarding the connection between v, L_n and $L_n(v)$; Formally, we have proved for some $v > 1$ that $L_n \lesssim (L_n(v))^4$ for all n. However, by adjusting the dimensions of the subspaces E and F, we can reduce the power of $L_n(v)$, at the expense of increasing the volume ratio constant, v. The dependence obtained using this method is quite poor: For any $0 < \theta < 1$,

$$L_n \leq e^{\frac{c}{1-\theta}} L(e^{\frac{c}{1-\theta}})^{\frac{1}{\theta}}.$$

5 The isotropic position and an M-ellipsoid

Proof of Proposition 1.4: Denote $\mathcal{D}_m = \{x \in \mathbb{R}^m; |x| \leq \kappa_m^{-1/m}\}$, a Euclidean ball of volume one. Let $K \subset \mathbb{R}^n$ be a convex isotropic body of volume one. Denote,

$$K' = \left\{(x_1, x_2); x_1 \in \sqrt{\frac{L_{\mathcal{D}_n}}{L_K}} K, x_2 \in \sqrt{\frac{L_K}{L_{\mathcal{D}_n}}} \mathcal{D}_n\right\} \subset \mathbb{R}^{2n}.$$

Let $E \subset \mathbb{R}^{2n}$ be the subspace spanned by the first n standard unit vectors, and let $F = E^{\perp}$. We claim that K' is an isotropic body. By a reasoning similar to that in Lemma 2.2, the subspaces E and F are invariant under the action of the matrix $M_{K'}$. In addition, $M_{K'}$ acts as a multiple of the identity in both subspaces. Let us show that it is the same multiple of the identity in both subspaces, and hence $M_{K'}$ is a scalar matrix. For any $v \in E$,

$$\int_{K'} \langle x, v\rangle^2 dx = \frac{L_{\mathcal{D}_n}}{L_K} \int_K \langle x, v\rangle^2 dx = L_{\mathcal{D}_n} L_K.$$

Also, for any $v \in F$,

$$\int_{K'} \langle x, v\rangle^2 dx = \frac{L_K}{L_{\mathcal{D}_n}} \int_{\mathcal{D}_n} \langle x, v\rangle^2 dx = L_K L_{\mathcal{D}_n}.$$

Therefore K' is isotropic. According to our assumption, a reverse Brunn-Minkowski inequality holds. Hence by (1),

$$Vol(K' + \mathcal{D}_{2n})^{1/2n} < C\left(Vol(K')^{1/2n} + Vol(\mathcal{D}_{2n})^{1/2n}\right) = 2C. \tag{5}$$

But $\sqrt{\frac{L_K}{L_{\mathcal{D}_n}}}\mathcal{D}_n + \mathcal{D}_{2n} \subset K' + \mathcal{D}_{2n}$. Hence,

$$Vol(K' + \mathcal{D}_{2n})^{1/2n} > Vol\left(\sqrt{\frac{L_K}{L_{\mathcal{D}_n}}}\mathcal{D}_n + \mathcal{D}_{2n}\right)^{1/2n} > c\left(\frac{L_K}{L_{\mathcal{D}_n}}\right)^{1/4}. \tag{6}$$

Combining (5) and (6), and using the fact that $L_{\mathcal{D}_n} < c'$ we get

$$L_K < (\tilde{c}C)^4$$

and since K is arbitrary, the isotropic constant of an arbitrary convex body K in $\mathbb{R}^n$ is universally bounded. □

Remark: The proof of Proposition 1.1 uses the close relation between an M-ellipsoid and the isotropy ellipsoid of the body whose isotropic constant is as large as possible. As follows from Proposition 1.4, if we could deduce such a relation between an M-ellipsoid and the isotropy ellipsoid of an arbitrary convex body $K \subset \mathbb{R}^n$, then a universal bound for the isotropic constant will follow.

6 Appendix: Concave functions

This section proves Lemma 2.3 in a way similar to the proofs presented in [Ba1], [F]. The following lemma reflects the fact that among all concave functions on the line, the linear function is extremal.

Lemma 6.1 *Let $f : [0, \infty) \to [0, \infty)$ be a compactly supported function such that $f^{1/k}$ is concave on its support and $a = f(0) > 0$. Let $n > 0$ and choose b such that*

$$\int_0^\infty f(x)x^n dx = \int_0^\infty \left(a^{1/k} - bx\right)_+^k x^n dx$$

where $x_+ = \max\{x, 0\}$. *Then for any* $p > 1$

$$\int_0^\infty f(x)^p x^n dx \geq \int_0^\infty \left(a^{1/k} - bx\right)_+^{pk} x^n dx. \tag{7}$$

Proof: Since f has a compact support, $\int_0^\infty f(x)x^n dx < \infty$, so $b > 0$. Denote $h(x) = a^{1/k} - f(x)^{1/k}$. Then h is a convex function and $h(0) = 0$. Therefore $\tilde{h}(x) = \frac{h(x)}{x}$ is increasing. Since

$$\int_0^\infty (a^{1/k} - x\tilde{h})_+^k x^n dx = \int_0^\infty (a^{1/k} - bx)_+^k x^n dx$$

it is impossible that $\tilde{h}$ is always smaller or always larger than b. The function $\tilde{h}$ is increasing, so there exists $x_0 \in [0, \infty)$ such that $\tilde{h} \leq b$ on $[0, x_0]$ and $\tilde{h} \geq b$ on $[x_0, \infty)$. Denote $g(x) = \left(a^{1/k} - bx\right)_+^k$. In order to obtain (7) we need to prove that

$$p \int_0^\infty \int_0^{f(x)} y^{p-1} dy x^n dx \geq p \int_0^\infty \int_0^{g(x)} y^{p-1} dy x^n dx.$$

Since $(g(x) - f(x))(x - x_0) \geq 0$, and g^{p-1} is a decreasing function,

$$\int_0^{x_0} \int_{g(x)}^{f(x)} y^{p-1} dy x^n dx \geq \int_0^{x_0} \int_{g(x)}^{f(x)} g(x_0)^{p-1} dy x^n dx, \tag{8}$$

$$\int_{x_0}^\infty \int_{f(x)}^{g(x)} y^{p-1} dy x^n dx \leq \int_{x_0}^\infty \int_{f(x)}^{g(x)} g(x_0)^{p-1} dy x^n dx. \tag{9}$$

Subtracting (9) from (8), we obtain

$$\int_0^\infty \int_{g(x)}^{f(x)} y^{p-1} dy x^n dx \geq g(x_0)^{p-1} \int_0^\infty (f(x) - g(x)) x^n dx = 0$$

and the lemma is proven. $\square$

Proof of Lemma 2.3: The inequality on the right has nothing to do with log-concavity: Since $\int_{\mathbb{R}^n} f = 1$,

$$\int_{\mathbb{R}^n} f^{1+\frac{2}{k}} = \int_{\mathbb{R}^n} f \cdot f^{\frac{2}{k}} \leq \int_{\mathbb{R}^n} f \cdot M^{\frac{2}{k}} = M^{\frac{2}{k}}.$$

Let us prove the left-most inequality. By translating f if necessary, we may assume that $f(0) = M$. We shall begin by integrating in polar coordinates:

$$\int_{\mathbb{R}^n} f(x)^{1+\frac{2}{k}} dx = \int_{S^{n-1}} \int_0^\infty f(r\theta)^{1+\frac{2}{k}} r^{n-1} dr d\theta.$$

Fix $\theta \in S^{n-1}$, and denote $g(r) = f(r\theta)$. Then $g^{1/k}$ is concave, as a restriction of a concave function to a straight line. Also, f must have a compact support, as it has a finite mass and $f^{1/k}$ is concave. Therefore g is compactly supported and by Lemma 6.1 for $p = 1 + \frac{2}{k}$,

$$\int_0^\infty g(x)^{1+\frac{2}{k}} x^{n-1} dx \geq \int_0^\infty \left(a^{1/k} - bx\right)_+^{k+2} x^{n-1} dx \tag{10}$$

where $a = g(0)$ and b is chosen as in Lemma 6.1, i.e. $\int_0^\infty g(x)x^{n-1}dx = \int_0^\infty \left(a^{1/k} - bx\right)_+^k x^{n-1}dx$. An elementary calculation yields that

$$\frac{\int_0^\infty \left(a^{1/k} - bx\right)_+^{k+2} x^{n-1}dx}{\int_0^\infty \left(a^{1/k} - bx\right)_+^{k} x^{n-1}dx} = a^{2/k}\frac{(k+1)(k+2)}{(n+k+1)(n+k+2)} \tag{11}$$

where we used the fact that $\int_0^1 x^a(1-x)^b dx = \frac{a!b!}{(a+b+1)!}$. Denote $c_{n,k} = \frac{(k+1)(k+2)}{(n+k+1)(n+k+2)}$. Combining (10) and (11) we obtain

$$\begin{aligned}\int_0^\infty g(x)^{1+\frac{2}{k}}x^{n-1}dx &\geq a^{2/k}c_{n,k}\int_0^\infty \left(a^{1/k} - bx\right)_+^k x^{n-1}dx \\ &= c_{n,k}g(0)^{2/k}\int_0^\infty g(x)x^{n-1}dx\end{aligned}$$

or in other words, for every $\theta \in S^{n-1}$,

$$\int_0^\infty f(r\theta)^{1+\frac{2}{k}}r^{n-1}dr \geq c_{n,k}f(0)^{2/k}\int_0^\infty f(r\theta)r^{n-1}dr.$$

By integrating this inequality over the sphere S^{n-1},

$$\int_{\mathbb{R}^n} f(x)^{1+\frac{2}{k}}dx \geq c_{n,k}f(0)^{2/k}\int_{\mathbb{R}^n} f(x)dx = c_{n,k}f(0)^{2/k}.$$

□

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An isomorphic version of the slicing problem[6]

Abstract. We show that any centrally-symmetric convex body $K \subset \mathbb{R}^n$ has a perturbation $T \subset \mathbb{R}^n$ which is convex and centrally-symmetric, such that the isotropic constant of T is universally bounded. T is close to K in the sense that the Banach-Mazur distance between T and K is $O(\log n)$. If K is a body of a non-trivial type then the distance is universally bounded. The distance is also universally bounded if the perturbation T is allowed to be non-convex. Our technique involves the use of mixed volumes and Alexandrov-Fenchel inequalities. Some additional applications of this technique are presented here.

1 Introduction

Let $K \subset \mathbb{R}^n$ be a centrally-symmetric (i.e. $K = -K$) convex set with a non-empty interior. Such sets are referred to here as "bodies". We denote by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ the standard scalar product and Euclidean norm in $\mathbb{R}^n$. We also define D as the unit Euclidean ball and $S^{n-1} = \partial D$. The body K has a linear image $\tilde{K}$ with $Vol(\tilde{K}) = 1$ such that

$$\int_{\tilde{K}} \langle x, \theta \rangle^2 dx \tag{1}$$

does not depend on the choice of $\theta \in S^{n-1}$. We say that $\tilde{K}$ is an isotropic linear image of K or that $\tilde{K}$ is in isotropic position. The isotropic linear image of K is unique, up to orthogonal transformations (e.g. [MP1]). The quantity in (1), for any $\theta \in S^{n-1}$ and any $\tilde{K}$ an isotropic linear image of K, is usually referred to as L_K^2 or as the square of the isotropic constant of K. An equivalent definition of L_K is the following:

$$nL_K^2 = \inf_T \int_K |Tx|^2 dx \tag{2}$$

where the infimum is over all matrices T such that $det(T) = 1$. For a comprehensive discussion of the isotropic position and the isotropic constant we refer the reader to [MP1].

L_K is an important linearly invariant parameter associated with K. A major conjecture is whether there exists a universal constant $c > 0$ such that $L_K < c$ for all convex centrally-symmetric bodies in all dimensions. A proof of this conjecture will have various consequences. Among others (see [MP1]), it will establish the fact that any body of volume one has at least one $n-1$ dimensional section whose volume is greater than some positive universal constant. This conjecture is known as the slicing problem or the hyperplane conjecture. The

[6] This chapter corresponds to the paper [Kl6].

best estimate known to date is $L_K < cn^{1/4} \log n$ for $K \subset \mathbb{R}^n$ and is due to Bourgain [Bou2] (see also the presentation in [Dar]). In addition, the conjecture was verified for large classes of bodies (some examples of references are [Ba2], [Bou1], [Ju], [KMP], [MP1]).

In this note we deal with a known relaxation of this conjecture, which we call the "isomorphic slicing problem". It was suggested to the author by V. Milman. For two sets $K, T \subset \mathbb{R}^n$, we define their "geometric distance" as

$$d_G(K,T) = \inf\left\{ab;\ \frac{1}{a}K \subset T \subset bK,\ a,b > 0\right\}.$$

The Banach-Mazur distance between K and T is

$$d_{BM}(K,T) = \inf\{d_G(K, L(T))\ ;\ L \textit{ is a linear operator}\}.$$

Let $K_n, T_n \subset \mathbb{R}^n$ for $n = 1, 2, ...$ be a sequence of bodies such that $d_{BM}(K_n, T_n) < Const$ independent of the dimension n. In this case we say that the families $\{K_n\}$ and $\{T_n\}$ are uniformly isomorphic. Indeed, the norms defined by K_n and T_n are uniformly isomorphic. The isomorphic slicing problem asks whether the slicing problem is correct, at least up to a uniform isomorphism. Formally:

Question 1.1 *Do there exist constants $c_1, c_2 > 0$ such that for any dimension n, for any centrally-symmetric convex body $K \subset \mathbb{R}^n$, there exists a centrally-symmetric convex body $T \subset \mathbb{R}^n$ with $d_{BM}(K,T) < c_1$ and $L_T < c_2$?*

In this note we answer this question affirmatively, up to a logarithmic factor. The following is proven here:

Theorem 1.2 *For any centrally-symmetric convex body $K \subset \mathbb{R}^n$ there exists a centrally-symmetric convex body $T \subset \mathbb{R}^n$ with $d_{BM}(K,T) < c_1 \log n$ and*

$$L_T < c_2$$

where $c_1, c_2 > 0$ are numerical constants.

The $\log n$ factor in Theorem 1.2 stems from the use of the l-position and Pisier's estimate for the norm of the Rademacher projection (see [Pi]). In fact, in the notation of Theorem 1.2 we prove that $d_{BM}(K,T) < c_1 M(K) M^*(K)$ (see definitions in Section 3). Therefore we verify the validity of the isomorphic slicing conjecture for bodies that have a linear image with bounded MM^*. This large class of bodies includes all bodies of a non trivial type (e.g. [MS]). In addition, Proposition 5.2 and Proposition 5.3 provide other classes of bodies for which Question 1.1 has a positive answer.

There exist some connections between the slicing problem and its isomorphic versions. An example is provided in the following lemma.

Lemma 1.3 *Assume that there exist $c_1, c_2 > 0$ such that for any integer n and an isotropic body $K \subset \mathbb{R}^n$ there exists an isotropic body $T \subset \mathbb{R}^n$ with $d_G(K,T) < c_1$ and $L_T < c_2$. Then there exists $c_3 > 0$ such that for any integer n and body $K \subset \mathbb{R}^n$, we have $L_K < c_3$.*

Proof: $L_T < c_2$, therefore T is in M-position (as observed by K. Ball, see definitions and proofs in [MP1]). Since $d_G(K,T) < c_1$, then K is also in M-position. Using Proposition 1.4 from [BKM2] we obtain a universal bound for the isotropic constant. $\square$

A set $K \subset \mathbb{R}^n$ is star-shaped if for any $0 \leq t \leq 1$ and $x \in K$ we have $tx \in K$. A star shaped set $K \subset \mathbb{R}^n$ is quasi-convex with constant $C > 0$ if $K + K \subset CK$, where $K + T = \{k + t; k \in K, t \in T\}$ for any $K, T \subset \mathbb{R}^n$. For centrally-symmetric quasi-convex sets, the isomorphic slicing problem has an affirmative answer. Formally, as is proven in Section 4,

Theorem 1.4 *For any $C > 1$ there exist $c_1, c_2 > 0$ with the following property: If $K \subset \mathbb{R}^n$ is centrally-symmetric and quasi-convex with constant C, then there exists a centrally-symmetric $T \subset \mathbb{R}^n$ such that $d_{BM}(K,T) < c_1$ and $L_T < c_2$. (Note that T is necessarily c_1C-quasi convex).*

Our proof has a number of consequences which are formulated and proved in Section 5. Among these are an improvement of an estimate from [BKM2], and a connection between the isotropic position and an M-position of order α for bodies with a small isotropic constant. Throughout this paper the letters $c, C, c', c_1, c_2, Const$ etc. denote positive numerical constants, whose value may differ in various appearances. The same goes for $c(\varphi), C(\varphi)$ etc. which denote some positive functions that depend purely on their arguments. We ignore measurability issues as they are not essential to our discussion. All sets and functions used here are assumed to be measurable.

2 Log concave functions

In this section we mention some facts regarding log-concave functions, most of which are known and appear in [Ba1] or [MP1], yet our versions are slightly different. $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if $\log f$ is concave on its support. f is s-concave, for $s > 0$, if $f^{1/s}$ is concave on its support. Any s-concave function is also log-concave (see e.g. [Bo], also for the connection with log-concave measures). Given a non-negative function f on $\mathbb{R}^n$ we define for $x \in \mathbb{R}^n$,

$$\|x\|_f = \left(\int_0^\infty f(rx) r^{n+1} dr \right)^{-1/n+2}.$$

We also define $K_f = \{x \in \mathbb{R}^n; \|x\|_f \leq 1\}$. The following Busemann-type theorem appears in [Ba1] (see also [MP1]):

Theorem 2.1 *Let f be an even log-concave function on $\mathbb{R}^n$. Then K_f is convex and centrally-symmetric and $\|\cdot\|_f$ is a norm.*

In what follows we repeatedly use two well known facts. The first is that for any $1 \leq k \leq n$,

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} < \left(e\frac{n}{k}\right)^k. \tag{3}$$

The second is that for any integers $a, b \geq 0$,

$$\int_0^1 s^a (1-s)^b ds = \frac{1}{(a+b+1)\binom{a+b}{a}}. \tag{4}$$

Lemma 2.2 *Let $f : \mathbb{R}^n \to [0, \infty)$ be an even function whose restriction to any straight line through the origin is s-concave. If $s > n$ then*

$$d_G(K_f, Supp(f)) < c\frac{s}{n}$$

where $c > 0$ is a numerical constant, and $Supp(f) = \{x; f(x) > 0\}$.

Proof: Multiplying f by a constant if necessary, we may assume that $f(0) = 1$. Fix $\theta \in S^{n-1}$. Denote $M_\theta = \sup\{r > 0; f(r\theta) > 0\}$. Since $f|_{\theta\mathbb{R}}$ is s-concave and $f(0) = 1$, for all $0 \leq r \leq M_\theta$,

$$f(r\theta) \geq \left(1 - \frac{r}{M_\theta}\right)^s.$$

By the definition of $\|\theta\|_f$ and by (4),

$$\|\theta\|_f^{-(n+2)} \geq \int_0^{M_\theta} \left(1 - \frac{r}{M_\theta}\right)^s r^{n+1} dr = \frac{M_\theta^{n+2}}{(n+s+2)\binom{n+s+1}{n+1}}.$$

In addition, since $f|_{\theta\mathbb{R}}$ is even, its maximum is $f(0) = 1$ and

$$\|\theta\|_f^{-(n+2)} \leq \int_0^{M_\theta} r^{n+1} dr = \frac{1}{n+2} M_\theta^{n+2}.$$

Combining this with the estimate (3),

$$\frac{(n+2)^{1/(n+2)}}{M_\theta} \leq \|\theta\|_f \leq \frac{e(n+s+2)^{1/n+2} \left(\frac{n+s+1}{n+1}\right)^{\frac{n+1}{n+2}}}{M_\theta}$$

and since $s > n$,

$$\forall \theta \in S^{n-1}, \ \frac{c_1}{M_\theta} < \|\theta\|_f < \frac{c_2}{M_\theta}\frac{s}{n} \quad \Rightarrow \quad \frac{n}{c_2 s} Supp(f) \subset K_f \subset \frac{1}{c_1} Supp(f)$$

and the lemma is proven. □

The isotropic constant and the isotropic position may also be defined for arbitrary measures or densities, not only for convex bodies. Let $f : \mathbb{R}^n \to [0, \infty)$ be an even function with $0 < \int_{\mathbb{R}^n} f < \infty$. The entries of its covariance matrix with respect to a fixed orthonormal basis $\{e_1, .., e_n\}$ are defined as

$$M_{i,j} = \frac{1}{\int_{\mathbb{R}^n} f(x)dx} \int_{\mathbb{R}^n} f(x)\langle x, e_i\rangle\langle x, e_j\rangle dx.$$

We define $L_f = \left(\frac{f(0)}{\int_{\mathbb{R}^n} f}\right)^{\frac{1}{n}} det(M)^{\frac{1}{2n}}$. One can verify that if $f = 1_K$ is the characteristic function a body $K \subset \mathbb{R}^n$, then $L_f = L_K$. Our next lemma claims that if f is log-concave, then the body K_f shares the isotropic constant of the function f, up to a universal constant. This fact appears in [MP1] and in [Ba1], but our formulation is slightly different. For completeness we present a proof here.

Lemma 2.3 *Let f be an even function on $\mathbb{R}^n$ whose restriction to any straight line through the origin is log-concave. Assume that $\int_{\mathbb{R}^n} f < \infty$. Then,*

$$c_1 L_f < L_{K_f} < c_2 L_f$$

where $c_1, c_2 > 0$ are universal constants.

Proof: We may assume that $f(0) = 1$. Integrating in polar coordinates, for any $y \in \mathbb{R}^n$,

$$\begin{aligned}
&\int_{K_f} \langle x, y\rangle^2 dx \\
&= \int_{S^{n-1}} \int_0^{1/\|\theta\|_f} \langle y, r\theta\rangle^2 r^{n-1} dr d\theta = \frac{1}{n+2} \int_{S^{n-1}} \langle y, \theta\rangle^2 \frac{1}{\|\theta\|_f^{n+2}} d\theta \\
&= \frac{1}{n+2} \int_0^\infty \int_{S^{n-1}} f(r\theta)\langle y, \theta\rangle^2 r^{n+1} dr d\theta = \frac{1}{n+2} \int_{\mathbb{R}^n} \langle x, y\rangle^2 f(x) dx
\end{aligned}$$

where $d\theta$ is the induced surface area measure on S^{n-1}. Denote by $M(f)$ and $M(K_f)$ the inertia matrices of f and of 1_{K_f}, respectively. We conclude that $Vol(K_f)M(K_f) = \frac{1}{n+2}\left(\int_{\mathbb{R}^n} f\right) M(f)$. To compare the isotropic constants, we need to estimate $\frac{\int f}{Vol(K_f)}$. Now,

$$Vol(K_f) = \frac{1}{n} \int_{S^{n-1}} \left(\int_0^\infty f(r\theta) r^{n+1} dr\right)^{\frac{n}{n+2}} d\theta. \tag{5}$$

We shall use the following one-dimensional lemma, which is proven at the end of this section (see also [Ba1], [BKM2] or [MP1]).

Lemma 2.4 *Let $g : [0,\infty) \to [0,\infty)$ be a non-increasing log-concave function with $g(0) = 1$ and $\int_0^\infty g(t)t^{n-1}dt < \infty$. Then, for any integer $n \geq 1$,*

$$\frac{n^{\frac{n+2}{n}}}{n+2} \leq \frac{\int_0^\infty g(t)t^{n+1}dt}{\left(\int_0^\infty g(t)t^{n-1}dt\right)^{\frac{n+2}{n}}} \leq \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}}.$$

(the left-most inequality - which is more important to us - holds also without the log-concavity assumption).

Since f is even and log-concave on any line through the origin, it is non-increasing on any ray that starts at the origin. From the left-most inequality in Lemma 2.4, for any $\theta \in S^{n-1}$ (except for a set of measure zero where the integral diverges),

$$\int_0^\infty f(r\theta)\, r^{n+1}dr \geq \frac{n^{\frac{n+2}{n}}}{n+2}\left(\int_0^\infty f(r\theta)\, r^{n-1}dr\right)^{\frac{n+2}{n}}$$

and according to (5),

$$Vol(K_f) \geq \frac{1}{n}\frac{n^{\frac{n+2}{n}}}{n+2}\int_{S^{n-1}}\int_0^\infty f(r\theta)\, r^{n-1}drd\theta = \frac{n^{2/n}}{n+2}\int_{\mathbb{R}^n} f.$$

Since $M(K_f) = \frac{1}{n+2}\frac{\int_{\mathbb{R}^n} f}{Vol(K_f)}M(f)$,

$$\frac{L_{K_f}^2}{L_f^2} = \frac{1}{n+2}\left(\frac{\int_{\mathbb{R}^n} f}{Vol(K_f)}\right)^{1+\frac{2}{n}} \leq \frac{1}{n+2}\left(\frac{n+2}{n^{2/n}}\right)^{\frac{n+2}{n}} < c_2.$$

This completes the proof of one part of the lemma. The proof of the other inequality is similar. Using the right-most inequality in Lemma 2.4,

$$\frac{L_{K_f}^2}{L_f^2} = \frac{1}{n+2}\left(\frac{\int_{\mathbb{R}^n} f}{Vol(K_f)}\right)^{1+\frac{2}{n}} \geq \frac{1}{n+2}\left(\frac{n\left((n-1)!\right)^{\frac{n+2}{n}}}{(n+1)!}\right)^{\frac{n+2}{n}} > c_1$$

and the lemma is proven. □

Proof of Lemma 2.4: Begin with the left-most inequality. Define $A > 0$ such that $\int_0^\infty g(t)t^{n-1}dt = \int_0^A t^{n-1}dt$. Then,

$$\begin{aligned}&\int_0^A (1-g(t))t^{n+1}dt - \int_A^\infty g(t)t^{n+1}dt \\ \leq\ & A^2\left[\int_0^A (1-g(t))t^{n-1}dt - \int_A^\infty g(t)t^{n-1}dt\right] = 0.\end{aligned}$$

Since $\int_0^A t^{n+1}dt = \frac{n^{\frac{n+2}{n}}}{n+2}\left(\int_0^A t^{n-1}dt\right)^{\frac{n+2}{n}}$, we get that

$$\int_0^\infty g(t)t^{n+1}dt \geq \int_0^A t^{n+1}dt = \frac{n^{\frac{n+2}{n}}}{n+2}\left(\int_0^\infty g(t)t^{n-1}dt\right)^{\frac{n+2}{n}}.$$

To obtain the other inequality we need to use the log-concavity of the function. Define $B > 0$ such that $h(t) = e^{-Bt}$ satisfies

$$\int_0^\infty g(t)t^{n-1}dt = \int_0^\infty h(t)t^{n-1}dt.$$

It is impossible that $g < h$ always or $g > h$ always, hence necessarily $t_0 = \inf\{t > 0; h(t) \geq g(t)\}$ is finite. $-\log g$ is convex and vanishes at zero, so $\tilde{g}(t) = \frac{-\log g(t)}{t}$ is non-decreasing. Thus $(B - \tilde{g}(t))(t - t_0) \geq 0$ or equivalently $(h(t) - g(t))(t - t_0) \geq 0$ for all $t > 0$. Therefore,

$$\begin{aligned} & \int_0^{t_0}(g(t) - h(t))t^{n+1}dt - \int_{t_0}^\infty (h(t) - g(t))t^{n+1}dt \\ \leq \; & t_0^2\left[\int_0^{t_0}(g(t) - h(t))t^{n-1}dt - \int_{t_0}^\infty (h(t) - g(t))t^{n-1}dt\right] = 0. \end{aligned}$$

Since $\int_0^\infty e^{-tB}t^{n+1}dt = \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}}\left(\int_0^\infty e^{-tB}t^{n-1}dt\right)^{\frac{n+2}{n}}$,

$$\int_0^\infty g(t)t^{n+1}dt \leq \int_0^\infty h(t)t^{n+1}dt = \frac{(n+1)!}{((n-1)!)^{\frac{n+2}{n}}}\left(\int_0^\infty g(t)t^{n-1}dt\right)^{\frac{n+2}{n}}.$$

□

3 Constructing a function on K

Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. In this section we find an αn-concave function F supported on K whose isotropic constant is bounded. From Lemma 2.3 it follows that $L_{K_F} < Const$. According to Lemma 2.1, K_F is a convex body, and by Lemma 2.2 we get that $d_G(K, K_F) < c\alpha$. If good estimates on α were obtained, Theorem 1.2 would follow. Let $\|\cdot\|$ be the norm for which K is its unit ball, and denote by σ the unique rotation invariant probability measure on S^{n-1}. The median of $\|x\|$ on S^{n-1} with respect to σ is referred to as $M'(K)$. We abbreviate $M' = M'(K)$ and define the following function on K:

$$f_K(x) = \inf\left\{0 \leq t \leq 1; x \in (1-t)\left[K \cap \frac{1}{M'}D\right] + tK\right\}.$$

Then f_K is a convex function which equals zero on $K \cap \frac{1}{M}D$. Define also

$$M(K) = \int_{S^{n-1}} \|x\| d\sigma(x), \quad M^*(K) = \int_{S^{n-1}} \|x\|_* d\sigma(x)$$

where $\|x\|_* = \sup_{y \in K}\langle x, y\rangle$ is the dual norm.

Proposition 3.1 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body, and let $\alpha = cM(K)M^*(K)$. Then,*

$$\int_K (1 - f_K(x))^{\alpha n}\, dx < 2Vol\left(K \cap \frac{1}{M'}D\right)$$

where $c > 0$ is some numerical constant.

Proof: We denote $F(x) = (1 - f(x))^{\alpha n}$. Then,

$$\int_K F(x)dx = \int_0^1 Vol\{x \in K; F(x) \geq t\}dt$$
$$= \int_0^1 Vol\{x \in K; f(x) \leq 1 - t^{\frac{1}{\alpha n}}\}dt$$

and substituting $s = 1 - t^{\frac{1}{\alpha n}}$ yields

$$\int_K F(x)dx = \alpha n \int_0^1 (1-s)^{\alpha n - 1} Vol\left((1-s)\left[K \cap \frac{1}{M'}D\right] + sK\right) ds.$$

Expand the volume term into a polynomial whose coefficients are mixed volumes (see e.g. [Schn]):

$$Vol\left((1-s)\left[K \cap \frac{1}{M'}D\right] + sK\right) = \sum_{i=0}^n \binom{n}{i} V_i s^i (1-s)^{n-i}$$

where $V_i = V(K, i; \left[K \cap \frac{1}{M'}D\right], n-i)$. Then,

$$\int_K F(x)dx = \alpha n \sum_{i=0}^n V_i \binom{n}{i} \int_0^1 s^i (1-s)^{(\alpha+1)n - i - 1} ds$$

and by (4),

$$\int_K F(x)dx = \frac{\alpha}{\alpha+1} V_0 \sum_{i=0}^n \frac{\binom{n}{i}}{\binom{(1+\alpha)n - 1}{i}} \frac{V_i}{V_0}.$$

Using (3) we may write

$$\int_K F(x)dx = \frac{\alpha}{\alpha+1} V_0 \left[1 + \sum_{i=1}^n \left(c_{n,i} \frac{n}{(1+\alpha)n - 1} \left(\frac{V_i}{V_0}\right)^{1/i}\right)^i\right] \tag{6}$$

where $\frac{1}{e} \leq c_{n,i} \leq e$. By Alexandrov-Fenchel inequalities, $V_i^2 \geq V_{i-1}V_{i+1}$ for $i \geq 1$ (e.g. [Schn]). It follows that for $1 \leq i \leq j$,

$$\left(\frac{V_i}{V_0}\right)^{1/i} \geq \left(\frac{V_j}{V_0}\right)^{1/j}. \tag{7}$$

In particular, if $\alpha + 1 > 4e\frac{V_1}{V_0}$, then by (7),

$$c_{n,i} \frac{n}{(1+\alpha)n - 1} \left(\frac{V_i}{V_0}\right)^{1/i} < \frac{2e}{1+\alpha} \frac{V_1}{V_0} \leq \frac{1}{2}.$$

Substituting into (6) we obtain

$$\int_K F(x)dx < V_0 \sum_{i=0}^n \frac{1}{2^i} < 2V_0 = 2Vol\left(K \cap \frac{1}{M'}D\right).$$

We still need to show that our $\alpha = cM(K)M^*(K)$ is greater than $4e\frac{V_1}{V_0}$. Since $\frac{1}{M'}D \cap K \subset \frac{1}{M'}D$,

$$V_1 = V(K, 1; \left[K \cap \frac{1}{M'}D\right], n-1)$$
$$\leq V\left(K, 1; \frac{1}{M'}D, n-1\right) = \frac{1}{(M')^{n-1}} Vol(D) M^*(K)$$

because $Vol(D)M^*(K) = V(K, 1; D, n-1)$ (see e.g. [Schn]). Regarding V_0, since M' is the median,

$$\sigma\left(M'K \cap S^{n-1}\right) \geq \frac{1}{2} \quad \Rightarrow \quad Vol\left(K \cap \frac{1}{M'}D\right) \geq \frac{Vol\left(\frac{1}{M'}D\right)}{2}.$$

In conclusion,

$$\frac{V_1}{V_0} \leq \frac{1}{(M')^{n-1}} Vol(D) M^*(K) \frac{2}{\frac{1}{(M')^n} Vol(D)} = 2M'(K)M^*(K).$$

The median of a positive function is not larger than twice its expectation. Therefore, $M'(K) \leq 2M(K)$, and we get that for $\alpha = cM(K)M^*(K)$, it is true that $\alpha + 1 > 4e\frac{V_1}{V_0}$ for a suitable numerical constant $c > 0$. $\square$

Corollary 3.2 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body, $\alpha = cM(K)M^*(K)$ and denote $F(x) = (1 - f_K(x))^{\alpha n}$. Then,*

$$L_F < c'$$

where $c, c' > 0$ are universal constants.

Proof: Consider F as a density on K, i.e. consider the probability measure $\mu_F(A) = \frac{\int_A F(x)dx}{\int_K F(x)dx}$. Since $F \equiv 1$ on $K \cap \frac{1}{M'}D$, by Proposition 3.1,

$$\mu\left(K \cap \frac{1}{M'}D\right) > \frac{1}{2}.$$

In other words, the median of the Euclidean norm with respect to μ is not larger than $\frac{1}{M'}$. Since F is αn-concave,

$$\mathbb{E}_\mu |x|^2 < \frac{c}{(M')^2}$$

by standard concentration inequalities for the Euclidean norm with respect to log-concave measures (it follows, e.g. from Theorem III.3 in [MS], due to Borell). Combining definition (2) and the fact that $L_F^2 = \left(\frac{F(0)}{\int_K F}\right)^{\frac{2}{n}} det(M_F)^{\frac{1}{n}}$ where M_F is the covariance matrix, we get that

$$\left(\frac{\int_K F(x)dx}{F(0)}\right)^{\frac{2}{n}} nL_F^2 \leq \mathbb{E}_\mu |x|^2 < \frac{c}{(M')^2}.$$

Since $\int_K F(x)dx \geq Vol\left(\frac{1}{M'}D \cap K\right) \geq \frac{1}{2}Vol(\frac{1}{M'}D)$ and $F(0) = 1$, we obtain that $L_F^2 < \frac{c'}{nVol(D)^{2/n}} < Const.$ $\square$

Proof of Theorem 1.2: We shall use the notion of l-ellipsoid, and Pisier's estimate for $M(K)M^*(K)$. We refer the reader to [Pi] or [MS] for definitions and proofs. Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. There exists a linear image $\tilde{K}$ of K such that its l-ellipsoid is the standard Euclidean ball. By Pisier's estimate,

$$M^*(\tilde{K})M(\tilde{K}) < c \log d_{BM}(K, D) < c' \log n.$$

According to Corollary 3.2, there exists an αn-concave function F supported exactly on $\tilde{K}$, with $\alpha = cM(\tilde{K})M^*(\tilde{K})$ and $L_F < c_1$. By Lemma 2.3 we get that $L_{K_F} < c_2$. From Lemma 2.2,

$$d_{BM}(K, K_F) \leq d_G(\tilde{K}, K_F) < c\alpha < c' M(\tilde{K})M^*(\tilde{K}) < C \log n.$$

This completes the proof. □

4 The quasi-convex case

We define the covering number of $K \subset \mathbb{R}^n$ by $T \subset \mathbb{R}^n$ as

$$N(K,T) = \min \left\{ N > 0; \exists x_1, .., x_N \in \mathbb{R}^n,\ K \subset \bigcup_{i=1}^{N} x_i + T \right\}.$$

Every convex body $K \subset \mathbb{R}^n$ is associated with a special ellipsoid, called a Milman ellipsoid or an M-ellipsoid. An M-ellipsoid may be defined by the following theorem, which was proved for the convex case in [M4] (see also chapter 7 in [Pi]). The extension to the quasi convex case appears in [BBS].

Theorem 4.1 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric quasi-convex body with constant β. Then there exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ with $Vol(\mathcal{E}) = Vol(K)$ such that*

$$N(K, \mathcal{E}) < e^{cn}, \quad N(\mathcal{E}, K) < e^{cn}$$

where $c = c(\beta) > 0$ depends solely on β. We say that $\mathcal{E}$ is an M-ellipsoid of K (with constant c).

If a Euclidean ball of appropriate radius is an M-ellipsoid of K, we say that K is in M-position (with some constant). The following lemma is standard:

Lemma 4.2 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric quasi-convex body with constant β such that $Vol(K) = 1$, and which is in M-position with constant $c = c(\beta)$. Then,*

1. $Vol(K \cap \sqrt{n}D)^{1/n} > c' Vol(D)^{1/n}$.

2. $K \subset e^{\tilde{c}n} D$

where $c' = c'(\beta) > 0$, $\tilde{c} = \tilde{c}(\beta) > 0$ *depend solely on* β.

Proof: All constants in this proof depend on β. Let $\mathcal{D}_n$ be a Euclidean ball of volume one in $\mathbb{R}^n$. Then $N(K, \mathcal{D}_n) < e^{\bar{c}n}$. Since $c < Vol(\sqrt{n}D)^{1/n} < C$, then also $N(K, \sqrt{n}D) < e^{cn}$ (e.g. Lemma 7.5 in [Pi]). Hence there exists a point $x \in \mathbb{R}^n$ such that $Vol\,(K \cap (x + \sqrt{n}D)) > e^{-cn}$. Since K is centrally-symmetric, $K \cap (-x + \sqrt{n}D) \neq \emptyset$. By quasi-convexity,

$$\emptyset \neq \left[K \cap \left(x + \sqrt{n}D\right)\right] + \left[K \cap \left(-x + \sqrt{n}D\right)\right] \subset \beta K \cap 2\sqrt{n}D$$

and hence $Vol(\beta K \cap 2\sqrt{n}D) > e^{-cn}$, as it contains a translation of $K \cap (x + \sqrt{n}D)$. Since $\beta \geq 2$,

$$Vol(K \cap \sqrt{n}D) \geq \frac{1}{\beta^n} Vol(\beta K \cap 2\sqrt{n}D) > e^{-(c+\log\beta)n}.$$

To obtain that $K \subset e^{\tilde{c}n}D$, we just use the fact that K is a star body, and that a segment of length larger than $2\sqrt{n}e^{cn}$ cannot be covered by e^{cn} balls of radius $\sqrt{n}$. $\square$

Let $K \subset \mathbb{R}^n$ be a centrally-symmetric quasi-convex body with constant β (in short "a β-quasi-body"). Assume that $Vol(K) = 1$ and that K is in M-position. Let us construct the following function on K:

$$F_K(x) = \begin{cases} 1 & |x| \leq \sqrt{n} \\ \left(1 - \frac{|x| - \sqrt{n}}{M_x - \sqrt{n}}\right)^{\alpha n} & |x| > \sqrt{n} \end{cases}$$

for some $\alpha > 0$ to be determined later, where

$$M_x = \sup\left\{r > 0; r\frac{x}{|x|} \in K\right\}.$$

F_K is not log-concave, yet we may still consider the centrally-symmetric set $K_{F_K} \subset \mathbb{R}^n$, defined in Section 2. Note that the restriction of F_K to any straight line through the origin is αn-concave on its support, hence it is possible to apply Lemma 2.2 or Lemma 2.3. We begin with a one-dimensional lemma.

Lemma 4.3 *Let* $0 < a < b$ *and* $\alpha > 1$ *be such that* $b > 2a\left(1 + \frac{\alpha}{e}\right)$. *Let* n *be a positive integer. Then,*

$$\int_a^b \left(1 - \frac{t-a}{b-a}\right)^{\alpha n} t^n dt < \left(\frac{c_1}{\alpha}\right)^n \int_a^b t^n dt$$

where $c_1 > 0$ *is a numerical constant.*

Proof: Denote the integral on the left by I and the integral on the right by $J = \frac{1}{n+1}\left[b^{n+1} - a^{n+1}\right]$. Substituting $s = \frac{t-a}{b-a}$ obtains

$$\begin{aligned} I &= (b-a)\int_0^1 (1-s)^{\alpha n}\,(a+(b-a)s)^n\,ds \\ &= (b-a)\sum_{i=0}^{n}\binom{n}{i} a^{n-i}(b-a)^i \int_0^1 (1-s)^{\alpha n} s^i ds \end{aligned}$$

and using (4),

$$I = (b-a)a^n \sum_{i=0}^{n} \frac{\binom{n}{i}}{(\alpha n + i + 1)\binom{\alpha n + i}{i}} \left(\frac{b-a}{a}\right)^i.$$

The estimate (3) along with some trivial inequalities, yields that

$$I \leq \frac{b-a}{\alpha n} a^n \sum_{i=0}^{n} \left(\frac{e}{\alpha}\right)^i \left(\frac{b-a}{a}\right)^i = \frac{b-a}{\alpha n} a^n \frac{q^{n+1}-1}{q-1}$$

where $q = \frac{e(b-a)}{\alpha a}$. We assumed that $q \geq 2$, and hence

$$I \leq \frac{2}{en}(aq)^{n+1} = \frac{2}{en}\left(\frac{e}{\alpha}\right)^n (b-a)^{n+1} < \left(\frac{c}{\alpha}\right)^n J.$$

$\square$

Next we show that for a suitable value of α, which is just a numerical constant, most of the mass of F_K is not far from the origin.

Lemma 4.4 *For any $\alpha > 1$,*

$$\int_{\mathbb{R}^n \setminus c_2\alpha\sqrt{n}D} F_K(x)dx < \left(\frac{c_1}{\alpha}\right)^{n-1} Vol(K)$$

where c_1 is the constant from Lemma 4.3 and $0 < c_2 \leq 2 + \frac{2}{e}$ is a numerical constant.

Proof: Note that

$$\int_{\mathbb{R}^n \setminus \sqrt{n}D} F_K(x)dx = \int_{S^{n-1}} \int_{\sqrt{n}}^{\max\{M_\theta, \sqrt{n}\}} \left(1 - \frac{r-\sqrt{n}}{M_\theta - \sqrt{n}}\right)^{\alpha n} r^{n-1} dr d\theta$$

where $d\theta$ is the induced surface area measure on the sphere. Let $E = \{\theta \in S^{n-1}; M_\theta > c_2\alpha\sqrt{n}\}$. By Lemma 4.3,

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus c_2\alpha\sqrt{n}D} F_K(x)dx \\ &< \int_E \int_{\sqrt{n}}^{M_\theta} \left(1 - \frac{r-\sqrt{n}}{M_\theta - \sqrt{n}}\right)^{\alpha n} r^{n-1} dr d\theta \\ &< \left(\frac{c_1}{\alpha}\right)^{n-1} \int_E \int_{\sqrt{n}}^{M_\theta} r^{n-1} dr d\theta < \left(\frac{c_1}{\alpha}\right)^{n-1} Vol(K). \end{aligned}$$

$\square$

Lemma 4.5 *Assume that $K \subset \mathbb{R}^n$ is a β-quasi-body of volume one in M-position. Then for $\alpha = c_3(\beta)$,*

$$L_{F_K} < c_4(\beta)$$

where $c_3(\beta), c_4(\beta)$ depend solely on β, not on K or on n.

Proof: By Lemma 4.2,

$$Vol\left(K \cap \sqrt{n}D\right)^{1/n} > c'(\beta).$$

If $\alpha = c_3(\beta)$ is suitably chosen, then by Lemma 4.4,

$$\int_{\mathbb{R}^n \setminus c_2\alpha\sqrt{n}D} F_K(x)dx < \left(\frac{c_1}{\alpha}\right)^{n-1} < \frac{\alpha}{c_1}\left(\frac{1}{e^{2\tilde{c}(\beta)}}\right)^n Vol\left(K \cap \sqrt{n}D\right).$$

Define a measure by $\mu(E) = \frac{\int_E F_K(x)dx}{\int_{\mathbb{R}^n} F_K(x)dx}$. Since F_K equals 1 on $K \cap \sqrt{n}D$, we get that

$$\mu(\mathbb{R}^n \setminus c_2\alpha\sqrt{n}D) < \frac{\alpha}{c_1}\left(\frac{1}{e^{2\tilde{c}(\beta)}}\right)^n.$$

Since $K \subset e^{\tilde{c}(\beta)n}D$, then

$$\mathbb{E}_\mu|x|^2 < (c_2\alpha)^2 n + \frac{\alpha}{c_1}\left(\frac{1}{e^{2\tilde{c}(\beta)}}\right)^n \cdot e^{2\tilde{c}(\beta)n} < c(\beta)n.$$

Therefore, as in Corollary 3.2, $L_{F_K}^2 < c(\beta)\left(\frac{F_K(0)}{\int F_K}\right)^{\frac{2}{n}}$. Note that $F_K(0) = 1$. Since $\int F_K \geq Vol(K \cap \sqrt{n}D)$, we conclude that

$$L_{F_K}^2 < c_4(\beta).$$

$\square$

Proof of Theorem 1.4: Let $K \subset \mathbb{R}^n$ be a C-quasi-body. Let $\tilde{K}$ be a linear image of K such that $Vol(\tilde{K}) = 1$ and $\tilde{K}$ is in M-position (with a constant that depends only on C). Consider the function $F_{\tilde{K}}$ for $\alpha = c_3(C)$. By Lemma 2.2, the body $T = K_{F_{\tilde{K}}}$ satisfies

$$d_G(\tilde{K}, T) < c'(C)$$

for some function $c'(C) > 0$. Also, by Lemma 2.3 and Lemma 4.5,

$$L_T < \tilde{c}L_{F_{\tilde{K}}} < \bar{c}(C)$$

for some $\bar{c}(C)$, a function of C. This completes the proof. $\square$

Remark: There exist quasi-bodies with large isotropic constants. For example, fix $\{e_1, .., e_n\}$ an orthonormal basis in $\mathbb{R}^n$, and let $K = B_1^n \cup \bigcup_{i=1}^n e_i + B_1^n$ where $B_1^n = \{x; \sum_i |\langle x, e_i\rangle| \leq 1\}$. The quasi-convex body K has an isotropic constant of order $\sqrt{n}$, the largest possible order. However, if a quasi-body is close to an ellipsoid, then its isotropic constant is controlled by the distance to the ellipsoid. Also, a quasi-body with a small outer volume ratio has a universally bounded isotropic constant.

5 Consequences of the proof

Here we present a few results which are byproducts of our methods. Our first two propositions enrich the family of convex bodies for which Question 1.1 has an affirmative answer. In this section $Vol(T)$ denotes the volume of a set $T \subset \mathbb{R}^n$ relative to its affine hull.

Lemma 5.1 *Let $K \subset \mathbb{R}^n$ be an isotropic centrally-symmetric convex body of volume one, $0 < \lambda < 1$ and $L_K < A$ for some $A > 1$. Then for any subspace E of dimension λn,*

$$Vol(K \cap E)^{\frac{1}{n}} < c(A)$$

where $c(A)$ depends solely on A, and is independent of the body K and of the dimension n.

Proof: Since $\mathbb{E}_K |x|^2 < nA^2$, the median of the function $|x|$ on K is smaller than $2\sqrt{n}A$. Then $K' = K \cap 2\sqrt{n}AD$ satisfies $Vol(K') > \frac{1}{2}$. Also, given any subspace $E \subset \mathbb{R}^n$ of dimension λn,

$$Vol(K' \cap E) \leq Vol(2\sqrt{n}AD \cap E) \leq \left(c\frac{A}{\sqrt{\lambda}} \right)^{\lambda n}.$$

Since K' is symmetric, $Vol(K') \leq Vol(K' \cap E) Vol(Proj_{E^\perp} K')$, where $E^\perp$ is the orthogonal complement of E and $Proj_{E^\perp}$ is the orthogonal projection onto $E^\perp$ in $\mathbb{R}^n$. Therefore,

$$Vol\left(Proj_{E^\perp} K\right) \geq Vol\left(Proj_{E^\perp} K'\ \right) \geq \frac{Vol(K')}{Vol(K' \cap E)} \geq \left(c\frac{\sqrt{\lambda}}{A} \right)^{\lambda n}.$$

We denote the polar body of K by $K^\circ = \{y \in \mathbb{R}^n ; \forall x \in K, \langle x, y \rangle \leq 1\}$. By Santaló's inequality [Sa] and reverse Santaló [BM] (recall that projection and section are dual operations),

$$\begin{aligned} & Vol(K \cap E) Vol\left(Proj_{E^\perp} K\right) \qquad (8) \\ < \ & \left(\frac{c}{\lambda n}\right)^{\lambda n} \left(\frac{c}{(1-\lambda)n}\right)^{(1-\lambda)n} \frac{1}{Vol\left(Proj_E K^\circ\right) Vol(K^\circ \cap E^\perp)} \\ < \ & \left(\frac{c'}{n}\right)^n \frac{1}{Vol(K^\circ)} < \left(\frac{c''}{n}\right)^n \frac{1}{Vol(D)^2} Vol(K) < \tilde{c}^n Vol(K). \end{aligned}$$

Hence,

$$Vol(K \cap E)^{\frac{1}{n}} < \tilde{c} \frac{Vol(K)^{\frac{1}{n}}}{Vol\left(Proj_{E^\perp} K\right)^{\frac{1}{n}}} < \tilde{c} \left(c\frac{A}{\sqrt{\lambda}} \right)^{\lambda} < c' A^\lambda$$

and the lemma is proven, with $c(A) = cA > cA^\lambda$. □

The next proposition states that the isomorphic slicing conjecture holds for all projections to proportional dimension of bodies with a bounded isotropic constant.

Proposition 5.2 *Let $K \subset \mathbb{R}^n$ be a body with $L_K < A$, and let $0 < \lambda < 1$. Then for any subspace E of dimension λn, there exists a convex body $T \subset E$ such that*

$$d_{BM}(Proj_E(K), T) < c'(\lambda), \quad L_T < c(\lambda, A)$$

where $Proj_E$ is the orthogonal projection onto E in $\mathbb{R}^n$, and $c'(\lambda), c(\lambda, A)$ are independent of K and of n.

Proof: We may assume that K is of volume one and in isotropic position. For $x \in E$, define

$$f(x) = Vol(K \cap [E^{\perp} + x]).$$

For any $\theta_1, \theta_2 \in E$,

$$\int_E \langle x, \theta_1 \rangle \langle x, \theta_2 \rangle f(x) dx = \int_K \langle x, \theta_1 \rangle \langle x, \theta_2 \rangle dx.$$

Hence by Lemma 5.1,

$$L_f = (f(0))^{\frac{1}{\lambda n}} L_K < Vol(K \cap E^{\perp})^{\frac{1}{\lambda n}} A < c(A)^{\frac{1}{\lambda}} A = c'(\lambda, A).$$

Set $T = K_f$. By Lemma 2.3 we know that $L_T < \tilde{c} L_f < c''(\lambda, A)$. Also, by Brunn-Minkowski (e.g. [Schn]) f is $(1-\lambda)n$-concave. By Lemma 2.2 $d_G(T, Proj_E(K)) < c\frac{1-\lambda}{\lambda}$, and the proof is complete. $\square$

Our next proposition verifies the isomorphic slicing conjecture under the condition that at least a small portion of K (say, of volume larger than $e^{-\sqrt{n}}$) is located not too far from the origin.

Proposition 5.3 *Let $K \subset \mathbb{R}^n$ be a body of volume one, such that $K \subset \beta n D$. Assume that $Vol(K \cap \gamma\sqrt{n}D) > e^{-\delta\sqrt{n}}$. Then there exists a body $T \subset \mathbb{R}^n$ such that*

$$d_{BM}(K, T) < c\left(1 + \frac{\beta\delta}{\gamma}\right), \quad L_T < c'\gamma$$

where $c, c' > 0$ are numerical constants.

Proof: If $K \subset 2\gamma\sqrt{n}D$, the proposition is trivial since $L_K < c'\gamma$. Assume the contrary, and denote $C = K \cap 2\gamma\sqrt{n}D$. As in Section 3, we define

$$f(x) = \inf\{0 \leq t \leq 1; x \in (1-t)C + tK\}$$

and consider the density $F(x) = (1 - f(x))^{\alpha n}$ on K for $\alpha = c'\frac{V(K,1;C,n-1)}{Vol(C)}$. As in Proposition 3.1, we get that $\int_C F(x)dx > \frac{1}{2}\int_K F(x)dx$. The same argument used in Corollary 3.2 shows that

$$L_{K_F} < c'\gamma, \quad d_G(K_F, K) < c\frac{V(K,1;C,n-1)}{Vol(C)}.$$

Hence, it remains to show that $\frac{V(K,1;C,n-1)}{Vol(C)} \leq 1 + \frac{\beta\delta}{\gamma}$. Define $f(t) = Vol(K \cap tD)$. According to our assumption, $\log f(\gamma\sqrt{n}) > -\delta\sqrt{n}$ and $\log f(2\gamma\sqrt{n}) < 0$. We conclude that there exists $\gamma\sqrt{n} < t_0 < 2\gamma\sqrt{n}$ with $(\log f(t_0))' < \frac{\delta}{\gamma}$. By Brunn-Minkowski inequality, $\log f$ is concave and $(\log f)'$ is decreasing. Therefore, for $t = 2\gamma\sqrt{n} \geq t_0$,

$$(\log f(t))' = \frac{Vol(K \cap tS^{n-1})}{Vol(K \cap tD)} < \frac{\delta}{\gamma}.$$

For $x \in \partial C$, we denote by ν_x the outer unit normal to C at x, if it is unique (it is unique except for a set of measure zero, see [Schn]). Let $h_K(x) = \sup_{y \in K}\langle x, y\rangle$. Then (see [Schn]),

$$\begin{aligned} V(K,1;C,n-1) &= \frac{1}{n}\int_{\partial C} h_K(\nu_x)dx \\ &= \frac{1}{n}\int_{K \cap tS^{n-1}} h_K(x)dx + \frac{1}{n}\int_{\partial C \setminus tS^{n-1}} h_C(\nu_x)dx \\ &\leq \frac{1}{n}\left(\frac{\delta}{\gamma}Vol(C)\right)\beta n + Vol(C) = \left(1 + \frac{\beta\delta}{\gamma}\right)Vol(C) \end{aligned}$$

where we used the fact that $h_K \leq \beta n$ and that $Vol(C) = \frac{1}{n}\int_{\partial C} h_C(\nu_x)dx$. This completes the proof. $\square$

Following Pisier (e.g. [Pi]), we say that K is in M-position of order α with constants c_α, c'_α if $Vol(K) = Vol(rD)$ and for all $t > 1$

$$max\{N(K, tc_\alpha rD), N(rD, tc_\alpha K)\} < e^{c'_\alpha \frac{n}{t^\alpha}}. \tag{9}$$

By a duality theorem [AMS], if K is in M-position of order α, then also

$$max\left\{N\left(K^\circ, c'c_\alpha t\frac{1}{r}D\right), N\left(\frac{1}{r}D, c'c_\alpha tK^\circ\right)\right\} < e^{\tilde{c}_\alpha \frac{n}{t^\alpha}}$$

for some numerical constant $c' > 0$. A fundamental theorem of Pisier [Pi] states that for any $\alpha < 2$, a centrally-symmetric convex body has a linear image in M-position of order α, with some constants that depend solely on α.

Next, we show that bodies with a relatively small isotropic constant satisfy half of the requirements of Pisier's M-position of order 1.

Proposition 5.4 *Let $K \subset \mathbb{R}^n$ be a convex isotropic body whose volume is one and such that $L_K < A$ for some number A. Then for any $t > 1$,*

$$N(K, ctA\sqrt{n}D) < exp\left(c'\frac{n}{t}\right)$$

where $c, c' > 0$ are numerical cosntants.

Proof: If $K \subset 4A\sqrt{n}D$, then trivially $N(K, 4At\sqrt{n}D) = 1$ and there is nothing to prove. Otherwise, denote $f(t) = Vol(K \cap tD)$. The median of the Euclidean norm on K is smaller than $2\sqrt{n}A$, hence $f(2\sqrt{n}A) \geq \frac{1}{2}$. Also, $f(4\sqrt{n}A) < 1$. Therefore, there exists a point $t_0 \in [2\sqrt{n}A, 4\sqrt{n}A]$ such that

$$\frac{Vol_{n-1}(K \cap t_0 S^{n-1})}{Vol_n(K \cap t_0 D)} = (\log f(t_0))' < \frac{\log 2}{4\sqrt{n}A - 2\sqrt{n}A} = \frac{c}{\sqrt{n}A}.$$

Denote $T = K \cap t_0 D$. For $x \in \partial T$, denote by ν_x the outer unit normal to T at x, if it is unique. Since K is isotropic, $K \subset \tilde{c}nAD$ (see [MP1]), and

$$\begin{aligned} &\int_{K \cap t_0 S^{n-1}} h_K(\nu_x)dx \qquad (10) \\ \leq\ & Vol_{n-1}(K \cap t_0 S^{n-1})\tilde{c}nA \leq \frac{c}{\sqrt{n}A} Vol(T)\tilde{c}nA = c'\sqrt{n}Vol(T). \end{aligned}$$

Because $Vol(T) = \frac{1}{n}\int_{\partial T} h_T(\nu_x)dx$,

$$\int_{\partial T \setminus t_0 S^{n-1}} h_K(\nu_x)dx = \int_{\partial T \setminus t_0 S^{n-1}} h_T(\nu_x)dx \leq nVol(T). \qquad (11)$$

Since $\partial T = \partial T \setminus t_0 S^{n-1} \cup [K \cap t_0 S^{n-1}]$, adding (10) to (11) obtains

$$nV(T, n-1; K, 1) = \int_{\partial T} h_K(\nu_x)dx \leq nVol(T)\left[1 + \frac{c'}{\sqrt{n}}\right].$$

Therefore $V(T, n-1; T + \varepsilon K, 1) \leq Vol(T)\left[1 + \varepsilon\left(1 + \frac{c'}{\sqrt{n}}\right)\right]$ for any $\varepsilon > 0$. By Minkowsi inequality (e.g. [Schn]),

$$Vol(T)^{\frac{n-1}{n}} Vol(T + \varepsilon K)^{\frac{1}{n}} \leq V(T, n-1; T + \varepsilon K, 1)$$

and hence

$$Vol(T + \varepsilon K)^{\frac{1}{n}} \leq Vol(T)^{\frac{1}{n}}\left[1 + \varepsilon\left(1 + \frac{c'}{\sqrt{n}}\right)\right].$$

Denote $t = \frac{1}{\varepsilon}$. Then for any $t > 0$ (see e.g. Lemma 4.16 in [Pi]),

$$N(K, 2tT) \leq \frac{Vol(K + tT)}{Vol(tT)} \leq \left[1 + \frac{1}{t}\left(1 + \frac{c'}{\sqrt{n}}\right)\right]^n < e^{c_1 \frac{n}{t}}$$

where $c_1 < 1 + \frac{c'}{\sqrt{n}}$ is in fact very close to one. For $t \geq 1$,

$$N(K, 4At\sqrt{n}D) \leq N(K, 2tt_0 D) \leq N(K, 2t[K \cap t_0 D]) \leq e^{c_1 \frac{n}{t}}$$

since $t_0 \geq 2\sqrt{n}A$ and the proposition is proven. □

Remark: As is evident from the proof, Proposition 5.4 also holds for any $A > 0$ that satisfies $Vol(K \cap 2\sqrt{n}A) > e^{-\sqrt{n}}$. This is a much weaker requirement than $L_K < A$.

The next Proposition follows immediately from Proposition 2.2 in [KM2] and Theorem 5.2 in [Pi] (due to Carl [Ca]).

Proposition 5.5 *Assume that there exists $c > 0$ such that for any dimension n and for any centrally symmetric convex body $K \subset \mathbb{R}^n$ we have $L_K < c$. Then for any isotropic centrally symmetric convex body $K \subset \mathbb{R}^n$ of volume one,*

$$N(\sqrt{n}D, c'tK) < exp\left(c'\frac{n}{t^{\frac{1}{3}}}\right)$$

where $c' = c'(c)$ depends only on c. Furthermore, the exponent "$\frac{1}{3}$" may be replaced by number smaller than $\frac{1}{2}$.

Proposition 5.4 and Proposition 5.5 together imply that if the hyperplane conjecture is correct, then the isotropic position is an M-position of order α for any $\alpha < \frac{1}{2}$. This information adds to the result of K. Ball, which states that the isotropic position is an M-position under the slicing hypothesis.

For $K \subset \mathbb{R}^n$, the volume ratio of K is defined as

$$v.r.(K) = \sup_{\mathcal{E} \subset K} \left(\frac{Vol(K)}{Vol(\mathcal{E})}\right)^{\frac{1}{n}}$$

where the supremum is over all ellipsoids contained in K. We denote

$$L_n = \sup\{L_K \ ; \ K \subset \mathbb{R}^n \ is \ a \ centrally - symmetric \ convex \ body\},$$

$$L_n(a) = \sup\{L_K \ ; \ K \subset \mathbb{R}^n, \ v.r.(K) \leq a\}.$$

In [BKM2] it is proven that for any $\delta > 0$,

$$L_n < c(\delta) \ L_n(v(\delta))^{1+\delta} \tag{12}$$

where $c(\delta), v(\delta) \approx e^{\frac{c}{1-\delta}}$. Next, we improve the dependence in (12).

Corollary 5.6 *There exist $c_1, c_2 > 0$, such that for all n,*

$$L_n < c_1 L_n(c_2).$$

Proof: Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body of volume one. Assume that K is in M-position. Then there exists a rotation $U \in O(n)$ such that the body $K + UK$ satisfies $v.r.(K+UK) < c$, for some numerical constant $c > 0$ (see [M6]). Define the following function:

$$f(x) = (1_K * 1_{UK})(x) = \int_{\mathbb{R}^n} 1_K(t) 1_{UK}(x - t) dt = Vol(K \cap (x + UK))$$

where $1_K, 1_{UK}$ are the characteristic functions of K and UK. It is straightforward to validate that $\int_{\mathbb{R}^n} f = 1$ and that $supp(f) = K + UK$. For any $\theta \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 f(x) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle t + x - t, \theta \rangle^2 1_K(t) 1_{UK}(x - t) dt dx$$

$$= \int_K \langle x, \theta \rangle^2 dx + \int_{UK} \langle x, \theta \rangle^2 dx$$

and hence $M(f) = M(K) + M(UK)$. In addition, since $det(M(K)) = det(M(U(K))$ and the matrices are positive,

$$det(M(f))^{1/n} \geq det(M(K))^{1/n} + det(M(UK))^{1/n} = 2det(M(K))^{1/n}.$$

Since $f(0) = Vol(K \cap UK) > c^n$ (e.g. [M6]), it follows that $L_K < c' L_f$. The function f is also n-concave, for it is a convolution of characteristic functions of convex bodies (e.g. the appendix of [GrM2]). Therefore, the body $T = K_f$ satisfies $d_G(T, K + UK) < c$, and $v.r.(T) < c_2$. Since $L_K < cL_f < c_1 L_T$, the corollary follows. □

Remarks.

1. At present, there is no good proven bound for $M(K)M^*(K)$ in the non-symmetric case, and hence the central symmetry assumption of the body is crucial to the proof of Theorem 1.2. However, some of the statements in this paper may be easily generalized to non-symmetric bodies. In particular, Theorem 1.4, Propositions 5.2–5.5 and Corollary 5.6 also hold in the non-symmetric case.

2. The proof of Corollary 5.6 reduces the problem of bounding the isotropic constant of K, to the problem of bounding the isotropic constant of a body close to $K+UK$, where $U \in O(n)$ and K is in M-position. If K is not centrally-symmetric, yet its barycenter is at the origin, then $Vol(K \cap (-K)) > c^n$ (see [MP2]). Choosing $U = -Id$ we find a centrally-symmetric body T, close to $K - K$, with $L_K < cL_T$. Hence, universal boundness of the isotropic constant of convex, centrally-symmetric bodies would imply the universal boundness of the isotropic constant of non-symmetric convex bodies as well. We also conclude Bourgain's estimate $L_K < cn^{1/4} \log n$ for $K \subset \mathbb{R}^n$ being a non-symmetric convex body. This was previously proved in [Pa].

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Rapid Steiner symmetrization of most of a convex body[7]

Abstract. For an arbitrary n-dimensional convex body, at least almost n Steiner symmetrizations are required in order to symmetrize the body into an isomorphic ellipsoid. We say that a body $T \subset \mathbb{R}^n$ is "quickly symmetrizable" if for any $\varepsilon > 0$ there exist only $\lfloor \varepsilon n \rfloor$ symmetrizations that transform T into a body which is $c(\varepsilon)$-isomorphic to an ellipsoid, where $c(\varepsilon)$ depends solely on ε. In this note we ask, given a body $K \subset \mathbb{R}^n$, whether it is possible to remove a small portion of its volume and obtain a body $T \subset K$ which is quickly symmetrizable? We show that this question, for a large variety of $c(\varepsilon)$, is equivalent to the slicing problem.

1 Introduction

We work in $\mathbb{R}^n$, endowed with the usual scalar product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $| \cdot |$. Let $K \subset \mathbb{R}^n$ be a convex body, and let $H = \{x \in \mathbb{R}^n ; \langle x, h \rangle = 0\}$ be a hyperplane through the origin in $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ there exists a unique decomposition $x = y + th$ where $y \in H, t \in \mathbb{R}$, so we can refer to (y, t) as coordinates in $\mathbb{R}^n$. The result of a "Steiner symmetrization of K with respect to h" is the body:

$$S_H(K) = \left\{ (x,t) \ ; \ K \cap (x + \mathbb{R}h) \neq \emptyset \ , \ |t| \leq \frac{1}{2} Meas\{K \cap (x + \mathbb{R}h)\} \right\}$$

where $Meas$ is the one-dimensional Lebesgue measure in the line $x + \mathbb{R}h$. Steiner symmetrization is a well-known operation in convexity. It preserves the volume of a body and transforms convex sets to convex sets (e.g. [BF]). A suitably chosen finite sequence of Steiner symmetrizations may transform an arbitrary convex body into a body that is close to a Euclidean ball. Less expected is the fact that relatively few symmetrizations suffice for obtaining a body that is close to a Euclidean ball. The following theorem, which improves a previous result of [BLM2], appears in [KM1] ($D = \{x \in \mathbb{R}^n ; |x| \leq 1\}$ is the standard Euclidean ball in $\mathbb{R}^n$):

Theorem 1.1 *For any $n \geq 2$ and any convex body $K \subset \mathbb{R}^n$ with $Vol(K) = Vol(D)$, there exist $3n$ Steiner symmetrizations that transform the body K into $\tilde{K}$ such that:*

$$\frac{1}{c} D \subset \tilde{K} \subset cD$$

where $c > 0$ is a numerical constant.

[7] This chapter corresponds to the paper [KM2].

Given a convex body $K \subset \mathbb{R}^n$, we define its "geometric distance" from a convex body $T \subset \mathbb{R}^n$ as

$$d_G(K,T) = \inf\left\{ab; \frac{1}{a}T \subset K \subset bT,\ a,b>0\right\}$$

and we set $d_G(K) = d_G(K,D)$, the geometric distance of K from a Euclidean ball. The Banach-Mazur distance of K from a Euclidean ball is $d_{BM}(K) = \inf_T d_G(TK)$, where the infimum runs over all invertible linear transformations. d_{BM} measures the geometric distance of K from an ellipsoid. Notice that we do not allow translations of the convex body when defining the distances.

The constant "3" in Theorem 1.1 is not optimal (see more accurate results in [KM1]). However, for bodies such as the cross-polytope $B_1^n = \{x \in \mathbb{R}^n; \sum |x_i| \leq 1\}$, at least $n - C\log n$ symmetrizations are required in order to symmetrize B_1^n into a body which is $\sqrt{C/2}$-close to an ellipsoid (see [KM1]). Therefore it is impossible to symmetrize a general convex body in $\mathbb{R}^n$ into an isomorphic ellipsoid, using significantly less than n symmetrizations. Let us consider another example: the cube $B_\infty^n = \{x \in \mathbb{R}^n; \forall i\ |x_i| \leq 1\}$ has a very short symmetrization process. For any $\varepsilon > 0$, there exist $\lfloor \varepsilon n \rfloor$ symmetrizations that transform B_∞^n into a body whose distance from a Euclidean ball is smaller than $c\sqrt{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}}$ for some numerical constant $c > 0$. Given a convex body $K \subset \mathbb{R}^n$, we say that "K is $c(\varepsilon)$-symmetrizable" if for any $\varepsilon > 0$, there exist $\lfloor \varepsilon n \rfloor$ symmetrizations that transform K into $\tilde{K}$ with $d_{BM}(K) < c(\varepsilon)$. Using this terminology, the cube is $c(\varepsilon)$-symmetrizable for $c(\varepsilon) = c\sqrt{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}}$. Note that here $c(\varepsilon)$ does not depend on the dimension n, and grows polynomially in $\frac{1}{\varepsilon}$ as ε tends to zero. Here we ask whether an arbitrary convex body $K \subset \mathbb{R}^n$ contains a large part which is $c(\varepsilon)$-symmetrizable, with $c(\varepsilon)$ being a polynomial in $\frac{1}{\varepsilon}$, whose coefficients do not depend on the dimension n.

Question 1.2 *Does there exist a function $c(\varepsilon)$, which is a polynomial in $\frac{1}{\varepsilon}$, such that for any dimension n, for any convex body $K \subset \mathbb{R}^n$, there exists a convex body $T \subset K$ with $Vol(T) > \frac{9}{10}Vol(K)$ such that T is $c(\varepsilon)$-symmetrizable?*

The number "$\frac{9}{10}$" has no special meaning, and may be replaced with any $\alpha < 1$. An apriori unrelated question is concerned with the isotropic constant. Let $K \subset \mathbb{R}^n$ be a convex body. K has an affine image $\tilde{K}$, which is unique up to orthogonal transformations, such that the barycenter of $\tilde{K}$ is at the origin, $Vol(\tilde{K}) = 1$, and

$$\int_{\tilde{K}} \langle x, \theta\rangle^2 dx = L_K^2 |\theta|^2$$

for any $\theta \in \mathbb{R}^n$, where L_K does not depend on θ (see [MP1]). We say that L_K is the isotropic constant of K. A fundamental question in asymptotic convex geometry is the following:

Question 1.3 *Does there exist a constant $c > 0$ such that for any integer n, for any convex body $K \subset \mathbb{R}^n$ we have $L_K < c$?*

The main goal of this note is to show that Question 1.3 and Question 1.2 are equivalent.

Theorem 1.4 *Question 1.2 and Question 1.3 have the same answer.*

Theorem 1.4 connects two properties of the class of all convex bodies in all dimensions, yet formally it does not say anything about an individual body $K \subset \mathbb{R}^n$. We also obtain here results that are applicable to individual bodies. Proposition 4.4 states that given a body $K \subset \mathbb{R}^n$ that contains a large portion which is $c(\varepsilon)$-symmetrizable for some polynomial $c(\varepsilon)$, the isotropic constant of K may be bounded by a quantity that depends solely on the polynomial $c(\varepsilon)$. See also Proposition 3.2 for the opposite direction.

Before turning to the details of the proofs, let us shed some light on the concept of a $c(\varepsilon)$-symmetrizable body $T \subset \mathbb{R}^n$, for a polynomial $c(\varepsilon)$. Assume that T is such a body, for $c(\varepsilon) < c_1 \left(\frac{1}{\varepsilon}\right)^{c_2}$, where $c_1, c_2 > 0$ do not depend on n. Then for any $\varepsilon > 0$ there exist $\lfloor \varepsilon n \rfloor$ symmetrizations of T with respect to special $v_1, .., v_{\lfloor \varepsilon n \rfloor}$, that transform T into $\tilde{T}$ that is $c(\varepsilon)$-close to an ellipsoid. Denote by E the subspace $\left\{v_1, .., v_{\lfloor \varepsilon n \rfloor}\right\}^{\perp}$. By Lemma 2.4 in [KM1],

$$Proj_E(T) = Proj_E(\tilde{T}) \quad \Longrightarrow \quad d_{BM}(Proj_E(T)) < c_1 \left(\frac{1}{\varepsilon}\right)^{c_2}$$

where $Proj_E$ is the orthogonal projection onto E in $\mathbb{R}^n$. Therefore $T \subset \mathbb{R}^n$ has, for any $\varepsilon > 0$, projections to subspaces of dimension $\lceil (1-\varepsilon)n \rceil$ whose distance from an ellipsoid is smaller than some polynomial in $\frac{1}{\varepsilon}$. In fact, as will be explained later, by Theorem 1.1 a body is $c(\varepsilon)$-symmetrizable for a polynomial $c(\varepsilon)$ if and only if it has large projections which are polynomially close to an ellipsoid. Since the latter notion is clearly linearly invariant, then also $c(\varepsilon)$-symmetrizability with a polynomial $c(\varepsilon)$ is a linearly invariant property.

Throughout the paper we denote by $c, c', \tilde{c}, C$ etc. some positive universal constants whose value is not necessarily the same on different appearances. Whenever we write $A \approx B$, we mean that there exist universal constants $c, c' > 0$ such that $cA < B < c'A$. Also, $Vol(T)$ denotes the volume of a set $T \subset \mathbb{R}^n$ relative to its affine hull. A random k-dimensional subspace in $\mathbb{R}^n$ is chosen according to the unique rotation invariant probability measure in the Grassman manifold $G_{n,k}$.

2 An M-position of order α and the isotropic position

For $K, T \subset \mathbb{R}^n$ denote the covering number of K by T as

$$N(K,T) = \min \left\{ N; \exists x_1, .., x_N \in \mathbb{R}^n, K \subset \bigcup_{i=1}^{N} x_i + T \right\}.$$

Let $K \subset \mathbb{R}^n$ be a convex body. An ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ is an M-ellipsoid of K with constant $c > 0$ if

$$\max\{N(K,\mathcal{E}), N(\mathcal{E}, K)\} < e^{cn}.$$

If $\mathcal{E} = D$, we say that K is in M-position with constant $c > 0$. A result by Milman states that any centrally symmetric (i.e. $K = -K$) convex body has a linear image in M-position with some absolute constant (see [M4], or chapter 7 in the book [Pi]). Furthermore, we say that K is in M-position of order α with constants c_α, c'_α if for all $t > 1$

$$max\{N(K, tc_\alpha D), N(D, tc_\alpha K)\} < e^{c'_\alpha \frac{n}{t^\alpha}}.$$

Another common terminology to describe this property is α-regular M-position with the appropriate constants. By a duality theorem [AMS], if K is centrally-symmetric and is in M-position of order α, then also

$$max\{N(K^\circ, c_\alpha t D), N(D, c_\alpha t K^\circ)\} < e^{\tilde{c}_\alpha \frac{n}{t^\alpha}}$$

where $K^\circ = \{y \in \mathbb{R}^n; \forall x \in K, \ \langle x, y \rangle \leq 1\}$. A theorem of Pisier [Pi] states that given a centrally-symmetric $K \subset \mathbb{R}^n$, for any $\alpha < 2$, there exists a linear image of K which is in M-position of order α with some constants that depend solely on α.

The assumption of central symmetry in the above discussion is not crucial. In [M5, MP2] it is proven that any convex body whose barycenter lies at the origin has a linear image in M-position with some absolute constant. However, the literature seems to contain no discussion on the existence of regular M-positions for non-symmetric convex bodies. Next, we deduce that a regular M-position exists for any convex body. Begin with a lemma in the spirit of [Kl6].

Lemma 2.1 *Let $K \subset \mathbb{R}^n$ be a convex body whose barycenter is at the origin. Let $E \subset \mathbb{R}^n$ be a subspace, $dim(E) = \lambda n$. Then there exists a convex body $T \subset E$ whose barycenter is at the origin such that*

$$c_1 \lambda Proj_E(K) \subset T \subset c_2 Proj_E(K)$$

where $c_1, c_2 > 0$ are universal constants.

Proof: The proof is just a minor adaptation of the proof of Proposition 5.2 in [Kl6] and we omit its details. For $x \in E$ we define $f(x) = Vol(K \cap [E^\perp + x])$. Then f is a log-concave function, and hence $T = \left\{x \in E; \int_0^\infty f(rx) r^n dr \geq 1\right\}$ is a convex set whose barycenter is at the origin. Since f is a $(1-\lambda)n$-concave function on a λn dimensional space, and its support is $Proj_E(K)$, Lemma 2.2 in [Kl6] completes the proof. □

Proposition 2.2 *Let $K \subset \mathbb{R}^n$ be a convex body whose barycenter is at the origin. Then there exists a linear transformation L such that $\tilde{K} = L(K)$ satisfies, for any $t > 1$,*

$$\max\{N(K, tD), N(D, tK), N(K^\circ, tD), N(D, tK^\circ)\} < exp\left(c\frac{n}{t^{1/6}}\right)$$

where $c > 0$ is a numerical constant.

Proof: Consider the centrally-symmetric convex bodies $\underline{K} = K \cap (-K)$ and $\overline{K} = conv(K, -K)$ where *conv* denotes convex hull. Then $\underline{K} \subset K \subset \overline{K}$ and also $\overline{K^\circ} = (\underline{K})^\circ$. Let $E \subset \mathbb{R}^n$ be a subspace, and denote $k = \lambda n = dim(E)$. Since $Proj_E(\overline{K}) = \overline{Proj_E(K)}$, by [RS]

$$Vol(Proj_E(\overline{K}))^{\frac{1}{k}} \leq 4 Vol(Proj_E(K))^{\frac{1}{k}}. \tag{1}$$

The barycenter of $Proj_E(K)$ may be different from zero. However, by Lemma 2.1 there exists a convex body $T \subset E$ whose barycenter lies at the origin, such that

$$c_1 \lambda Proj_E(K) \subset T \subset c_2 Proj_E(K).$$

For a k-dimensional body T we denote $v.rad.(T) = \left(\frac{Vol(T)}{Vol(D_k)}\right)^{\frac{1}{k}}$ where D_k is a k-dimensional Euclidean unit ball. By Santalo inequality (e.g. [MeP1]) $v.rad.(T)v.rad.(T^\circ) < C$ and hence

$$v.rad.(Proj_E(K^\circ))v.rad.(K \cap E) < \frac{c}{\lambda} \tag{2}$$

for any convex body K whose barycenter is at the origin. Next, by [Ru] and Theorem 1 in [F],

$$v.rad.(\overline{K} \cap E) < C\frac{1}{\lambda} \sup_{x \in E^\perp} v.rad.(K \cap (E + x)) < C'\left(\frac{1}{\lambda}\right)^2 v.rad.(K \cap E).$$

By the reverse Santalo inequality [BM],

$$\begin{aligned} & v.rad.(Proj_E(\underline{K})) \\ & \quad > \frac{c}{v.rad.((\underline{K})^\circ \cap E)} = \frac{c}{v.rad.(\overline{K^\circ} \cap E)} > c'\lambda^2 \frac{1}{v.rad.(K^\circ \cap (E + x))} \end{aligned}$$

where $x \in \mathbb{R}^n$ is the barycenter of K°. By (2) and (1),

$$v.rad.(Proj_E(\underline{K})) > \tilde{c}\lambda^3 v.rad.(Proj_E(K)) > C\lambda^3 v.rad.(Proj_E(\overline{K})).$$

Let us assume that $\underline{K}$ is in 1-regular position. Denote by $\mathcal{E}$ a 1-regular ellipsoid of $\overline{K}$ (i.e. if $L(\overline{K})$ is in M-position of order 1, then $\mathcal{E}$ is defined so that $L(\mathcal{E}) = D$). Since $N(Proj_E(\mathcal{E}), \frac{1}{\lambda}Proj_E(\overline{K})) < exp(c\lambda n)$, we get that

$$v.rad.(Proj_E(\mathcal{E})) < \frac{c}{\lambda} v.rad.(Proj_E(\overline{K})) < \frac{C}{\lambda^4} v.rad.(Proj_E(\underline{K})).$$

Because $N(Proj_E(\underline{K}), \frac{1}{\lambda}Proj_E(D)) < exp(c\lambda n)$, we conclude that

$$v.rad.(Proj_E(\mathcal{E})) < \frac{C}{\lambda^4} v.rad.(Proj_E(\underline{K})) < \frac{C'}{\lambda^5}.$$

This is true for any λn-dimensional subspace E, for any $0 < \lambda < 1$ such that λn is an integer. By standard estimates for the covering number of an ellipsoid by Euclidean balls (e.g. Remark 5.15 in [Pi]), we get that for any $t > 1$,

$$N(\mathcal{E}, tD) < exp\left(cnt^{-\frac{1}{5}}\right), \quad N(D, t\mathcal{E}^\circ) < exp\left(cnt^{-\frac{1}{5}}\right)$$

and hence

$$N(K, tD) \leq N(\overline{K}, tD) \leq N\left(\overline{K}, t^{\frac{1}{6}}\mathcal{E}\right) N\left(\mathcal{E}, t^{\frac{5}{6}}D\right) < exp\left(c\frac{n}{t^{\frac{1}{6}}}\right),$$

$$N(D, tK^\circ) \leq N\left(D, t(\overline{K})^\circ\right) \leq N\left(D, t^{\frac{1}{6}}\mathcal{E}^\circ\right) N\left(\mathcal{E}^\circ, t^{5/6}(\overline{K})^\circ\right) < exp\left(c\frac{n}{t^{\frac{1}{6}}}\right).$$

Trivially $N(D, tK) < N(D, t\underline{K}) < exp(c\frac{n}{t})$ and also $N(K^\circ, tD) < N((\underline{K})^\circ, tD) < exp(c\frac{n}{t})$. We conclude that D is an M-ellipsoid of K of order $\frac{1}{6}$. □

Remark: The power "$\frac{1}{6}$" in Proposition 2.2 is clearly non-optimal and may be improved. We do not know what the optimal power is.

If K is in M-position, then proportional sections of K typically have a small diameter, and proportional projections of K typically contain a large Euclidean ball. If K is also in M-position of order α, then typical sections of dimension $\lfloor(1-\varepsilon)n\rfloor$ have a diameter which is smaller than some polynomial in $\frac{1}{\varepsilon}$, as follows from the next theorem (see e.g. [GM2]).

Theorem 2.3 *Let $K \subset \mathbb{R}^n$ be a convex body in M-position of order α with constants c_α, c'_α. Let E be a random subspace of dimension $(1-\varepsilon)n$. Then with probability larger than $1-e^{-c'\varepsilon n}$,*

$$K \cap E \subset \left(\frac{c(c_\alpha, c'_\alpha)}{\varepsilon^{\frac{1}{2}+\frac{1}{\alpha}}}\right) D$$

$$\left(\frac{\varepsilon^{\frac{1}{2}+\frac{1}{\alpha}}}{c(c_\alpha, c'_\alpha)}\right) D \cap E \subset Proj_E(K)$$

where $c' > 0$ is a numerical constant, and $c(c_\alpha, c'_\alpha)$ depends neither on K nor on n, but solely on its arguments.

Assume that Question 1.3 has an affirmative answer. Our next proposition proves the existence of large projections that contain large Euclidean balls as in Theorem 2.3, for bodies in isotropic position (compare with Proposition 5.4 in [Kl6]).

Proposition 2.4 *Assume a positive answer to Question 1.3. Let $K \subset \mathbb{R}^n$ be a convex isotropic body with volume one whose barycenter is at the origin. Then for any integer $k = (1-\varepsilon)n$ where $0 < \varepsilon < 1$, there exists a subspace E of dimension k with*

$$c\varepsilon^{\beta}\sqrt{n}D \cap E \subset Proj_E(K)$$

where $c > 0$ depends only on the constant in Question 1.3, and $\beta \leq 13$ is a numerical constant. If in addition K is centrally-symmetric, then $\beta \leq 3$.

Proof: We shall use the following observation which appears in [MP1] (Proposition 3.11 there) and in [Ba1]. Although it is stated there for centrally-symmetric bodies, the generalization to the non-symmetric case is straightforward (a formulation appears in [BKM2]). A positive answer to Question 1.3 yields that for any subspace F of dimension k,

$$c_1 < Vol(K \cap F)^{\frac{1}{n-k}} < c_2 \tag{3}$$

where c_1, c_2 depend only on the constant in Question 1.3. Since the barycenter of K is at the origin, then $Vol(K \cap F)Vol(Proj_{F^{\perp}}(K)) \geq Vol(K) = 1$ for any subspace F (see [Sp]). By (3),

$$Vol(Proj_{F^{\perp}}(K))^{\frac{1}{n-k}} > \frac{1}{c_2}. \tag{4}$$

Assume for simplicity that K is centrally-symmetric. Let $\mathcal{E}$ be an M-ellipsoid of order 1 of K, i.e.

$$\max\{N(K, t\mathcal{E}), N(\mathcal{E}, tK)\} < e^{c\frac{n}{t}}$$

where $c > 0$ is a numerical constant. Let $0 < \lambda_1 \leq ... \leq \lambda_n$ be the axes of $\mathcal{E}$. Let $0 < \delta < 1$, and denote by F_1 the subspace spanned by the shortest $\lfloor \delta n \rfloor$ axes of $\mathcal{E}$. Since $N(Proj_{F_1}(K), tProj_{F_1}(\mathcal{E})) < e^{c\frac{n}{t}}$, we obtain that

$$Vol(Proj_{F_1}(K)) < e^{\frac{cn}{t}} \left(t\lambda_{\lfloor \delta n \rfloor}\right)^{\lfloor \delta n \rfloor} Vol(D \cap E)$$

for any $t > 0$. Using (4) and the fact that $Vol(D \cap E)^{\frac{1}{\lfloor \delta n \rfloor}} \approx \frac{1}{\sqrt{\delta n}}$, when we set $t = \frac{1}{\delta}$ we get that $\lambda_{\lfloor \delta n \rfloor} > c'\sqrt{n}\delta^{3/2}$. Assume that $\lfloor \delta n \rfloor = \lfloor \frac{\varepsilon n}{2} \rfloor$ (hence $\delta \leq \frac{1}{2}$), and let F_2 denote the subspace of the longest $\lceil (1-\delta)n \rceil$ axes of $\mathcal{E}$. Since

$$N(Proj_{F_2}(K), t(\mathcal{E} \cap F_2)) \leq N(K, t\mathcal{E}) < e^{c\frac{n}{t}} < e^{2c\frac{(1-\delta)n}{t}}$$

and since a similar inequality holds for $N(\mathcal{E}\cap F_2, tProj_{F_2}(K))$, then $\mathcal{E}\cap F_2$ is an M-ellipsoid of order 1 of $Proj_{F_2}(K)$. Also $c'\delta^{3/2}\sqrt{n}D \cap F_2 \subset \mathcal{E}\cap F_2$. By Theorem 2.3, there exists a subspace $E \subset F_2$ of dimension $(1-2\delta)n \geq (1-\varepsilon)n$, with

$$c\varepsilon^3\sqrt{n}D\cap E \subset c'\varepsilon^{3/2}\mathcal{E}\cap E \subset Proj_E(Proj_{F_2}(K)) = Proj_E(K)$$

and the proposition is proved for centrally-symmetric bodies. Regarding non-symmetric convex bodies, we may repeat the argument using a $\frac{1}{6}$-regular M-ellipsoid, whose existence is guaranteed by Proposition 2.2. We obtain the same conclusion as in the symmetric case, but with a different power of ε. $\square$

Remark: Even in the centrally-symmetric case, our bound $\beta \leq 3$ in Proposition 2.4 is not optimal, and may be improved by considering M-ellipsoids of higher order. We do not know what the best β is.

3 Slicing implies rapid symmetrization

The following lemma is standard. For completeness, we include its proof, which is trivial for centrally-symmetric bodies. We would like to remind the reader that our definitions of distances forbid translations of the bodies.

Lemma 3.1 *Let $K \subset \mathbb{R}^n$ be a convex body and let $E \subset \mathbb{R}^n$ be a subspace such that:*

1. *$Proj_E(K) = K\cap E$ and $d_{BM}(K\cap E) < A$ for some $A \geq 1$.*
2. *$d_{BM}(K\cap E^\perp) < B$ for some $B \geq 1$.*

Then $d_{BM}(K) < cAB$ where $c > 0$ is a numerical constant.

Proof: Applying a linear transformation inside E if necessary, we may assume that $D \subset K\cap E \subset AD$. Let $x \in Proj_E(K)$ be any point. We claim that

$$-x + \left[K\cap\left(x+E^\perp\right)\right] \subset (A+1)K\cap E^\perp.$$

Indeed, since $|x| \leq A$ and since $-\frac{x}{A} \in K$, by convexity of K

$$\frac{-x+\left[K\cap\left(x+E^\perp\right)\right]}{A+1} \subset conv\left[-\frac{x}{A}, K\cap\left(x+E^\perp\right)\right]\cap E^\perp \subset K\cap E^\perp.$$

Therefore, $Proj_{E^\perp}(K) \subset (A+1)K\cap E^\perp$. Now, let $\mathcal{E}$ be an ellipsoid, symmetric with respect to E, such that $\mathcal{E}\cap E \subset K\cap E \subset A\mathcal{E}\cap E$ and $\mathcal{E}\cap E^\perp \subset K\cap E^\perp \subset B\mathcal{E}\cap E^\perp$. Then,

$$\frac{1}{\sqrt{2}}\mathcal{E} \subset conv(K\cap E, K\cap E^\perp) \subset K \subset Proj_E(K)\times Proj_{E^\perp}(K)$$

$$\subset K \cap E \times (A+1)K \cap E^{\perp} \subset (A\mathcal{E} \cap E) \times \left[(A+1)B\mathcal{E} \cap E^{\perp}\right] \subset \sqrt{2}(A+1)B\mathcal{E}$$

and the lemma is proven. □

Let $K \subset \mathbb{R}^n$ be a convex body, and assume that for any $\varepsilon > 0$ there exists a subspace E of dimension $\lfloor \varepsilon n \rfloor$ such that $d_{BM}(Proj_{E^{\perp}}(K)) < c(\varepsilon)$, for some function $c(\varepsilon)$. Consider the body $K \cap E$. According to Theorem 1.1, after $\lfloor 3\varepsilon n \rfloor$ symmetrizations $K \cap E$ may be transformed into an isomorphic Euclidean ball. Apply the same symmetrizations to K, to obtain $\tilde{K}$. Since these symmetrizations include symmetrizations with respect to an orthogonal basis of E, elementary properties of the symmetrization (e.g. [KM1]) together with Lemma 3.1 imply that

$$d_{BM}(\tilde{K}) < c' c(\varepsilon).$$

We conclude, as was mentioned in the introduction, that a convex body is $c(\varepsilon)$-symmetrizable with a $c(\varepsilon)$ which is polynomial in $\frac{1}{\varepsilon}$ if and only if it has projections to dimension $\lfloor (1-\varepsilon)n \rfloor$ whose distance from an ellipsoid is smaller than some polynomial in $\frac{1}{\varepsilon}$.

Before proving one direction of Theorem 1.4, which assumes a positive answer to Question 1.3, let us prove a weaker statement (with an exponential dependence, rather than polynomial), one that is applicable to an individual body $K \subset \mathbb{R}^n$, and does not require uniform boundness of the isotropic constant.

Proposition 3.2 *Let $\varepsilon > 0$, and let $K \subset \mathbb{R}^n$ be a convex body with $L_K < A$ for some $A > 0$. Then there exists a body $T \subset K$ with $Vol(T) > \frac{9}{10} Vol(K)$ and $\lfloor \varepsilon n \rfloor$ Steiner symmetrizations that transform T into $\tilde{T}$ such that*

$$d_{BM}(\tilde{T}, D) < (cA)^{\frac{1}{\varepsilon}}$$

where $c > 0$ is a universal constant.

Proof: Assume that the barycenter of K is at the origin. Let $\mathcal{E}$ be the isotropy ellipsoid of K normalized so that $Vol(\mathcal{E}) = Vol(K)$ (i.e. if $\tilde{K} = L(K)$ is isotropic for a linear operator L, then $\mathcal{E}$ is defined so that $L(\mathcal{E})$ is a Euclidean ball of volume one). Let $T = K \cap cA\mathcal{E}$. By Borell lemma (e.g. Theorem III.3 in [MS]) $Vol(T) > \frac{9}{10} Vol(K)$, if $c > 0$ is suitably chosen. Note that $T \subset cA\mathcal{E}$, and

$$\left(\frac{Vol(cA\mathcal{E})}{Vol(T)} \right)^{1/n} < c'A.$$

By a theorem of Szarek and Tomczak-Jaegermann ([Sz], [SzT]) there exists a subspace E of dimension $\lceil (1-\varepsilon)n \rceil$ such that

$$d_G(Proj_E(T), Proj_E(\mathcal{E})) < (cA)^{\frac{1}{\varepsilon}}.$$

Let us apply the $3\varepsilon n$ symmetrizations that suit $T \cap E^{\perp}$ according to Theorem 1.1, to the body T, and obtain the body $\tilde{T}$. By Lemma 2.1 and Lemma 2.4 from [KM1] (these $3\varepsilon n$ symmetrizations include symmetrizations with respect to an orthogonal basis),

$$\tilde{T} \cap E = Proj_E(\tilde{T}) = Proj_E(T)$$

and also $\tilde{T} \cap E^{\perp}$ has a universally bounded distance from a Euclidean ball. By Lemma 3.1,

$$d_{BM}(\tilde{T}, D) < (c'A)^{\frac{1}{\varepsilon}}$$

which completes the proof. □

The following proposition proves one part of Theorem 1.4.

Proposition 3.3 *Assume that Question 1.3 has a positive answer. Let $\varepsilon > 0$, and let $K \subset \mathbb{R}^n$ be a convex body. Then there exists a body $T \subset K$ with $Vol(T) > \frac{9}{10}Vol(K)$ and $\lfloor \varepsilon n \rfloor$ Steiner symmetrizations that transform T into $\tilde{T}$ such that*

$$d_{BM}(\tilde{T}) < c\frac{1}{\varepsilon^{\beta}}$$

where $c > 0$ is a constant that depends only on the constant in Question 1.3 and $0 < \beta < 13$ is a numerical constant.

Proof: Assume that $Vol(K) = 1$ and that the barycenter of K is at the origin. Let $\mathcal{E}$ be the isotropy ellipsoid of K normalized so that $Vol(\mathcal{E}) = Vol(K)$, and denote $T = K \cap c\mathcal{E}$, where $c > 0$ depends linearly on the constant in Question 1.3. As before, by Borell lemma, $Vol(T) > \frac{9}{10}$. Also, if $c > 0$ is chosen properly, then the isotropy ellipsoid $\mathcal{F}$ of T satisfies $d_G(\mathcal{F}, \mathcal{E}) < c_1$ (e.g. [Bou3]). By Proposition 2.4 there exists a subspace E of dimension $> (1-\varepsilon)n$ with

$$c\varepsilon^{\beta} Proj_E(\mathcal{E}) \subset c'\varepsilon^{\beta} Proj_E(\mathcal{F}) \subset Proj_E(T) \subset c'' Proj_E(\mathcal{E})$$

where $c, c'' > 0$ depend only on the constant in Question 1.3. By Theorem 1.1 there exist some special $\lfloor 3\varepsilon n \rfloor$ symmetrizations designed specific to the body $T \cap E^{\perp}$. Apply these $\lfloor 3\varepsilon n \rfloor$ symmetrizations to T itself. Reasoning as in Proposition 3.2, we obtain a body $\tilde{T}$ with

$$d_{BM}(\tilde{T}) < c\frac{1}{\varepsilon^{\beta}}.$$

□

3.1 Dual symemtrization

Let $K \subset \mathbb{R}^n$ be a convex body and let H be a hyperplane in $\mathbb{R}^n$. For simplicity, assume that K is centrally-symmetric. The result of a dual Steiner symmetrization of K is the body

$$S_H^\circ(K) = [S_H(K^\circ)]^\circ ,$$

i.e. we symmetrize the dual body with respect to H. Next, we propose an alternative short symmetrization process for an arbitrary convex body $K \subset \mathbb{R}^n$. Rather than cutting a small portion of the volume, we combine symmetrizations of two kinds: Steiner symmetrization and dual Steiner symmetrization.

Theorem 3.4 *Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. Then there exists $\tilde{K}$, a linear image of K, such that for any $0 < \varepsilon < 1$ there exist εn Steiner symmetrizations that transform $\tilde{K}$ into K_1, and εn dual Steiner symmetrizations that transform K_1 into K_2 such that*

$$d_G(K_2) < \frac{c}{\varepsilon^3}$$

where $c > 0$ is a numerical constant.

Proof: Assume that $\varepsilon < \frac{1}{2}$. Let $\tilde{K}$ be a linear image of K which is in M-position of order 1. By Theorem 2.3 there exists a subspace E of dimension $\lfloor \varepsilon n \rfloor$ such that

$$c\varepsilon^{3/2} D \cap E^\perp \subset Proj_{E^\perp}(\tilde{K}). \tag{5}$$

Also, since $N\left(D \cap E, \frac{1}{\varepsilon} Proj_E(\tilde{K})\right) < exp(c\varepsilon n)$, then

$$\left(\frac{Vol(Proj_E(\tilde{K}))}{Vol(\varepsilon D \cap E)}\right)^{\frac{1}{dim(E)}} > C.$$

We apply $\lfloor 3\varepsilon n \rfloor$ symmetrization to $\tilde{K}$, all in the subspace E according to Theorem 1.1, to obtain the body K_1. The body K_1 satisfies

$$c\varepsilon D \cap E \subset K_1 \cap E.$$

In addition, $K_1 \cap E^\perp = Proj_{E^\perp}(K_1) = Proj_{E^\perp}(\tilde{K})$ (see e.g. [KM1]). By (5) we conclude that

$$c\varepsilon^{3/2} D \subset K_1. \tag{6}$$

Note that (6) also remains true if we replace K_1 with a dual Steiner symmetrization of K_1. Next, as in the proof of Proposition 2.4, we have that $D \cap E^\perp$ is an M-ellipsoid of order 1 for

$Proj_{E^\perp}\tilde{K} = Proj_{E^\perp}K_1 = K_1 \cap E^\perp$. By Theorem 2.3 there exists a subspace F of dimension $\lfloor 2\varepsilon n \rfloor$ that contains E such that

$$K_1 \cap F^\perp \subset \frac{c}{\varepsilon^{3/2}} D \cap F^\perp.$$

Note that all Steiner symmetrizations were carried out with respect to vectors inside F and hence the volume of $\tilde{K} \cap F$ is preserved. Reasoning as before, since $\tilde{K}$ is in M-position of order 1,

$$\left(\frac{Vol(K_1 \cap F)}{Vol\left(\frac{1}{\varepsilon} D \cap F\right)} \right)^{\frac{1}{dim(F)}} = \left(\frac{Vol(\tilde{K} \cap F)}{Vol\left(\frac{1}{\varepsilon} D \cap F\right)} \right)^{\frac{1}{dim(F)}} < C.$$

We apply $\lfloor 2\varepsilon n \rfloor$ dual Steiner symmetrizations to K_1, all in the subspace F according to Theorem 1.1, to obtain the body K_2. As before, we obtain that the body K_2 satisfies

$$Proj_F K_2 \subset \frac{c}{\varepsilon} D \cap F, \quad Proj_{F^\perp} K_2 \subset \frac{c}{\varepsilon^{3/2}} D \cap F^\perp.$$

Combining this with (6) we get that

$$c\varepsilon^{3/2} D \subset K_2 \subset \frac{C}{\varepsilon^{3/2}} D$$

and the proof is complete. $\square$

Remark: It is possible to avoid the use of a linear image in Theorem 3.4, at the cost of replacing the geometric distance with a Banach-Mazur distance. i.e. For any centrally-symmetric convex body $K \subset \mathbb{R}^n$ there exist $\lfloor \varepsilon n \rfloor$ Steiner symmetrizations followed by $\lfloor \varepsilon n \rfloor$ dual Steiner symmetrizations that transform K into a body which is $\frac{c}{\varepsilon^3}$ close to an ellipsoid.

4 Rapid symmetrization implies slicing

It remains to prove the second implication in Theorem 1.4, that a positive answer to Question 1.2 implies a positive answer to Question 1.3. We begin with a few lemmas, the first of which is standard and well-known, and is proved here only for completeness.

Lemma 4.1 *Let $\mathcal{E}$ be an ellipsoid in $\mathbb{R}^n$. Then among all k-dimensional sections of $\mathcal{E}$, the intersection of $\mathcal{E}$ with the subspace spanned by the shortest k axes of the ellipsoid has a minimal volume.*

Proof: Choose orthogonal coordinates such that $\mathcal{E} = TD$ for a diagonal matrix T. Let $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the numbers on the diagonal. Let V be a matrix of k rows and n

columns such that its rows are orthonormal vectors in $\mathbb{R}^n$. Writing volumes as determinants, we need to show that

$$\sqrt{det(VT^2V^t)} \geq \prod_{i=1}^{k} \lambda_i.$$

We will use the Cauchy-Binet formula. The sums in the next formula are over all subsets $A \subset \{1,..,n\}$ with exactly k elements. For such A, we write V_A for the matrix obtained from V by taking the columns whose indices are in A. Then,

$$det(VT^2V^t) = \sum_A det(V_A T^2 (V_A)^t) = \sum_A \left(\prod_{i\in A} \lambda_i^2\right) det(V_A(V_A)^t)$$

$$\geq \left(\prod_{i=1}^{k} \lambda_i^2\right) \sum_A det(V_A(V_A)^t) = \left(\prod_{i=1}^{k} \lambda_i^2\right) det(VV^t) = \prod_{i=1}^{k} \lambda_i^2.$$

□

Lemma 4.2 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin. Assume that K is in isotropic position, and denote $d = d_{BM}(K)$. Then for any subspace E of dimension εn,*

$$Vol(K \cap E)^{1/n} > \left(\frac{c}{d}\right)^{\varepsilon}$$

where $c > 0$ is a numerical constant.

Proof: Let $\mathcal{E}$ be such that $\mathcal{E} \subset K \subset d\mathcal{E}$, and select an orthonormal basis $\{e_1,..,e_n\}$ and $0 < \lambda_1 \leq ... \leq \lambda_n$ such that $\mathcal{E} = \left\{x \in \mathbb{R}^n; \sum \frac{\langle x,e_i\rangle^2}{\lambda_i^2} \leq 1\right\}$. Since $K \subset d\mathcal{E}$,

$$c\sum_{i=1}^{n} \frac{1}{\lambda_i^2} < L_K^2 \sum_{i=1}^{n} \frac{1}{\lambda_i^2} = \int_K \sum_{i=1}^{n} \frac{\langle x, e_i\rangle^2}{\lambda_i^2} dx \leq d^2.$$

Therefore, by the Geometric-Harmonic means inequality,

$$\left(\prod_{i=1}^{\varepsilon n} \lambda_i\right)^{\frac{1}{\varepsilon n}} \geq \sqrt{\frac{\varepsilon n}{\sum_{i=1}^{\varepsilon n} \frac{1}{\lambda_i^2}}} \geq \sqrt{\frac{c\varepsilon n}{d^2}} > c' \frac{\sqrt{\varepsilon n}}{d}.$$

Let E_ε denote the subspace spanned by the shortest εn axes, $e_1,..,e_{\varepsilon n}$. By Lemma 4.1, $Vol(\mathcal{E} \cap E) \geq Vol(\mathcal{E} \cap E_\varepsilon)$ and

$$Vol(K \cap E)^{1/n} \geq Vol(\mathcal{E} \cap E)^{1/n} \geq Vol(\mathcal{E} \cap E_\varepsilon)^{1/n}.$$

Since

$$Vol(\mathcal{E} \cap E_\varepsilon)^{1/n} > \left(\prod_{i=1}^{\varepsilon n} \lambda_i\right)^{1/n} \frac{c}{(\sqrt{\varepsilon n})^{\varepsilon}} > \left(c' \frac{\sqrt{\varepsilon n}}{d\sqrt{\varepsilon n}}\right)^{\varepsilon}$$

the lemma is proven. □

Let $K \subset \mathbb{R}^n$ be a convex body, and let $E \subset \mathbb{R}^n$ be a subspace of dimension k. We define the Schwartz symmetrization of K with respect to E, as the unique body $S_E(K)$ such that:

(i) For any $x \in E^{\perp}$, $Vol(K \cap (x+E)) = Vol(S_E(K) \cap (x+E))$.

(ii) For any $x \in E^{\perp}$, the body $S_E(K) \cap (x+E)$ is a Euclidean ball centered at $E^{\perp}$.

We replace any section of K parallel to E with a Euclidean ball of the same volume. Schwartz symmetrization is a limit of a sequence of Steiner symmetrizations, and preserves volume and convexity. The following lemma is a reformulation of Theorem 2.5 in [BKM2]. For a convex body $K \subset \mathbb{R}^n$ of volume one whose barycenter is at the origin, denote by M_K the operator defined by

$$\forall u, v \in \mathbb{R}^n, \quad \langle u, M_K v \rangle = \int_K \langle x, u \rangle \langle x, v \rangle dx.$$

Define also $Iso(K) = L_K M_K^{-1/2} K$. Then $Iso(K)$ is the unique isotropic image of K under a positive definite linear transformation.

Lemma 4.3 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin, and let $E \subset \mathbb{R}^n$ be a k-dimensional invariant subspace of M_K. Then,*

$$\left(\frac{1}{c}\frac{k}{n}\right)^{\frac{k}{n}} < \frac{L_{S_E(K)}}{L_K^{1-\frac{k}{n}}\ Vol(Iso(K) \cap E)^{\frac{1}{n}}} < \left(c\frac{n}{k}\right)^{\frac{k}{n}}$$

where $c > 0$ is a numerical constant.

Proof: If K is isotropic, then the lemma is just a particular case of Theorem 2.5 in [BKM2]. Otherwise, since E is an invariant subspace of M_K,

$$Iso(S_E(Iso(K))) = Iso(S_E(K))$$

and hence the isotropic constant of $S_E(K)$ equals the isotropic constant of $S_E(Iso(K))$, and the lemma follows. $\square$

Assume that there exist k Steiner symmetrizations that transform T into a body $\tilde{T}$ with $d_{BM}(\tilde{T}) < A$. Then also a Schwartz symmetrization of T with respect to a k-dimensional subspace that contains these k symmetrization vectors, transforms T into $\tilde{\tilde{T}}$ with $d_{BM}(\tilde{\tilde{T}}) < A$.

Proposition 4.4 *Let $K \subset \mathbb{R}^n$ be a convex body. Assume that there exists $T \subset K$ with $Vol(T) > \frac{9}{10} Vol(K)$, such that for any $\varepsilon > 0$ there exist $\lfloor \varepsilon n \rfloor$ symmetrizations, that transform T into $\tilde{T}$ with $d_{BM}(\tilde{T}) < c_1 \frac{1}{\varepsilon^{c_2}}$, where c_1, c_2 are independent of ε.*
Then $L_K < c(c_1, c_2)$ where $c(c_1, c_2)$ depends solely on its arguments.

Proof: By the discussion at the end of Section 1, we may assume that the barycenter of T is at the origin, that $Vol(T) = 1$ and that T is isotropic (symmetrizability is an affine invariant property). Also, for any $\varepsilon > 0$, there exists a subspace $E_{\varepsilon n} \subset \mathbb{R}^n$ of dimension $\lfloor \varepsilon n \rfloor$ such that the Schwartz symmetrization of T with respect to any subspace that contains $E_{\varepsilon n}$ is $\frac{c_1}{\varepsilon^{c_2}}$-close to an ellipsoid. Let us denote $\log^{(0)} n = n$ and $\log^{(i+1)} n = \log \max\{\log^{(i)} n, e\}$. Substitute $\delta_i = \frac{1}{(\log^{(i)} n)^2}$, and for i such that $\delta_i < \frac{1}{2}$ let

$$F_i = sp\left\{E_{\delta_1 n}, ..., E_{\delta_i n}\right\}$$

where sp denotes linear span. Denote $\varepsilon_i = \frac{1}{n} dim(F_i)$. Then $\frac{1}{(\log^{(i)} n)^2} \leq \varepsilon_i \leq \sum_{j=1}^{i} \frac{1}{(\log^{(j)} n)^2} < \frac{2}{(\log^{(i)} n)^2}$. Let T_i denote the Schwartz symmetrization of T with respect to F_i. Since $F_{i-1} \subset F_i$ we can think of T_i as the Schwartz symmetrization of T_{i-1} with respect to F_i. According to our assumptions,

$$cL_{T_i} \leq d_{BM}(T_i) < c_1 \left(\frac{1}{\varepsilon_i}\right)^{c_2} < c_1 \left(\log^{(i)} n\right)^{2c_2}$$

where the left-most inequality appears in [MP1]. By Lemma 4.2, since $\varepsilon_{i+1} < \frac{2}{\left(\log^{(i+1)} n\right)^2}$,

$$Vol(Iso(T_i) \cap E_{i+1})^{1/n} > \left(\frac{c}{c_1 (\log^{(i)} n)^{2c_2}}\right)^{\frac{2}{(\log^{(i+1)} n)^2}} > C^{\frac{1}{\log^{(i+1)} n}}$$

and hence by Lemma 4.3, since F_{i+1} is an invariant subspace of M_{T_i} (recall that T is isotropic, and symmetrizations were applied only with respect to subspaces contained in F_{i+1}),

$$L_{T_{i+1}} > \left(\frac{c}{\left(\log^{(i+1)} n\right)^2}\right)^{\frac{2}{\left(\log^{(i+1)} n\right)^2}} L_{T_i}^{1 - \frac{2}{\left(\log^{(i+1)} n\right)^2}} C^{\frac{1}{\log^{(i+1)} n}}$$

and since $L_{T_i} < c\left(\log^{(i)} n\right)^{2c_2}$,

$$L_{T_{i+1}} > c^{\frac{1}{\log^{(i+1)} n}} L_{T_i} > ... > c^{\sum_{j=1}^{i+1} \frac{1}{\log^{(j)} n}} L_T.$$

Let i^* be the largest integer such that $\varepsilon_i < \frac{1}{2}$. Then T_{i^*} has a bounded distance from an ellipsoid, and $L_{T_{i^*}} < c(c_1, c_2)$ (see [MP1]). Therefore,

$$L_T < c^{\sum_{j=1}^{t^*} \frac{1}{\log^{(j)} n}} c(c_1, c_2) < c'(c_1, c_2)$$

and since $L_K \approx L_T$ (e.g. [Bou3] or Borell lemma), the proposition is proved. □

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Part III: Additional Results

A geometric inequality and a low M-estimate[8]

Abstract. We present an integral inequality connecting volumes and diameters of sections of a convex body. We apply this inequality to obtain some new inequalities concerning diameters of sections of convex bodies, among which is our "Low M-estimate". Also, we give novel, alternative proofs to some known results, such as the fact that a finite volume ratio body has proportional sections which are isomorphic to a Euclidean ball.

1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex body that contains the origin and has a non empty interior. Denote $D_n = \{x \in \mathbb{R}^n; |x| \leq 1\}$ the unit Euclidean ball in $\mathbb{R}^n$, where $|\cdot|$ is the standard Euclidean norm. We denote the volume-radius of $K \subset \mathbb{R}^n$ by

$$v.rad.(K) = \left(\frac{Vol(K)}{Vol(D_n)}\right)^{1/n}.$$

The meaning of Vol here is clear - standard Lebesgue measure in $\mathbb{R}^n$. However in general, when we write $Vol(K)$ or $v.rad.(K)$ for a body K of smaller dimension, the quantity should be interpreted in the corresponding ambient subspace (i.e. the affine hull of K). Denote also the diameter of K as $diam(K) = \sup_{x,y \in K} |x - y|$. A particular case of our inequality states that if $Vol(K) = Vol(D_n)$ and n is even, then

$$\int_{G_{n,n/2}} v.rad.(K \cap E) diam(K \cap E) d\mu(E) < c \tag{1}$$

where $G_{n,n/2}$ is the Grassmanian of $n/2$-dimensional subspaces in $\mathbb{R}^n$, and μ is the unique rotation invariant probability measure on $G_{n,n/2}$. By c, c', c_1, C etc. we denote numerical constants, whose value is not necessarily equal in various appearances. In addition, we obtain strong concentration of the integrand in (1):

$$\mu \left\{ E \in G_{n,n/2} \ ; \ v.rad.(K \cap E) diam(K \cap E) > c_1 \right\} < e^{-n}.$$

The complete version of our inequality is presented in Proposition 1.1. This inequality has many interesting applications in asymptotic convex geometry, as demonstrated in Section 3. For instance, it suggests a "low M-estimate", which is dual in some sense to Milman's low M^*-estimate ([M1], [PT], [Go]), and provides a global proof of the statement due to Szarek/Tomczak-Jaegermann ([Sz], [SzT]), that a finite volume ratio body has sections of proportional dimension which are isomorphic to a Euclidean ball.

[8]This section corresponds to the paper [Kl3].

Proposition 1.1 *Let $K \subset \mathbb{R}^n$ be a convex body with a non empty interior, that contains the origin, and let $1 \le k \le n$ be an integer. Denote $k = \lambda n$. Then,*

$$\int_{G_{n,k}} v.rad.(K \cap E)^{\lambda} diam(K \cap E)^{1-\lambda} d\mu(E) < Cv.rad.(K) \tag{2}$$

where μ is the unique rotation invariant probability measure on $G_{n,k}$ and $C > 0$ is a numerical constant. Moreover,

$$\mu\left\{E \in G_{n,k}; \frac{v.rad.(K \cap E)^{\lambda} diam(K \cap E)^{1-\lambda}}{v.rad.(K)} > C\right\} < e^{-n}. \tag{3}$$

2 Proof of the inequality

The central component of the proof is the following simple formula, which follows from integration in polar coordinates.

Lemma 2.1 *For any star-shaped $K \subset \mathbb{R}^n$ (i.e. $tK \subset K$ for $0 \le t \le 1$), and any integer $1 \le k \le n$,*

$$\int_{G_{n,k}} \int_{K \cap E} |x|^{n-k} dx d\mu(E) = \frac{kVol(D_k)}{nVol(D_n)} Vol(K)$$

where μ is the unique rotation invariant probability measure on $G_{n,k}$, and the measure dx is the natural Lebesgue measure in the appropriate subspace.

Proof: Denote $\kappa_m = Vol(D_m)$. Let χ_K be the characteristic function of $K \subset \mathbb{R}^n$. Integrating in polar coordinates yields,

$$\begin{aligned} &\int_{G_{n,k}} \int_{K \cap E} |x|^{n-k} dx d\mu(E) \\ &= \int_{G_{n,k}} k\kappa_k \int_{S(E)} \int_0^{\infty} \chi_K(r\theta) r^{n-k} r^{k-1} dr d\sigma_E(\theta) d\mu(E) \end{aligned}$$

where $S(E)$ is the unit sphere in E and σ_E is the Haar probability measure on that sphere. Denote $S^{n-1} = S(\mathbb{R}^n)$ and $\sigma = \sigma_{\mathbb{R}^n}$. Now,

$$\int_{G_{n,k}} \int_{S(E)} \int_0^{\infty} \chi_K(r\theta) r^{n-1} dr d\sigma_E(\theta) d\mu(E) = \int_{S^{n-1}} \int_0^{\infty} \chi_K(r\theta) r^{n-1} dr d\sigma_n(\theta)$$

because of the rotation invariancy of both probability measures. With an additional application of polar integration we obtain

$$k\kappa_k \int_{S^{n-1}} \int_0^{\infty} \chi_K(r\theta) r^{n-1} dr d\sigma_n(\theta) = \frac{k\kappa_k}{n\kappa_n} \int_{\mathbb{R}^n} \chi_K(x) dx = \frac{k\kappa_k}{n\kappa_n} Vol(K)$$

and the lemma is proved. □

The following lemma is somewhat standard, following from simple convexity principles. Yet, for completeness we shall state it and prove it here.

Lemma 2.2 *Let $K \subset \mathbb{R}^n$ be a convex compact body that contains the origin, and let $r = \max_{x \in K} |x|$. Then,*

$$Vol\left\{x \in K; |x| > \frac{r}{2}\right\} > c^n Vol(K)$$

where $c > 0$ is a numerical constant.

Proof: Clearly, we may assume that $Vol(K) = 1$. There exists $x_0 \in K$ which satisfies $|x_0| = r$. Denote,

$$K_t = K \cap \left\{x \in \mathbb{R}^n; \left\langle x, \frac{x_0}{|x_0|}\right\rangle = t\right\}, \quad \varphi(t) = Vol_{n-1}(K_t).$$

By maximality of $|x_0|$, the support of φ is contained in $[-r, r]$ and $\int_{-r}^{r} \varphi = 1$. It is sufficient to prove that

$$\int_{r/2}^{r} \varphi > c^n \tag{4}$$

since in that case,

$$Vol\left\{x \in K; |x| > \frac{r}{2}\right\} \geq Vol\left\{x \in K; \left\langle x, \frac{x_0}{|x_0|}\right\rangle > \frac{r}{2}\right\} > c^n.$$

Let us prove inequality (4). Denote $M = \max \varphi$ and let $-r \leq y_0 \leq r$ be a point such that $\varphi(y_0) = M$. Let $\frac{1}{2}r \leq x \leq \frac{3}{4}r$ be an arbitrary point. Then either $x \in [y_0, r]$ or $x \in [0, y_0]$. In the first case, by convexity $\frac{r-x}{r-y_0} K_{y_0} + \frac{x-y_0}{r-y_0} x_0 \subset K_x$, and

$$\varphi(x) = Vol_{n-1}(K_x) \geq \left(\frac{r-x}{r-y_0}\right)^{n-1} Vol_{n-1}(K_{y_0}) \geq \frac{1}{8^{n-1}} \varphi(y_0) = \frac{M}{8^{n-1}}$$

since $r - x \geq \frac{r}{4}$ and $r - y_0 \leq 2r$. In the second case, $\frac{x}{y_0} K_{y_0} \subset K_x$ since $0 \in K$, and

$$\varphi(x) \geq \left(\frac{x}{y_0}\right)^{n-1} \varphi(y_0) \geq \frac{M}{2^{n-1}}.$$

To conclude, for any $\frac{1}{2}r \leq x \leq \frac{3}{4}r$ we have $\varphi(x) \geq \frac{M}{8^{n-1}}$ and

$$\int_{r/2}^{r} \varphi(x)dx > \int_{\frac{1}{2}r}^{\frac{3}{4}r} \varphi(x)dx \geq \frac{M}{8^{n-1}} \frac{r}{4} \geq \frac{1}{8^n}$$

since $1 = \int_{-r}^{r} \varphi \leq 2rM$, and (4) is proved with $c = \frac{1}{8}$. □

Proof:[Proof of Proposition 1.1.] By Lemma 2.1,

$$\left(\int_{G_{n,\lambda n}} \int_{K \cap E} |x|^{(1-\lambda)n} dx d\mu(E)\right)^{1/n} = \left(\frac{\lambda n \kappa_{\lambda n}}{n \kappa_n} Vol(K)\right)^{1/n} \tag{5}$$

where again we use the notation $\kappa_m = Vol(D_m)$. Regarding the left hand side of (5), from Lemma 2.2,

$$c^{\lambda n} Vol(K \cap E) \left(\frac{diam(K \cap E)}{4}\right)^{(1-\lambda)n} \leq \int_{K \cap E} |x|^{(1-\lambda)n} dx$$

$$\leq Vol(K \cap E) diam(K \cap E)^{(1-\lambda)n}$$

since $\frac{diam(K\cap E)}{2} \leq \max_{x\in K\cap E} |x| \leq diam(K \cap E)$. Let us introduce a simplifying notation: when we write $A \approx B$, we mean that there exist two numerical constants $c_1, c_2 > 0$ such that $c_1 A < B < c_2 A$. Hence,

$$\left(\int_{G_{n,\lambda n}} \int_{K\cap E} |x|^{(1-\lambda)n} dx d\mu(E)\right)^{1/n} \tag{6}$$

$$\approx \left(\int_{G_{n,\lambda n}} Vol(K \cap E) diam(K \cap E)^{(1-\lambda)n} d\mu(E)\right)^{1/n}.$$

Combining (5) and (6), and using $Vol(K) = v.rad.(K)^n \kappa_n$, $Vol(K\cap E) = v.rad.(K\cap E)^{\lambda n}\kappa_{\lambda n}$ and $\lambda^{1/n} \approx 1$ (since $\frac{1}{n} \leq \lambda \leq 1$), we obtain

$$\left(\int_{G_{n,\lambda n}} v.rad.(K \cap E)^{\lambda n} diam(K \cap E)^{(1-\lambda)n} d\mu(E)\right)^{1/n} \approx v.rad.(K).$$

Using Markov inequality $Prob\{f > t\|f\|_p\} \leq t^{-p}$ for $p = n$ and $t = e$ yields

$$\mu \left\{E \in G_{n,k}; v.rad.(K \cap E)^{\lambda} diam(K \cap E)^{1-\lambda} > eCv.rad.(K)\right\} < e^{-n}$$

which proves (3). Obtaining (2) is now easy. By Jensen inequality

$$\int_{G_{n,\lambda n}} v.rad.(K \cap E)^{\lambda} diam(K \cap E)^{1-\lambda} d\mu(E) < C' v.rad.(K).$$

□

Remark 2.3 *Lemma 2.2 uses the convexity of the body K. Nevertheless, weaker notions than convexity are sufficient for obtaining the conclusion of this lemma. For example, quasi convexity is enough. Since the convexity assumption is used only in the proof of Lemma 2.2, the conclusions of Proposition 1.1 hold for quasi convex bodies as well.*

3 Corollaries of the inequality

A few corollaries of Proposition 1.1 are presented in this section. Some of these are known, yet our proof is completely different from their usual proofs, and some are new, like the "Low M-estimate" for diameters of proportional sections, and others.

3.1 The volume of proportional sections is not large

Corollary 3.1 *Let $K \subset \mathbb{R}^n$ be a convex body that contains the origin and has a non empty interior, and let $1 \leq k \leq n$ be an integer. Then a random subspace $E \in G_{n,k}$ satisfies:*

$$v.rad.(K \cap E) < Cv.rad.(K)$$

with probability greater than $1 - e^{-n}$*, where* C *is a numerical constant, independent of* k *and* n*.*

Proof: Clearly $v.rad.(K \cap E) \leq \frac{1}{2} diam(K \cap E)$ by an isoperimetric inequality in $\mathbb{R}^n$ (see [BF], page 77). Therefore, (3) transforms to

$$\mu \{E \in G_{n,k}; v.rad.(K \cap E) > Cv.rad.(K)\} < e^{-n}.$$

□

3.2 Diameters of sections via orthogonal projection

Lemma 3.2 *Let* $K \subset \mathbb{R}^n$ *be a convex body such that the origin is its barycenter. Let* E *be any subspace of dimension* $k = \lambda n$*. Then,*

$$v.rad.(K \cap E) > c(\lambda) \frac{v.rad.(K)^{\frac{1}{\lambda}}}{v.rad.(Proj_{E^\perp}(K))^{\frac{1-\lambda}{\lambda}}}$$

where $Proj_{E^\perp}$ *is the orthogonal projection onto* $E^\perp$*, and* $c(\lambda) \approx C^{\frac{1}{\lambda}}$*.*

Proof: By [Sp],

$$Vol(K) \leq Vol(Proj_{E^\perp}(K)) Vol(K \cap E)$$

and raising to the power of $\frac{1}{\lambda n}$ we obtain,

$$v.rad.(K \cap E) \geq \left(\frac{\kappa_n}{\kappa_{\lambda n} \kappa_{(1-\lambda)n}} \right)^{\frac{1}{\lambda n}} \frac{v.rad.(K)^{\frac{1}{\lambda}}}{v.rad.(Proj_{E^\perp}(K))^{\frac{1-\lambda}{\lambda}}}.$$

Since $\sqrt{m}\kappa_m^{1/m} \approx 1$, we have $\left(\frac{\kappa_n}{\kappa_{\lambda n} \kappa_{(1-\lambda)n}} \right)^{1/n} \approx \lambda^{\frac{\lambda}{2}} (1-\lambda)^{\frac{1-\lambda}{2}} \approx 1$ and

$$v.rad.(K \cap E) > C^{1/\lambda} \frac{v.rad.(K)^{\frac{1}{\lambda}}}{v.rad.(Proj_{E^\perp}(K))^{\frac{1-\lambda}{\lambda}}}.$$

and the lemma is proved. □

Corollary 3.3 *Let* $K \subset \mathbb{R}^n$ *be a convex body such that the origin is its barycenter. Let* $1 \leq k \leq n$ *be an integer,* $k = \lambda n$*. Then a random subspace* $E \in G_{n,k}$ *satisfies:*

$$diam(K \cap E) < c(\lambda) v.rad.(Proj_{E^\perp}(K))$$

with probability greater than $1 - e^{-n}$*, where* $Proj_{E^\perp}$ *is the orthogonal projection onto* $E^\perp$*, and* $c(\lambda) \approx C^{\frac{1}{1-\lambda}}$*.*

Proof: By Lemma 3.2,

$$\frac{v.rad.(K)^{\frac{1}{1-\lambda}}}{v.rad.(K\cap E)^{\frac{\lambda}{1-\lambda}}} < C^{\frac{1}{1-\lambda}} v.rad.(Proj_{E^\perp}(K))$$

and by Proposition 1.1, with probability greater than $1-e^{-n}$,

$$diam(K\cap E) < \tilde{C}^{\frac{1}{1-\lambda}} \frac{v.rad.(K)^{\frac{1}{1-\lambda}}}{v.rad.(K\cap E)^{\frac{\lambda}{1-\lambda}}} < c(\lambda) v.rad.(Proj_{E^\perp}(K))$$

where $c(\lambda) = (C\tilde{C})^{\frac{1}{1-\lambda}}$. □

Remark 3.4 *Note that the Low M^*-estimate (Theorem 3.8 here) follows from Corollary 3.3, with a poorer dependence on λ. This follows from*

$$v.rad.(Proj_{E^\perp}(K)) \le M^*(Proj_{E^\perp}(K)) \le c\sqrt{\frac{1}{1-\lambda}} M^*(K)$$

by Urysohn inequality and a contraction principle ([GM4], Section 4.2).

3.3 Euclidean sections of finite volume ratio bodies

Here we provide another proof for the fact discovered by Szarek and Tomczak-Jaegermann ([Sz], [SzT]), regarding sections of bodies with a finite volume ratio. The common proof of this fact uses some concentration inequalities, and an argument involving nets. Let us give an alternative proof, based on Proposition 1.1, which is a "global" one and avoids the use of nets (see also [Ba4] or [Ku]).

Corollary 3.5 *Let $K \subset \mathbb{R}^n$ be a convex body, such that $D_n \subset K$ and such that $v.rad.(K) < \alpha$. Let $1 \le k \le n$ be an integer, $k = \lambda n$. Then a random subspace $E \in G_{n,k}$ satisfies:*

$$D_k \subset K\cap E \subset c(\lambda,\alpha) D_k$$

with probability greater than $1-e^{-n}$, where $c(\lambda,\alpha) \approx (C\alpha)^{\frac{1}{1-\lambda}}$.

Proof: Since $D_n \subset K$, then also $D_k \subset K \cap E$ for any subspace E, and $v.rad.(K\cap E) \ge 1$. It is enough to bound the diameter of a random section. According to (3),

$$\begin{aligned} &\mu\left\{E\in G_{n,k}; diam(K\cap E) > (C\alpha)^{\frac{1}{1-\lambda}}\right\} \\ \le\ &\mu\left\{E\in G_{n,k}; \frac{v.rad.(K\cap E)^{\lambda} diam(K\cap E)^{1-\lambda}}{v.rad.(K)} > C\right\} < e^{-n} \end{aligned}$$

and the corollary is proved. Note that we get $c(\lambda,\alpha) < (C\alpha)^{\frac{1}{1-\lambda}}$, which is the same dependence obtained in [Sz], [SzT]. □

3.4 Bounded sections in M-position

Let $K \subset \mathbb{R}^n$ be a convex body, such that $Vol(K) = Vol(D_n)$ and such that the covering number

$$N(K, D_n) = \inf\{\sharp A; K \subset A + D_n\} < e^{cn} \tag{7}$$

for some constant $c > 0$, where $\sharp A$ is the number of elements in the set A. Under these conditions, we say that K is in M-position with constant c (see [M4], [M5] or [Pi], chapter 7, for different proofs of the fact that any convex body has a linear image which is in M-position with some absolute constant). Our next corollary provides a simple proof for a result in the spirit of [GM1], section 2.3 and [GM2], section 5.

Corollary 3.6 *Let $K \subset \mathbb{R}^n$ be a convex body in M-position, such that the origin is its barycenter, and such that $Vol(K) = Vol(D_n)$. Let $1 \leq k \leq n$ be an integer, $k = \lambda n$. Then a random subspace $E \in G_{n,k}$ satisfies:*

$$c_1(\lambda) < diam(K \cap E) < c_2(\lambda) \tag{8}$$

with probability greater than $1 - e^{-n}$, where $c_1(\lambda) \approx c^{\frac{1}{\lambda}}$ and $c_2(\lambda) \approx C^{\frac{1}{1-\lambda}}$.

Proof: Begin by estimating volumes of projections. From (7),

$$v.rad.(Proj_{E^\perp}(K)) \leq N(K, D_n)^{\frac{1}{(1-\lambda)n}} v.rad.(Proj_{E^\perp}(D_n)) < e^{\frac{c}{1-\lambda}}.$$

By Lemma 3.2,

$$v.rad.(K \cap E) > c(\lambda) \frac{1}{v.rad.(Proj_{E^\perp}(K))^{\frac{1-\lambda}{\lambda}}} > c^{\frac{1}{\lambda}}.$$

This is true for any subspace $E \in G_{n,k}$. Hence, for any subspace E of dimension k, we have $diam(K \cap E) \geq 2v.rad.(K \cap E) > c_1(\lambda)$. We continue as in the proof of Corollary 3.5:

$$\begin{aligned} &\mu\left\{E \in G_{n,k}; diam(K \cap E) > c_2(\lambda)\right\} \\ \leq \;\; &\mu\left\{E \in G_{n,k}; \frac{v.rad.(K \cap E)^\lambda diam(K \cap E)^{1-\lambda}}{v.rad.(K)} > C\right\} < e^{-n}. \end{aligned}$$

$\square$

Remark 3.7 *The right-most inequality in (8) also follows from Corollary 3.5 by considering the body $K + D_n$, and observing that $v.rad.(K + D_n) < c$ when K is in M-position.*

3.5 Low M-estimate

Here we prove a dual estimate to the well-known Low M^*-estimate (Theorem 3.8 here), which involves the "dual parameter" M of the body. Let us define these parameters. Let $K \subset \mathbb{R}^n$ be a convex body such that the origin is in its interior. Denote the gauge function of K as

$$\|x\| = \inf\{\lambda > 0; x \in \lambda K\}.$$

Note that the gauge function is homogeneous in $\mathbb{R}^n$, and satisfies the triangle inequality. However, it is not symmetric (not necessarily $\|x\| = \|-x\|$), hence it may not be a norm. Define

$$M(K) = \int_{S^{n-1}} \|x\| d\sigma(x)$$

where σ is the unique rotation invariant probability measure on S^{n-1}. The polar body of K is $K^\circ = \{x \in \mathbb{R}^n; \forall y \in K, \langle x, y\rangle \le 1\}$, and we write $M^*(K) = M(K^\circ)$. The Low M^*-estimate is the following theorem (see [M1], [PT], or [Go] for best dependence on λ):

Theorem 3.8 *(Low M^*-estimate). Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. Let $1 \le k \le n$ be an integer, $k = \lambda n$. Then a random subspace $E \in G_{n,k}$ satisfies:*

$$diam(K \cap E) < c(\lambda) M^*(K)$$

with probability greater than $1 - e^{-\tilde{c}\cdot(1-\lambda)n}$, where $c(\lambda) \approx \frac{1}{\sqrt{1-\lambda}}$.

The Low M^*-estimate is a fundamental inequality in the study of high dimensional convex bodies. It is crucially involved, for example, in the proof of Milman's quotient of subspace theorem [M3]. A dual form of the Low M^*-estimate appears in [GM1], under the name "conditional low M-estimate", as it requires that $M(K)$ be very close to $\sup_{x\in S^{n-1}} \|x\|$. Here we present a different estimate which is valid for all levels of $M(K)$.

Proposition 3.9 *(Low M-estimate) Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. Let $1 \le k \le n$ be an integer, $k = \lambda n$. Then a random subspace $E \in G_{n,k}$ satisfies:*

$$diam(K \cap E) < c(\lambda) M(K)^{\frac{\lambda}{1-\lambda}} v.rad.(K)^{\frac{1}{1-\lambda}}$$

with probability greater than $1 - e^{-n}$, where $c(\lambda) \approx C^{\frac{1}{1-\lambda}}$.

Proof: By Proposition 1.1, a random section satisfies

$$diam(K \cap E)^{1-\lambda} < C \frac{v.rad.(K)}{v.rad.(K \cap E)^{\lambda}} \tag{9}$$

with the desired probability. According to Jensen inequality,

$$M(K \cap E) = \int_{S(E)} \|x\| d\sigma_E(x) \geq \left(\int_{S(E)} \|x\|^{-k} d\sigma_E(x) \right)^{-1/k} = \frac{1}{v.rad.(K \cap E)}$$

and by substituting into (9) we get,

$$diam(K \cap E)^{1-\lambda} < CM(K \cap E)^{\lambda} v.rad.(K).$$

By a contraction principle, for any subspace E we have $M(K \cap E) < c\sqrt{\frac{1}{\lambda}} M(K)$ (see e.g. [GM4], Section 4.2). Hence, with the appropriate probability,

$$diam(K \cap E)^{1-\lambda} < cC\lambda^{-\frac{\lambda}{2}} M(K)^{\lambda} v.rad.(K) < C' M(K)^{\lambda} v.rad.(K)$$

and raising to the power of $\frac{1}{1-\lambda}$ obtains the proposition. $\square$

Remark 3.10 *By imitating the technique of [GM1], [GM2], one can use the Low M-estimate to show that if $R > 0$ satisfies*

$$M(K \cap RD_n)^{\frac{\lambda}{1-\lambda}} v.rad.(K \cap RD_n)^{\frac{1}{1-\lambda}} \approx c(\lambda) R$$

then for a random λn-dimensional subspace E, we have $diam(K \cap E) \leq R$.

3.6 Distances to Euclidean ball

Given a convex body $K \subset \mathbb{R}^n$ that contains the origin in its interior, we introduce the outer and inner radius of K, respectively:

$$a(K) = \inf\{a; K \subset aD_n\} \ , \ b(K) = \inf\left\{b; \frac{1}{b} D_n \subset K\right\}.$$

Note that $a(K) \approx diam(K)$ and $b(K) \approx diam(K^{\circ})$. The "geometric distance" of K to a Euclidean ball is simply $d(K, D_n) = a(K)b(K)$. This can be written as follows:

$$d(K, D_n) = \frac{a(K)}{v.rad.(K)} \frac{v.rad.(K)}{1/b(K)}.$$

Note that both terms are not smaller than one. The first term measures to what extent the body "fills" the circumscribing Euclidean ball, while the second term measures how much volume of K is captured by the circumscribed Euclidean ball. We prove the following corollary, just to indicate a possible application of the combination of the Low M^*-estimate with our Low M-estimate.

Corollary 3.11 *Let $K \subset \mathbb{R}^n$ be a convex body that contains the origin in its interior and such that $Vol(K) = Vol(D_n)$. Let $E \subset \mathbb{R}^n$ be a random subspace of dimension $\lfloor \frac{n}{2} \rfloor$. Then,*

$$K \cap E \subset c \min\{M(K)M^*(K)\}D_n \subset c' \sqrt{d(K,D)}D_n$$

with probability greater than $1 - e^{-n}$, where $c, c' > 0$ are numerical constants.

Proof: Begin with the upper inclusion. According to Corollary 3.9, a random $\lfloor \frac{n}{2} \rfloor$-dimensional subspace $E \subset \mathbb{R}^n$ satisfies

$$K \cap E \subset cM(K)D_n.$$

Also, according to Theorem 3.8,

$$K \cap E \subset c'M^*(K)D_n$$

with high probability. Hence, a random section of dimension $\lfloor n/2 \rfloor$ satisfies

$$K \cap E \subset C \min\{M(K), M^*(K)\}D_n \subset C' \sqrt{M(K)M^*(K)}D_n$$

and since $M(K) \leq b(K), M^*(K) \leq a(K)$, we get that for a random section,

$$K \cap E \subset \tilde{c}\sqrt{d(K,D)}D_n.$$

$\square$

3.7 Diameters of low-dimensional sections

We will show that given a "non-degenerate" arbitrary convex body $K \subset \mathbb{R}^n$, its sections of small dimension (up to $\frac{n}{logn}$, almost proportional sections) typically have a small diameter, i.e. with high probability their diameter is not much greater than the volume radius of the body. By saying "non-degenerate" we mean that K is not essentially a body of smaller dimension. For example, it should contain some small n-dimensional ball.

Corollary 3.12 *Let $\alpha > 0$, and assume that $n > n(\alpha) = e^{2\alpha}$ is an integer. Let $K \subset \mathbb{R}^n$ be a convex body that contains the origin, and such that $Vol(K) = Vol(D_n)$. Assume also that $\frac{1}{n^t}D_n \subset K$ for some $t > 0$. Let $1 \leq k \leq \alpha\frac{n}{\log n}$ be an integer. Then, a random subspace $E \in G_{n,k}$ satisfies:*

$$diam(K \cap E) < c(\alpha, t)$$

with probability greater than $1 - e^{-n}$, where $c(\alpha, t) = C^{t\alpha}$.

Proof: Since $n > n(\alpha)$, we know that $\alpha < \frac{\log n}{2}$. By Proposition 1.1, a random subspace $E \in G_{n,k}$ satisfies

$$diam(K \cap E) < \frac{c}{v.rad.(K \cap E)^{\frac{k}{n-k}}} < \frac{c}{v.rad.(K \cap E)^{\frac{2\alpha}{\log n}}}$$

with probability greater than $1 - e^{-n}$. But since $\frac{1}{n^t} D_n \subset K$, we get that

$$v.rad.(K \cap E) \geq \frac{1}{n^t}.$$

Hence, for a random subspace $E \in G_{n,k}$,

$$diam(K \cap E) < cn^{\frac{2t\alpha}{\log n}} = ce^{2t\alpha}.$$

□

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On John-Type Ellipsoids[9]

Abstract. Given an arbitrary convex symmetric body $K \subset \mathbb{R}^n$, we construct a natural and non-trivial continuous map u_K which associates ellipsoids to ellipsoids, such that the Löwner-John ellipsoid of K is its unique fixed point. A new characterization of the Löwner-John ellipsoid is obtained, and we also gain information regarding the contact points of inscribed ellipsoids with K.

1 Introduction

We work in $\mathbb{R}^n$, yet we choose no canonical scalar product. A centrally-symmetric ellipsoid in $\mathbb{R}^n$ is any set of the form

$$\left\{ \sum_{i=1}^n \lambda_i u_i \ ; \ \sum_i \lambda_i^2 \leq 1, \ u_1, .., u_n \in \mathbb{R}^n \right\}.$$

If $u_1, .., u_n$ are linearly independent, the ellipsoid is non-degenerate. Whenever we mention an "ellipsoid" we mean a centrally-symmetric non-degenerate one. Given an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, denote by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ the unique scalar product such that $\mathcal{E} = \{x \in \mathbb{R}^n; \langle x, x \rangle_{\mathcal{E}} \leq 1\}$. There is a group $O(\mathcal{E})$ of linear isometries of $\mathbb{R}^n$ (with respect to the metric induced by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$), and a unique probability measure $\mu_{\mathcal{E}}$ on $\partial \mathcal{E}$ which is invariant under $O(\mathcal{E})$. A body in $\mathbb{R}^n$ is a centrally-symmetric convex set with a non-empty interior. Given a body $K \subset \mathbb{R}^n$, denote by $\| \cdot \|_K$ the unique norm on $\mathbb{R}^n$ such that K is its unit ball:

$$\|x\|_K = \inf\{\lambda > 0; x \in \lambda K\}.$$

Given a body $K \subset \mathbb{R}^n$ and an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, denote

$$M_{\mathcal{E}}^2(K) = M_{\mathcal{E}}^2(\| \cdot \|_K) = \int_{\partial \mathcal{E}} \|x\|_K^2 d\mu_{\mathcal{E}}(x).$$

This quantity is usually referred to as M_2. Let us consider the following parameter:

$$J_K(\mathcal{E}) = \inf_{\mathcal{F} \subset K} M_{\mathcal{E}}(\mathcal{F}) \tag{1}$$

where the infimum runs over all ellipsoids $\mathcal{F}$ that are contained in K. Since the set of all ellipsoids contained in K (including degenerate ellipsoids) is a compact set with respect to the Hausdorff metric, the infimum is actually attained. In addition, the minimizing ellipsoids must be non-degenerate, since otherwise $J_K(\mathcal{E}) = \infty$ which is impossible for a body K. We

[9] This section corresponds to the paper [Kl5].

are not so much interested in the exact value of $J_K(\mathcal{E})$, as in the ellipsoids where the minimum is obtained.

In Section 2 we prove that there exists a unique ellipsoid for which the minimum in (1) is attained. We shall denote this unique ellipsoid by $u_K(\mathcal{E})$, and we show that the map u_K is continuous. A finite measure ν on $\mathbb{R}^n$ is called $\mathcal{E}$-isotropic if for any $\theta \in \mathbb{R}^n$,

$$\int \langle x, \theta\rangle_{\mathcal{E}}^2 d\nu(x) = L_\nu^2 \langle \theta, \theta\rangle_{\mathcal{E}}$$

where L_ν does not depend on θ. One of the important properties of the map u_K is summarized in the following proposition, to be proved in Section 3.

Proposition 1.1 *Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E}$, $\mathcal{F} \subset \mathbb{R}^n$ be ellipsoids such that $\mathcal{F} \subset K$. Then $\mathcal{F} = u_K(\mathcal{E})$ if and only if there exists an $\mathcal{E}$-isotropic measure ν supported on $\partial\mathcal{F} \cap \partial K$.*

In particular, given any Euclidean structure (i.e. scalar product) in $\mathbb{R}^n$, there is always a unique ellipsoid contained in K with an isotropic measure supported on its contact points with K. This unexpected fact leads to a connection with the Löwner-John ellipsoid of K, which is the (unique) ellipsoid of maximal volume contained in K. By the characterization of the Löwner-John ellipsoid due to John [Jo] and Ball [Ba3] (see also [GM3]), $u_K(\mathcal{E}) = \mathcal{E}$ if and only if the ellipsoid $\mathcal{E}$ is the Löwner-John ellipsoid of K. Thus, we obtain the following:

Corollary 1.2 *Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E} \subset K$ be an ellipsoid such that for any ellipsoid $\mathcal{F} \subset K$,*

$$M_{\mathcal{E}}(\mathcal{F}) \geq 1.$$

Then $\mathcal{E}$ is the Löwner-John ellipsoid of K.

As a byproduct of our methods, we also obtain an extremality property of the mean width of the Löwner-John ellipsoid (Corollary 5.2). In Section 4 we show that the body K is determined by the map u_K. Further evidence for the naturalness of this map is demonstrated in Section 5, where we discuss optimization problems similar to the optimization problem in (1), and discover connections with the map u_K.

2 Uniqueness

Let D be a minimizing ellipsoid in (1). We will show that it is the only minimizing ellipsoid. We write $|x| = \sqrt{\langle x, x\rangle_D}$. An equivalent definition of $J_K(\mathcal{E})$ is the following:

$$J_K^2(\mathcal{E}) = \min\left\{ \int_{\partial\mathcal{E}} |T^{-1}(x)|^2 d\mu_{\mathcal{E}}(x) \ ; \ \|T : l_2^n \to X_K\| \leq 1 \right\} \qquad (2)$$

where $X_K = (\mathbb{R}^n, \|\cdot\|_K)$ is the normed space whose unit ball is K, and where $l_2^n = (\mathbb{R}^n, |\cdot|)$. The definitions are indeed equivalent; $T(D)$ is the ellipsoid from definition (1), as clearly $\|x\|_{T(D)} = |T^{-1}(x)|$. Since D is a minimizing ellipsoid, Id is a minimizing operator in (2). Note that in (2) it is enough to consider linear transformations which are self adjoint and positive definite with respect to $\langle\cdot,\cdot\rangle_D$. Assume on the contrary that T is another minimizer, where $T \neq Id$ is a self adjoint positive definite operator. Let $\{e_1, .., e_n\}$ be an orthogonal basis of eigenvectors of T, and let $\lambda_1, .., \lambda_n > 0$ be the corresponding eigenvalues. Consider the operator $S = \frac{Id+T}{2}$. Then S satisfies the norm condition in (2), and by the strict convexity of the function $x \mapsto \frac{1}{x^2}$ on $(0,\infty)$,

$$\int_{\partial\mathcal{E}} |S^{-1}(x)|^2 d\mu_{\mathcal{E}}(x) = \int_{\partial\mathcal{E}} \sum_{i=1}^n \left(\frac{1}{\frac{1+\lambda_i}{2}}\right)^2 \langle x, e_i\rangle_D^2 d\mu_{\mathcal{E}}(x)$$

$$< \int_{\partial\mathcal{E}} \sum_{i=1}^n \frac{1+\left(\frac{1}{\lambda_i}\right)^2}{2} \langle x, e_i\rangle_D^2 d\mu_{\mathcal{E}}(x)$$

$$= \frac{\int_{\partial\mathcal{E}} |x|^2 d\mu_{\mathcal{E}}(x) + \int_{\partial\mathcal{E}} |T^{-1}(x)|^2 d\mu_{\mathcal{E}}(x)}{2} = J_K^2(\mathcal{E})$$

since not all the λ_i's equal one, in contradiction to the minimizing property of Id and T. Thus the minimizer is unique, and we may define a map u_K which matches to any ellipsoid $\mathcal{E}$, the unique ellipsoid $u_K(\mathcal{E})$ such that $u_K(\mathcal{E}) \subset K$ and $J_K(\mathcal{E}) = M_{\mathcal{E}}(u_K(\mathcal{E}))$. It is easily verified that for any linear operator T, and $t \neq 0$,

$$u_{TK}(T\mathcal{E}) = Tu_K(\mathcal{E}), \tag{3}$$

$$u_K(t\mathcal{E}) = u_K(\mathcal{E}).$$

The second property means that the map u_K is actually defined over the "projective space" of ellipsoids. Moreover, the image of u_K is naturally a "projective ellipsoid" rather than an ellipsoid: If $\mathcal{E}$ and $t\mathcal{E}$ both belong to the image of u_K, then $t = \pm 1$. Nevertheless, we still formally define u_K as a map that matches an ellipsoid to an ellipsoid, and not as a map defined over the "projective space of ellipsoids".

Let us establish the continuity of the map u_K. One can verify that $M_{\mathcal{E}}(\mathcal{F})$ is a continuous function of $\mathcal{E}$ and $\mathcal{F}$ (using an explicit formula as in (4), for example). Fix a body $K \subset \mathbb{R}^n$, and denote by X the compact space of all (possibly degenerate) ellipsoids contained in K. Then $M_{\mathcal{E}}(\mathcal{F}) : X \times X \to [0,\infty]$ is continuous. The map u_K is defined only on a subset of X, the set of non-degenerate ellipsoids. The continuity of u_K follows from the following standard lemma.

Lemma 2.1 *Let X be a compact metric space, and $f : X \times X \to [0, \infty]$ a continuous function. Let $Y \subset X$, and assume that for any $y \in Y$ there exists a unique $g(y) \in X$ such that*

$$\min_{x \in X} f(x, y) = f(g(y), y).$$

Then $g : Y \to X$ is continuous.

Proof: Assume that $y_n \to y$ in Y. The function $\min_{x \in X} f(x, y)$ is continuous, and therefore

$$\min_{x \in X} f(x, y_n) = f(g(y_n), y_n) \stackrel{n \to \infty}{\longrightarrow} f(g(y), y) = \min_{x \in X} f(x, y).$$

Since $X \times X$ is compact, f is uniformly continuous and

$$\begin{aligned} & |f(g(y_n), y) - f(g(y), y)| \\ \leq \;& |f(g(y_n), y) - f(g(y_n), y_n)| + |f(g(y_n), y_n) - f(g(y), y)| \stackrel{n \to \infty}{\longrightarrow} 0. \end{aligned}$$

Therefore, for any convergent subsequence $g(y_{n_k}) \to z$, we must have $f(z, y) = f(g(y), y)$ and by uniqueness $z = g(y)$. Since X is compact, necessarily $g(y_n) \to g(y)$, and g is continuous. $\square$

3 Extremality conditions

There are several ways to prove the existence of the isotropic measure announced in Proposition 1.1. One can adapt the variational arguments from [GM3], or use the Lagrange multiplier technique due to John [Jo] (as suggested by O. Guedon). The argument we choose involves duality of linear programming (see e.g. [Bar]). For completeness, we state and sketch the proof of the relevant theorem ($\langle \cdot, \cdot \rangle$ is an arbitrary scalar product in $\mathbb{R}^m$):

Theorem 3.1 *Let $\{u_\alpha\}_{\alpha \in \Omega} \subset \mathbb{R}^m$, $\{b_\alpha\}_{\alpha \in \Omega} \subset \mathbb{R}$ and $c \in \mathbb{R}^m$. Assume that*

$$\langle x^0, c \rangle = \inf\{\langle x, c \rangle; \forall \alpha \in \Omega, \langle x, u_\alpha \rangle \geq b_\alpha\}$$

and also $\langle x^0, u_\alpha \rangle \geq b_\alpha$ for any $\alpha \in \Omega$. Then there exist $\lambda_1, .., \lambda_s > 0$ and $u_1, .., u_s \in \Omega' = \{\alpha \in \Omega; \langle x^0, u_\alpha \rangle = b_\alpha\}$ such that

$$c = \sum_{i=1}^{s} \lambda_i u_i.$$

Proof: $K = \{x \in \mathbb{R}^m; \forall \alpha \in \Omega, \langle x, u_\alpha \rangle \geq b_\alpha\}$ is a convex body. x^0 lies on its boundary, and $\{x \in \mathbb{R}^m; \langle x, c \rangle = \langle x^0, c \rangle\}$ is a supporting hyperplane to K at x^0. The vector c is an inner normal vector to K at x^0, hence $-c$ belongs to the cone of outer normal vectors to K

at x^0. The crucial observation is that this cone is generated by $-\Omega'$ (e.g. Corollary 8.5 in chapter II of [Bar]), hence

$$c \in \left\{ \sum_{i=1}^{s} \lambda_i u_i \ ; \ \forall i \ u_i \in \Omega', \lambda_i \geq 0 \right\}.$$

□

Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid. This ellipsoid induces a scalar product in the space of operators: if $T, S : \mathbb{R}^n \to \mathbb{R}^n$ are linear operators, and $\{e_1, .., e_n\} \subset \mathbb{R}^n$ is any orthogonal basis (with respect to $\langle \cdot, \cdot \rangle_{\mathcal{E}}$), then

$$\langle T, S \rangle_{\mathcal{E}} = \sum_{i,j} T_{i,j} S_{i,j}$$

where $T_{i,j} = \langle Te_i, e_j \rangle_{\mathcal{E}}$ and $S_{i,j} = \langle Se_i, e_j \rangle_{\mathcal{E}}$ are the entries of the corresponding matrix representations of T and S. This scalar product does not depend on the choice of the orthogonal basis. If $\mathcal{F} = \{x \in \mathbb{R}^n; \langle x, Tx \rangle_{\mathcal{E}} \leq 1\}$ is another ellipsoid, then

$$\begin{aligned} M^2_{\mathcal{E}}(\mathcal{F}) &= \int_{\partial \mathcal{E}} \langle x, Tx \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x) \\ &= \sum_{i,j=1}^{n} T_{i,j} \int_{\partial \mathcal{E}} \langle x, e_i \rangle_{\mathcal{E}} \langle x, e_j \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x) = \sum_{i,j=1}^{n} T_{i,j} \frac{\delta_{i,j}}{n} = \frac{1}{n} \langle T, Id \rangle_{\mathcal{E}}. \end{aligned} \tag{4}$$

The ellipsoid $\mathcal{F} = \{x \in \mathbb{R}^n; \langle x, Tx \rangle_{\mathcal{E}} \leq 1\}$ is contained in K if and only if for any $x \in \partial K$,

$$\langle x, Tx \rangle_{\mathcal{E}} = \langle x \otimes x, T \rangle_{\mathcal{E}} \geq 1$$

where $(x \otimes x)(y) = \langle x, y \rangle_{\mathcal{E}} x$ is a linear operator. Therefore, the optimization problem (1) is equivalent to the following problem:

$$nJ^2_K(\mathcal{E}) = \min\{\langle T, Id \rangle_{\mathcal{E}} \ ; \ T \ is \ \mathcal{E}\text{–}positive, \ \forall x \in \partial K \ \langle x \otimes x, T \rangle_{\mathcal{E}} \geq 1\}$$

where we say that T is $\mathcal{E}$-positive if it is self adjoint and positive definite with respect to $\langle \cdot, \cdot \rangle_{\mathcal{E}}$. Actually, the explicit positivity requirement is unnecessary. If K is non-degenerate and $\forall x \in \partial K \ \langle T, x \otimes x \rangle_{\mathcal{E}} \geq 1$ then T is necessarily positive definite with respect to $\mathcal{E}$. This is a linear optimization problem, in the space $\mathbb{R}^m = \mathbb{R}^{n^2}$. Let T be the unique self adjoint minimizer, and let $\mathcal{F} = \{x \in \mathbb{R}^n; \langle x, Tx \rangle_{\mathcal{E}} \leq 1\}$ be the corresponding ellipsoid. By Theorem 3.1, there exist $\lambda_1, .., \lambda_s > 0$ and vectors $u_1, .., u_s \in \partial K$ such that

1. For any $1 \leq i \leq s$ we have $\langle u_i \otimes u_i, T \rangle_{\mathcal{E}} = 1$, i.e. $u_i \in \partial K \cap \partial \mathcal{F}$.

2. $Id = \sum_{i=1}^{s} \lambda_i u_i \otimes u_i$. Equivalently, for any $\theta \in \mathbb{R}^n$,

$$\sum_{i=1}^{s} \lambda_i \langle u_i, \theta \rangle_{\mathcal{E}}^2 = \langle \theta, \theta \rangle_{\mathcal{E}}^2.$$

Hence we proved the following:

Lemma 3.2 *Let $K \subset \mathbb{R}^n$ be a body and let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid. If $u_K(\mathcal{E}) = \mathcal{F}$, then there exist contact points $u_1, .., u_s \in \partial K \cap \partial \mathcal{F}$ and positive numbers $\lambda_1, .., \lambda_s$ such that for any $\theta \in \mathbb{R}^n$,*

$$\sum_{i=1}^{s} \lambda_i \langle u_i, \theta \rangle_{\mathcal{E}}^2 = \langle \theta, \theta \rangle_{\mathcal{E}}.$$

The following lemma completes the proof of Proposition 1.1.

Lemma 3.3 *Let $K \subset \mathbb{R}^n$ be a body and let $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^n$ be ellipsoids. Assume that $\mathcal{F} \subset K$ and that there exists a measure ν supported on $\partial K \cap \partial \mathcal{F}$ such that for any $\theta \in \mathbb{R}^n$,*

$$\int \langle x, \theta \rangle_{\mathcal{E}}^2 d\nu(x) = \langle \theta, \theta \rangle_{\mathcal{E}}.$$

Then $u_K(\mathcal{E}) = \mathcal{F}$.

Proof: Since $\int x \otimes x d\nu(x) = Id$, for any operator T,

$$\int \langle Tx, x \rangle_{\mathcal{E}} d\nu(x) = \langle T, Id \rangle_{\mathcal{E}} = n \int_{\partial \mathcal{E}} \langle Tx, x \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x). \tag{5}$$

where the last equality follows by (4). Let T be such that $\mathcal{F} = \{x \in \mathbb{R}^n; \langle Tx, x \rangle_{\mathcal{E}} \leq 1\}$. By (5),

$$\nu(\partial \mathcal{F}) = \int_{\partial \mathcal{F}} \langle Tx, x \rangle_{\mathcal{E}} d\nu(x) = n \int_{\partial \mathcal{E}} \langle Tx, x \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x) = n M_{\mathcal{E}}^2(\mathcal{F}).$$

Suppose that $u_K(\mathcal{E}) \neq \mathcal{F}$. Then there exists a linear map $S \neq T$ such that $\langle Sx, x \rangle_{\mathcal{E}} \geq 1$ for all $x \in \partial K$ and such that

$$\int_{\partial \mathcal{E}} \langle Sx, x \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x) < M_{\mathcal{E}}^2(\mathcal{F}).$$

Therefore,

$$\nu(\partial K) \leq \int_{\partial K} \langle Sx, x \rangle_{\mathcal{E}} d\nu(x) = n \int_{\partial \mathcal{E}} \langle Sx, x \rangle_{\mathcal{E}} d\mu_{\mathcal{E}}(x) < n M_{\mathcal{E}}^2(\mathcal{F})$$

which is a contradiction, since $\nu(\partial K) = \nu(\partial \mathcal{F}) = n M_{\mathcal{E}}^2(\mathcal{F})$. $\square$

Remark: This proof may be modified to provide an alternative proof of John's theorem. Indeed, instead of minimizing the linear functional $\langle T, Id \rangle$ we need to minimize the concave functional $det^{1/n}(T)$. The minimizer still belongs to the boundary, and minus of the gradient at this point belongs to the normal cone.

4 Different bodies have different maps

Lemma 4.1 *Let $K, T \subset \mathbb{R}^n$ be two closed bodies such that $T \not\subset K$. Then there exists an ellipsoid $\mathcal{F} \subset T$ such that $\mathcal{F} \not\subset K$ and n linearly independent vectors $v_1, .., v_n$ such that for any $1 \leq i \leq n$,*

$$v_i \in \partial\mathcal{F} \cap \partial C$$

where $C = conv(K, \mathcal{F})$ and conv denotes convex hull.

Proof: Let $U \subset T \setminus K$ be an open set whose closure does not intersect K, and let $v_1^*, .., v_n^*$ be linearly independent functionals on $\mathbb{R}^n$ such that for any $y \in K,\ z \in U$,

$$v_i^*(y) < v_i^*(z)$$

for all $1 \leq i \leq n$. Let $\mathcal{F} \subset conv(U, -U)$ be an ellipsoid that intersects U, and let $v_1, .., v_n \in \partial\mathcal{F}$ be the unique vectors such that for $1 \leq i \leq n$,

$$v_i^*(v_i) = \sup_{v \in \mathcal{F}} v_i^*(v).$$

Then $v_1, .., v_n$ are linearly independent. Also, $v_i^*(v_i) = \sup_{v \in C} v_i^*(v)$ and hence $v_1, .., v_n$ belong to the boundary of $C = conv(K, \mathcal{F})$. □

Theorem 4.2 *Let $K, T \subset \mathbb{R}^n$ be two closed bodies, such that for any ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ we have $J_K(\mathcal{E}) = J_T(\mathcal{E})$. Then $K = T$.*

Proof: Assume $K \neq T$. Without loss of generality, $T \not\subset K$. Let $\mathcal{F}$ and $v_1, .., v_n$ be the ellipsoid and vectors from Lemma 4.1. Consider the following bodies:

$$L = conv\{\mathcal{F}, K \cap T\}, \quad C = conv\{\mathcal{F}, K\}.$$

Then $v_1, .., v_n \in \partial C$ and also $v_1, .., v_n \in \partial L$. Let $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ be the scalar product with respect to which these vectors constitute an orthonormal basis. Then the uniform measure on $\{v_1, .., v_n\}$ is $\mathcal{E}$-isotropic. By Proposition 1.1, $u_L(\mathcal{E}) = u_C(\mathcal{E}) = \mathcal{F}$, and

$$J_L(\mathcal{E}) = M_{\mathcal{E}}(u_L(\mathcal{E})) = M_{\mathcal{E}}(u_C(\mathcal{E})) = J_C(\mathcal{E}). \tag{6}$$

Since $K \subset C$, also $u_K(\mathcal{E}) \subset C$. Since $\mathcal{F} = u_C(\mathcal{E}) \not\subset K$, we have $\mathcal{F} \neq u_K(\mathcal{E})$. By the uniqueness of the minimizing ellipsoid for $J_C(\mathcal{E})$,

$$J_C(\mathcal{E}) = M_{\mathcal{E}}(\mathcal{F}) < M_{\mathcal{E}}(u_K(\mathcal{E})) = J_K(\mathcal{E}). \tag{7}$$

Since $L \subset T$ we have $J_T(\mathcal{E}) \leq J_L(\mathcal{E})$. Combining this with (6) and (7), we get

$$J_T(\mathcal{E}) \leq J_L(\mathcal{E}) = J_C(\mathcal{E}) < J_K(\mathcal{E})$$

and therefore $J_K(\mathcal{E}) \neq J_T(\mathcal{E})$. □

Corollary 4.3 *Let $K, T \subset \mathbb{R}^n$ be two closed bodies such that $u_K = u_T$. Then $K = T$.*

Proof: For any ellipsoid $\mathcal{E} \subset \mathbb{R}^n$,

$$J_K(\mathcal{E}) = M_{\mathcal{E}}(u_K(\mathcal{E})) = M_{\mathcal{E}}(u_T(\mathcal{E})) = J_T(\mathcal{E})$$

and the corollary follows from Theorem 4.2.

5 Various optimization problems

Given an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ and a body $K \subset \mathbb{R}^n$, define

$$K_{\mathcal{E}}^{\circ} = \{x \in \mathbb{R}^n; \forall x \in K, \ \langle x, y \rangle_{\mathcal{E}} \leq 1\}$$

and also $M_{\mathcal{E}}^*(K) = M_{\mathcal{E}}\left(K_{\mathcal{E}}^{\circ}\right)$. Consider the following optimization problem:

$$\inf_{K \subset \mathcal{F}} M_{\mathcal{E}}^*(\mathcal{F}) \tag{8}$$

where the infimum runs over all ellipsoids that contain K. Then (8) is simply the dual, equivalent formulation of problem (1) that was discussed above. Indeed, $\mathcal{F}$ is a minimizer in (8) if and only if $u_{K_{\mathcal{E}}^{\circ}}(\mathcal{E}) = \mathcal{F}_{\mathcal{E}}^{\circ}$. An apriori different optimization problem is the following:

$$I_K(\mathcal{E}) = \sup_{K \subset \mathcal{F}} M_{\mathcal{E}}(\mathcal{F}) \tag{9}$$

where the supremum runs over all ellipsoids that contain K. The characteristics of this problem are indeed different. For instance, the supremum need not be attained, as shown by the example of a narrow cylinder (in which there is a maximizing sequence of ellipsoids that tends to an infinite cylinder) and need not be unique, as shown by the example of a cube (any ellipsoid whose axes are parallel to the edges of the cube, and that touches the cube - is a maximizer. See also the proof of Corollary 5.2). Nevertheless, we define

$$\bar{u}_K(\mathcal{E}) = \left\{\mathcal{F} \subset \mathbb{R}^n \ ; \ \mathcal{F} \textit{ is an ellipsoid}, \ K \subset \mathcal{F}, \ \ M_{\mathcal{E}}(\mathcal{F}) = I_K(\mathcal{E})\right\}.$$

The dual, equivalent formulation of (9) means to maximize M^* among ellipsoids that are contained in K. Apriori, $u_K(\mathcal{E})$ and $\bar{u}_K(\mathcal{E})$ do not seem to be related. It is not clear why there should be a connection between minimizing M and maximizing M^* among inscribed ellipsoids. The following proposition reveals a close relation between the two problems.

Proposition 5.1 *Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^n$ be ellipsoids. Then*

$$\mathcal{F} \in \bar{u}_K(\mathcal{E}) \quad \Longleftrightarrow \quad u_{K_{\mathcal{F}}^{\circ}}(\mathcal{E}) = \mathcal{F}.$$

Proof:

$\Longrightarrow$: We write $\mathcal{F} = \{x \in \mathbb{R}^n; \langle x, Tx\rangle_{\mathcal{E}} \le 1\}$ for an $\mathcal{E}$-positive operator T. Since $\mathcal{F} \in \bar{u}_K(\mathcal{E})$, the operator T is a maximizer of

$$nI_K^2(\mathcal{E}) = \max\{\langle S, Id\rangle_{\mathcal{E}}\ ;\ S \in L(n),\ \forall x \in \partial K\ 0 \le \langle S, x\otimes x\rangle_{\mathcal{E}} \le 1\}.$$

where $L(n)$ is the space of linear operators acting on $\mathbb{R}^n$. Note that the requirement $\langle S, x\otimes x\rangle_{\mathcal{E}} \ge 0$ for any $x \in \partial K$ ensures that S is $\mathcal{E}$-non-negative definite. This is a linear optimization problem. Following the notation of Theorem 3.1, we rephrase our problem as follows:

$$-nI_K^2(\mathcal{E}) = \inf\{\langle S, -Id\rangle_{\mathcal{E}}; \forall x \in \partial K,\ \langle S, x\otimes x\rangle_{\mathcal{E}} \ge 0, \langle S, -x\otimes x\rangle_{\mathcal{E}} \ge -1\}.$$

According to Theorem 3.1, since T is a maximizer, there exist $\lambda_1, .., \lambda_s > 0$ and vectors $u_1, .., u_t, u_{t+1}, .., u_s \in \partial K$ such that

1. For any $1 \le i \le t$ we have $\langle T, -u_i \otimes u_i, \rangle_{\mathcal{E}} = -1$, i.e. $u_i \in \partial K \cap \partial\mathcal{F}$.
   For any $t+1 \le i \le s$ we have $\langle T, u_i \otimes u_i\rangle_{\mathcal{E}} = 0$.
2. $Id = \sum_{i=1}^t \lambda_i u_i \otimes u_i - \sum_{i=t+1}^s \lambda_i u_i \otimes u_i$.

Since we assumed that $\mathcal{F}$ is an ellipsoid, T is $\mathcal{E}$-positive, and it is impossible that $\langle Tu_i, u_i\rangle_{\mathcal{E}} = 0$. Hence, $t = s$ and there exists an $\mathcal{E}$-isotropic measure supported on $\partial K \cap \partial\mathcal{F}$. Since $K \subset \mathcal{F}$, then $\mathcal{F} \subset K_{\mathcal{F}}^\circ$ and $\partial K_{\mathcal{F}}^\circ \cap \partial\mathcal{F} = \partial K \cap \partial\mathcal{F}$. Therefore, there exists an $\mathcal{E}$-isotropic measure supported on $\partial K_{\mathcal{F}}^\circ \cap \partial\mathcal{F}$. Since $\mathcal{F} \subset K_{\mathcal{F}}^\circ$ we must have $u_{K_{\mathcal{F}}^\circ}(\mathcal{E}) = \mathcal{F}$, according to Proposition 1.1.

$\Longleftarrow$: Since $u_{K_{\mathcal{F}}^\circ}(\mathcal{E}) = \mathcal{F}$, then $K \subset \mathcal{F}$ and we can write $Id = \int x \otimes x d\nu(x)$ where *supp*$(\nu) \subset \partial K_{\mathcal{F}}^\circ \cap \partial\mathcal{F} = \partial K \cap \partial\mathcal{F}$. Reasoning as in Lemma 3.3, $\nu(\partial K) = nM_{\mathcal{E}}^2(\mathcal{F})$ and for any admissible operator S,

$$\langle S, Id\rangle_{\mathcal{E}} = \int \langle x, Sx\rangle_{\mathcal{E}} d\nu(x) \le \nu(\partial K) = nM_{\mathcal{E}}^2(\mathcal{F})$$

since $\langle x, Sx\rangle_{\mathcal{E}} \le 1$ for any $x \in \partial K$. Hence T is a maximizer, and $\mathcal{F} \in \bar{u}_K(\mathcal{E})$. $\square$

If $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^n$ are ellipsoids, then $K_{\mathcal{E}}^\circ$ is a linear image of $K_{\mathcal{F}}^\circ$. By (3), the map $u_{K_{\mathcal{E}}^\circ}$ is completely determined by $u_{K_{\mathcal{F}}^\circ}$. Therefore, the family of maps $\{u_{K_{\mathcal{F}}^\circ}; \mathcal{F}\ \textit{is an ellipsoid}\}$ is determined by a single map $u_{K_{\mathcal{E}}^\circ}$, for any ellipsoid $\mathcal{E}$. By Proposition 5.1, this family of maps determines $\bar{u}_K$. Therefore, for any ellipsoid $\mathcal{E}$, the map $u_{K_{\mathcal{E}}^\circ}$ completely determines $\bar{u}_K$. Additional consequence of Proposition 5.1 is the following:

Corollary 5.2 *Let $K \subset \mathbb{R}^n$ be a body, and let $\mathcal{E}$ be its Löwner-John ellipsoid. Then for any ellipsoid $\mathcal{F} \subset K$,*

$$M^*_{\mathcal{E}}(\mathcal{F}) \leq 1.$$

Equality occurs for $\mathcal{F} = \mathcal{E}$, yet there may be additional cases of equality.

Proof: If $\mathcal{E}$ is the Löwner-John ellipsoid, then $u_K(\mathcal{E}) = \mathcal{E}$. By Proposition 5.1, $\mathcal{E} \in \bar{u}_{K^\circ_{\mathcal{E}}}(\mathcal{E})$. Dualizing, we get that $\mathcal{E} \subset K$, and

$$1 = M^*_{\mathcal{E}}(\mathcal{E}) = \sup_{\mathcal{F} \subset K} M^*_{\mathcal{E}}(\mathcal{F})$$

where the supremum is over all ellipsoids contained in K. This proves the inequality. To obtain the remark about the equality cases, consider the cross-polytope $K = \{x \in \mathbb{R}^n; \sum_{i=1}^n |x_i| \leq 1\}$, where $(x_1, .., x_n)$ are the coordinates of x. By symmetry arguments, its Löwner-John ellipsoid is $D = \{x \in \mathbb{R}^n; \sum_i x_i^2 \leq \frac{1}{n}\}$. However, for any ellipsoid of the form

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n; \sum_i \frac{x_i^2}{\lambda_i} \leq 1 \right\}$$

where the λ_i are positive and $\sum_i \lambda_i = 1$, we get that $\mathcal{E} \subset K$, yet $M^*_D(\mathcal{E}) = M^*_D(D) = 1$. □

Remarks:

1. If $K \subset \mathbb{R}^n$ is smooth and strictly convex, then $\bar{u}_K$ is always a singleton. Indeed, if K is strictly convex and is contained in an infinite cylinder, it is also contained in a subset of that cylinder which is an ellipsoid, hence the supremum is attained. From the proof of Proposition 5.1 it follows that if $\mathcal{F}_1, \mathcal{F}_2$ are maximizers, then there exists an isotropic measure supported on their common contact points with K. Since K is smooth, it has a unique supporting hyperplane at any of these contact points, which is also a common supporting hyperplane of $\mathcal{F}_1$ and $\mathcal{F}_2$. Since these common contact points span $\mathbb{R}^n$, necessarily $\mathcal{F}_1 = \mathcal{F}_2$. Hence, if K is smooth and strictly convex, only John ellipsoid may cause an equality in Corollary 5.2.

2. If $\mathcal{E}$ is the Löwner-John ellipsoid of K, then for any other ellipsoid $\mathcal{F} \subset K$ we have $M_{\mathcal{E}}(\mathcal{F}) > M_{\mathcal{E}}(\mathcal{E}) = 1$. This follows from our methods, yet it also follows immediately from the fact that $\frac{1}{M_{\mathcal{E}}(\mathcal{F})} \leq \left(\frac{Vol(\mathcal{F})}{Vol(\mathcal{E})}\right)^{1/n}$, and from the uniqueness of the Löwner-John ellipsoid.

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