Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 254 (2008) 2275–2293
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Pointwise estimates for marginals of convex bodies

R. Eldan [a,1], B. Klartag [b,*,2]

[a] *School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel*
[b] *Department of Mathematics, Princeton University, Princeton, NJ 08544, USA*

Received 22 August 2007; accepted 27 August 2007

Available online 18 January 2008

Communicated by J. Bourgain

Abstract

We prove a pointwise version of the multi-dimensional central limit theorem for convex bodies. Namely, let μ be an isotropic, log-concave probability measure on $\mathbb{R}^n$. For a typical subspace $E \subset \mathbb{R}^n$ of dimension n^c, consider the probability density of the projection of μ onto E. We show that the ratio between this probability density and the standard Gaussian density in E is very close to 1 in large parts of E. Here $c > 0$ is a universal constant. This complements a recent result by the second named author, where the total variation metric between the densities was considered.

Keywords: Central limit theorem; Convex bodies; Marginal distribution

1. Introduction

Suppose X is a random vector in $\mathbb{R}^n$ that is distributed uniformly in some convex set $K \subset \mathbb{R}^n$. For a subspace $E \subset \mathbb{R}^n$ we denote by Proj_E the orthogonal projection operator onto E in $\mathbb{R}^n$. The central limit theorem for convex bodies [7,8] asserts that there exists a subspace $E \subset \mathbb{R}^n$,

* Corresponding author.
E-mail addresses: roneneldan@gmail.com (R. Eldan), bklartag@princeton.edu (B. Klartag).
1 Partly supported by the Israel Science Foundation.
2 Supported by the Clay Mathematics Institute and by NSF grant #DMS-0456590.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.08.014

with $\dim(E) > n^c$, such that the random vector $\mathrm{Proj}_E(X)$ is approximately Gaussian, in the total variation sense. This means that for a certain Gaussian random vector Γ in the subspace E,

$$\sup_{A \subseteq E} \left| \mathbb{P}\left\{\mathrm{Proj}_E(X) \in A\right\} - \mathbb{P}\{\Gamma \in A\} \right| \leqslant \frac{C}{n^c}, \tag{1}$$

where the supremum runs over all measurable subsets $A \subseteq E$. Here, and throughout this note, the letters $c, C, c_1, C_2, c', \tilde{C}$, etc. denote some positive universal constants, whose value may change from one appearance to the next.

The total variation estimate (1) implies that the density of $\mathrm{Proj}_E(X)$ is close to the density of Γ in the L^1-norm. In this note we observe that a stronger conclusion is within reach: One may deduce that the ratio between the density of $\mathrm{Proj}_E(X)$ and the density of Γ deviates from 1 by no more than Cn^{-c}, in the significant parts of the subspace E.

Let us introduce some notation. Write $|\cdot|$ for the standard Euclidean norm in $\mathbb{R}^n$. A random vector Z in $\mathbb{R}^n$ is isotropic if the following normalization holds:

$$\mathbb{E}Z = 0, \qquad \mathrm{Cov}(Z) = Id \tag{2}$$

where $\mathrm{Cov}(Z)$ stands for the covariance matrix of Z, and Id is the identity matrix. The Grassman manifold $G_{n,\ell}$ of all ℓ-dimensional subspaces of $\mathbb{R}^n$ carries a unique rotationally-invariant probability measure $\mu_{n,\ell}$. Whenever we say that E is a random ℓ-dimensional subspace in $\mathbb{R}^n$, we relate to the above probability measure $\mu_{n,\ell}$. Under the additional assumption that the random vector X is isotropic, the subspace E for which $\mathrm{Proj}_E(X)$ is approximately Gaussian may be chosen at random, and (1) will hold with high probability [7,8].

A function $f : \mathbb{R}^n \to [0,\infty)$ is log-concave if $\log f : \mathbb{R}^n \to [-\infty,\infty)$ is a concave function. The characteristic function of a convex set is log-concave. Throughout the entire discussion, the requirement that X be distributed uniformly in a convex body could have been relaxed to the weaker condition, that X has a log-concave density. Our main result in this paper reads as follows:

Theorem 1. *Let X be an isotropic random vector in $\mathbb{R}^n$ with a log-concave density. Let $1 \leqslant \ell \leqslant n^{c_1}$ be an integer. Then there exists a subset $\mathcal{E} \subseteq G_{n,\ell}$ with $\mu_{n,\ell}(\mathcal{E}) \geqslant 1 - C\exp(-n^{c_2})$ such that for any $E \in \mathcal{E}$, the following holds. Denote by f_E the density of the random vector* $\mathrm{Proj}_E(X)$. *Then,*

$$\left| \frac{f_E(x)}{\gamma(x)} - 1 \right| \leqslant \frac{C}{n^{c_3}} \tag{3}$$

for all $x \in E$ with $|x| \leqslant n^{c_4}$. Here, $\gamma(x) = (2\pi)^{-\ell/2}\exp(-|x|^2/2)$ is the standard Gaussian density in E, and $C, c_1, c_2, c_3, c_4 > 0$ are universal constants.

Note that almost the entire mass of a standard ℓ-dimensional Gaussian distribution is contained in a ball of radius $10\sqrt{\ell}$ about the origin. Therefore, (3) easily implies the total variation bound mentioned above. The history of the central limit theorem for convex bodies goes back to the conjectures and results of Brehm and Voigt [4] and Anttila, Ball and Perissinaki [2], see [7] and references therein. The case $\ell = 1$ of Theorem 1 was proved in [8] using the moderate deviation estimates of Sodin [13]. The generalization to higher dimensions is the main contribution of the present paper. See also [3] and [1].

The basic idea of the proof of Theorem 1 is the following. It is shown in [8], using concentration techniques, that the density of $\mathrm{Proj}_E(X+Y)$ is pointwise approximately radial, where Y is an independent small Gaussian random vector. It is furthermore proved that the random vector $X+Y$ is concentrated in a thin spherical shell. We combine these facts to deduce, in Section 2, that the density of $\mathrm{Proj}_E(X+Y)$ is not only radial, but in fact very close to the Gaussian density in E. Then, in Section 3, we show that the addition of the Gaussian random vector Y is not required. That is, we prove that when a log-concave density convolved with a small Gaussian is almost Gaussian—then the original density is also approximately Gaussian.

2. Convolved marginals are Gaussian

For a dimension n and $v>0$ we write

$$\gamma_n[v](x)=\frac{1}{(2\pi v)^{n/2}}\exp\left(-\frac{|x|^2}{2v}\right)\quad\left(x\in\mathbb{R}^n\right).\tag{4}$$

That is, $\gamma_n[v]$ is the density of a Gaussian random vector in $\mathbb{R}^n$ with mean zero and covariance matrix $v\,Id$. Let X be an isotropic random vector with a log-concave density in $\mathbb{R}^n$, and let Y be an independent Gaussian random vector in $\mathbb{R}^n$ whose density is $\gamma_n[n^{-\alpha}]$, for a parameter α to be specified later on. Denote by f_{X+Y} the density of the random vector $X+Y$. Our first step is to show that the density of the projection of $X+Y$ onto a typical subspace is pointwise approximately Gaussian.

We follow the notation of [8]. For an integrable function $f:\mathbb{R}^n\to[0,\infty)$, a subspace $E\subseteq\mathbb{R}^n$ and a point $x\in E$ we write

$$\pi_E(f)(x)=\int_{x+E^\perp}f(y)\,dy,\tag{5}$$

where $x+E^\perp$ is the affine subspace orthogonal to E that passes through the point x. In other words, $\pi_E(f):E\to[0,\infty)$ is the marginal of f onto E. The group of all orthogonal transformations of determinant one in $\mathbb{R}^n$ is denoted by $SO(n)$. Fix a dimension ℓ and a subspace $E_0\subset\mathbb{R}^n$ with $\dim(E_0)=\ell$. For $x_0\in E_0$ and a rotation $U\in SO(n)$, set

$$M_{f,E_0,x_0}(U)=\log\pi_{E_0}(f\circ U)(x_0).\tag{6}$$

Define

$$M\left(|x_0|\right)=\int_{SO(n)}M_{f_{X+Y},E_0,x_0}(U)\,d\mu_n(U),\tag{7}$$

where μ_n stands for the unique rotationally-invariant Haar probability measure on $SO(n)$. Note that $M(|x_0|)$ is independent of the direction of x_0, so it is well defined. We learned in [8] that the function $U\mapsto M_{f_{X+Y},E_0,x_0}(U)$ is highly concentrated with respect to U in the special orthogonal group $SO(n)$, around its mean value $M(|x_0|)$. This implies that the function $\pi_E(f_{X+Y})$ is almost spherically symmetric, for a typical subspace E. This information is contained in our next lemma, which is equivalent to [8, Lemma 3.3].

Lemma 2. *Let $1 \leqslant \ell \leqslant n$ be integers, let $0 < \alpha < 10^5$ and denote $\lambda = \frac{1}{5\alpha+20}$. Assume that $\ell \leqslant n^\lambda$. Suppose that X is an isotropic random vector with a log-concave density and that Y is an independent random vector with density $\gamma_n[n^{-\alpha\lambda}]$. Denote the density of $X+Y$ by f_{X+Y}.*

Let $E \in G_{n,\ell}$ be a random subspace. Then, with probability greater than $1 - Ce^{-cn^{1/10}}$ of selecting E, we have

$$\left|\log \pi_E(f_{X+Y})(x) - M\big(|x|\big)\right| \leqslant Cn^{-\lambda}, \tag{8}$$

for all $x \in E$ with $|x| \leqslant 5n^{\lambda/2}$. Here $c, C > 0$ are universal constants.

Sketch of proof. We need to follow the proof of Lemma 3.3 in [8], choosing for instance, $u = \frac{9}{10}$, $\lambda = \frac{1}{5\alpha+20}$, $k = n^\lambda$ and $\eta = 1$. Throughout the argument in [8], it was assumed that the dimension of the subspace is exactly $k = n^\lambda$, while in the present version of the statement, note that it could possibly be smaller, i.e., $\ell \leqslant k$ (note also that here, k need not be an integer). We re-run the proofs of Lemmas 2.7, 2.8, 3.1 and 3.3 from [8], allowing the dimension of the subspace we are working with to be smaller than k, noting that the reduction of the dimension always acts in our favor.

We refer the reader to the original argument in the proof of Lemma 3.3 in [8] for further details. □

Our main goal in this section is to show that $M(|x|)$ behaves approximately like $\log \gamma_n[1 + n^{-\alpha\lambda}](x)$. Once we prove this, it would follow from the above lemma that the density of $X+Y$ is pointwise approximately Gaussian. Next we explain why no serious harm is done if we take the logarithm outside the integral in the definition of $M(|x|)$. Denote, for $x \in E_0$,

$$\tilde{M}\big(|x|\big) = \int_{SO(n)} \pi_{E_0}(f_{X+Y} \circ U)(x)\, d\mu_n(U). \tag{9}$$

Lemma 3. *Under the notation and assumptions of Lemma 2, for $|x| \leqslant 5n^{\lambda/2}$ we have*

$$0 \leqslant \log \tilde{M}\big(|x|\big) - M\big(|x|\big) \leqslant \frac{C}{n^{1/5}}, \tag{10}$$

where $C > 0$ is a universal constant.

Proof. Recall that $E_0 \subset \mathbb{R}^n$ is some fixed ℓ-dimensional subspace with $\ell \leqslant n^\lambda$. Fix $x_0 \in E_0$ with $|x_0| \leqslant 5n^{\lambda/2}$. Lemma 3.1 of [8] states that for any $U_1, U_2 \in SO(n)$,

$$\left|M_{f_{X+Y},E_0,x_0}(U_1) - M_{f_{X+Y},E_0,x_0}(U_2)\right| \leqslant C_0 n^{\lambda(2\alpha+2)} \cdot d(U_1, U_2), \tag{11}$$

where $d(U_1, U_2)$ stands for the geodesic distance between U_1 and U_2 in $SO(n)$. As mentioned before, Lemma 3.1 is proved in [8] under the assumption that the dimension of the subspace E_0 is exactly n^λ. In our case, the dimension ℓ might be smaller than n^λ, but a close inspection of the proofs in [8] reveals that the reduction of the dimension can only improve the estimates. Hence (11) holds true.

We apply the Gromov–Milman concentration inequality on $SO(n)$, quoted as Proposition 3.2 in [8], and conclude from (11) that for any $\varepsilon > 0$,

$$\mu_n\bigl\{U \in SO(n);\ \bigl|M_{f_{X+Y},E_0,x_0}(U) - M\bigl(|x_0|\bigr)\bigr| \geqslant \varepsilon\bigr\} \leqslant \bar{C}\exp\bigl(-\bar{c}n\varepsilon^2/L^2\bigr), \tag{12}$$

with $L = C_0 n^{\lambda(2\alpha+2)}$. That is, the distribution of

$$F(U) = \frac{\sqrt{n}}{L}\bigl(M_{f_{X+Y},E_0,x_0}(U) - M\bigl(|x_0|\bigr)\bigr) \quad \bigl(U \in SO(n)\bigr)$$

on $SO(n)$ has a subgaussian tail. Note also that $\int_{SO(n)} F(U)\,d\mu_n(U) = 0$. A standard computation shows for any $p \geqslant 1$,

$$\int_{SO(n)} F^p(U)\,d\mu_n(U) \leqslant (C'\sqrt{p})^p, \tag{13}$$

where C' is a universal constant. Hence, for any $0 < t \leqslant C_0$,

$$\begin{aligned}
&\int_{SO(n)} \exp\bigl(tF(U)\bigr)\,d\mu_n(U) \\
&\quad\leqslant 1 + t\int_{SO(n)} F(U)\,d\mu_n(U) + \sum_{i=2}^{\infty}(C'\sqrt{i})^i\frac{t^i}{i!} \\
&\quad\leqslant 1 + \sum_{i=2}^{\infty}\frac{(\tilde{C}t^2)^{i/2}}{\lfloor i/2\rfloor!} \leqslant 1 + \bigl(\sqrt{C_0^2\tilde{C}} + 1\bigr)\sum_{j=1}^{\infty}\frac{(\tilde{C}t^2)^j}{j!} \leqslant \sum_{j=0}^{\infty}\frac{(\bar{C}t^2)^j}{j!} = \exp\bigl(\bar{C}t^2\bigr).
\end{aligned} \tag{14}$$

The left-hand side of (10) follows by Jensen's inequality. We use (14) for the value

$$t = \frac{L}{\sqrt{n}} = C_0 n^{\frac{2\alpha+2}{5\alpha+20}-\frac{1}{2}} \leqslant C_0 n^{-1/10} \leqslant C_0,$$

to conclude that

$$\begin{aligned}
\frac{\tilde{M}(|x_0|)}{\exp(M(|x_0|))} &= \frac{\int_{SO(n)}\exp(M_{f_{X+Y},E_0,x_0}(U))\,d\mu_n(U)}{\exp(M(|x_0|))} \\
&= \int_{SO(n)}\exp\bigl(M_{f_{X+Y},E_0,x_0}(U) - M\bigl(|x_0|\bigr)\bigr)\,d\mu_n(U) \leqslant \exp\bigl(\hat{C}n^{-1/5}\bigr).
\end{aligned}$$

Taking logarithms of both sides completes the proof. □

Let $X, Y, \alpha, \lambda, \ell$ be as in Lemma 2. We choose a slightly different normalization. Define

$$Z = \frac{X+Y}{\sqrt{1+n^{-\lambda\alpha}}}, \tag{15}$$

and denote by f_Z the corresponding density. Clearly f_Z is isotropic and log-concave. Next we define, for $x \in E_0$,

$$\tilde{M}_1\big(|x|\big) := \int_{SO(n)} \pi_{E_0}(f_Z \circ U)(x)\, d\mu_n(U). \tag{16}$$

Our goal is to show that the following estimate holds:

$$\left|\frac{\tilde{M}_1(|x|)}{\gamma_\ell[1](x)} - 1\right| < C_1 n^{-c_1} \tag{17}$$

for all $x \in \mathbb{R}^\ell$ with $|x| < c_2 n^{c_2}$ for some universal constants $C_1, c_1, c_2 > 0$.

We write $S^{n-1} = \{x \in \mathbb{R}^n;\ |x| = 1\}$, the unit sphere in $\mathbb{R}^n$. Define:

$$\tilde{f}_Z(x) = \int_{S^{n-1}} f_Z\big(|x|\theta\big)\, d\sigma_n(\theta) = \int_{SO(n)} f_Z(Ux)\, d\mu_n(U) \quad \big(x \in \mathbb{R}^n\big) \tag{18}$$

where σ_n is the unique rotationally-invariant probability measure on S^{n-1}. Since $\tilde{f}_Z$ is spherically symmetric, we shall also use the notation $\tilde{f}_Z(|x|) = \tilde{f}_Z(x)$. Clearly, for any $x \in E_0$,

$$\begin{aligned} \tilde{M}_1\big(|x|\big) &= \int_{SO(n)} \pi_{E_0}(f_Z \circ U)(x)\, d\mu_n(U) = \int_{SO(n)} \pi_{E_0}(\tilde{f}_Z \circ U)(x)\, d\mu_n(U) \\ &= \pi_{E_0}(\tilde{f}_Z)(x). \end{aligned} \tag{19}$$

We will use the following thin-shell estimate, proved in [8, Theorem 1.3].

Proposition 4. *Let $n \geqslant 1$ be an integer and let X be an isotropic random vector in $\mathbb{R}^n$ with a log-concave density. Then,*

$$\mathbb{P}\left\{\left|\frac{|X|}{\sqrt{n}} - 1\right| \geqslant \frac{1}{n^{1/15}}\right\} < C\exp\big(-cn^{1/15}\big) \tag{20}$$

where $C, c > 0$ are universal constants.

Applying the above for f_Z, denoting $\varepsilon = n^{-1/15}$, and defining

$$A = \big\{x \in \mathbb{R}^n;\ \sqrt{n}(1-\varepsilon) \leqslant |x| \leqslant \sqrt{n}(1+\varepsilon)\big\},$$

we get,

$$\int_A f_Z(x)\, dx > 1 - Ce^{-cn^{1/15}}. \tag{21}$$

From the definition of $\tilde{f}_Z$, it is clear that the above inequality also holds when we replace f_Z with $\tilde{f}_Z$. In other words, if we define

$$g(t) = t^{n-1}\omega_n \tilde{f}_Z(t) \quad (t \geqslant 0) \tag{22}$$

where ω_n is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$, and use integration in polar coordinates, we get

$$1 \geqslant \int_{\sqrt{n}(1-\varepsilon)}^{\sqrt{n}(1+\varepsilon)} g(t)\,dt > 1 - Ce^{-cn^{1/15}}. \tag{23}$$

Our next step is to apply the methods from Sodin's paper [13] in order to prove a generalization of [13, Theorem 2], for a multi-dimensional marginal rather than a one-dimensional marginal. Our estimate will be rather crude, but suitable for our needs.

Denote by $\sigma_{n,r}$ the unique rotationally-invariant probability measure on the Euclidean sphere of radius r around the origin in $\mathbb{R}^n$. A standard calculation shows that the density of an ℓ-dimensional marginal of $\sigma_{n,r}$ is given by the following formula:

$$\psi_{n,\ell,r}(x) = \psi_{n,\ell,r}\big(|x|\big) := \Gamma_{n,\ell}\frac{1}{r^\ell}\left(1 - \frac{|x|^2}{r^2}\right)^{\frac{n-\ell-2}{2}} 1_{[-r,r]}\big(|x|\big) \tag{24}$$

where

$$\Gamma_{n,\ell} = \left(\frac{1}{\sqrt{\pi}}\right)^\ell \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-\ell}{2})} \tag{25}$$

and where $1_{[-r,r]}$ is the characteristic function of the interval $[-r,r]$. (see for example [5, Remark 2.10]). When $\ell \ll \sqrt{n}$ we have $\Gamma_{n,\ell}(\frac{2\pi}{n})^{\ell/2} \approx 1$. By the definition (22) of g, and since $\tilde{f}_Z$ is spherically symmetric, we may write

$$\pi_{E_0}(\tilde{f}_Z)(x) = \int_0^\infty \psi_{n,\ell,r}\big(|x|\big) g(r)\,dr \quad (x \in E_0). \tag{26}$$

Indeed, the measure whose density is $\tilde{f}_Z$ equals $\int_0^\infty g(r)\sigma_{n,r}\,dr$, hence its marginal onto E_0 has density $x \mapsto \int_0^\infty \psi_{n,\ell,r}(x) g(r)\,dr$. We will show that the above density is approximately Gaussian for $x \in E_0$ when $|x|$ is not too large. But first we need the following technical lemma.

Lemma 5. *Let g be the density defined in* (22)*, and suppose that $n \geqslant C'$ and $\ell \leqslant n^{1/20}$. For $\varepsilon = n^{-1/15}$ denote $U = \{t > 0;\ t < (1-\varepsilon)\sqrt{n} \text{ or } t > (1+\varepsilon)\sqrt{n}\}$. Then,*

$$\int_U t^{-\ell} g(t)\,dt < C' \exp\big(-c' n^{1/15}\big). \tag{27}$$

Here, $c', C' > 0$ are universal constants.

Proof. Define for convenience,

$$h(t) = t^{-\ell} g(t). \tag{28}$$

Denote

$$A = \left[0, \frac{1}{n^2}\right], \qquad B = \left[\frac{1}{n^2}, \sqrt{n}(1-\varepsilon)\right] \cup \big[\sqrt{n}(1+\varepsilon), \infty\big),$$

and write

$$\int_U h(t)\,dt = \int_A h(t)\,dt + \int_B h(t)\,dt. \tag{29}$$

We estimate the two terms separately. For $t > \frac{1}{n^2}$ we have

$$h(t) \leqslant n^{2\ell} g(t) = e^{2\ell \log n} g(t). \tag{30}$$

Thus we can estimate the second term as follows:

$$\int_B h(t)\,dt \leqslant e^{2\ell \log n} \int_B g(t)\,dt < e^{2\ell \log n} C e^{-cn^{1/15}} < C e^{-\frac{1}{2}cn^{1/15}}, \tag{31}$$

where for the second inequality we apply the reformulation (23) of Proposition 4 (recall that $\varepsilon = n^{-1/15}$ and that $\ell \leqslant n^{1/20}$).

To estimate the first term on the right-hand side of (29), we use the fact that f_Z is isotropic and log concave, so we can use a crude bound for the isotropic constant (see e.g. [11, Theorem 5.14(e)] or [6, Corollary 4.3]) which gives $\sup_{\mathbb{R}^n} f_Z < e^{n \log n}$, thus, also $\sup_{\mathbb{R}^n} \tilde{f}_Z < e^{n \log n}$. Hence we can estimate

$$\begin{aligned}\int_A h(t)\,dt &= \int_0^{1/n^2} t^{-\ell} g(t)\,dt = \int_0^{1/n^2} t^{n-\ell-1} \omega_n \tilde{f}_Z(t)\,dt \\ &< n^{-2(n-\ell)} \omega_n \sup \tilde{f}_Z < e^{-1.5n \log n + n \log n} < e^{-n},\end{aligned} \tag{32}$$

as $\omega_n < C$. The combination of (31) and (32) completes the proof. □

We are now ready to show that the marginals of $\tilde{f}_Z$ are approximately Gaussian. Note that by (19) and (26),

$$\left|\frac{\tilde{M}_1(|x|)}{\gamma_\ell[1](x)} - 1\right| = \left|\frac{\int_0^\infty \psi_{n,\ell,r}(|x|) g(r)\,dr}{\gamma_\ell[1](x)} - 1\right|. \tag{33}$$

Our desired bound (17) is contained in the following lemma.

Lemma 6. *Let $1 \leqslant \ell \leqslant n$ be integers, with $n \geqslant C$ and $\ell \leqslant n^{1/20}$. Let $g:\mathbb{R}^+ \to \mathbb{R}^+$ be a function that satisfies* (23) *and* (27). *Then we have,*

$$\left| \frac{\int_0^\infty \psi_{n,\ell,r}(|x|) g(r)\, dr}{\gamma_\ell[1](x)} - 1 \right| < Cn^{-1/60} \tag{34}$$

for all $x \in \mathbb{R}^\ell$ with $|x| < 2n^{1/40}$ where $C > 0$ is a universal constant.

Proof. We begin by using a well-known fact, that follows from a straightforward computation using asymptotics of Γ-functions: for $|x| < n^{1/8}$,

$$\left| \frac{\psi_{n,\ell,\sqrt{n}}(|x|)}{\gamma_\ell[1](x)} - 1 \right| = \left| \left(\frac{2\pi}{n}\right)^{\ell/2} \Gamma_{n,\ell} \frac{\left(1 - \frac{|x|^2}{n}\right)^{(n-\ell-2)/2}}{e^{-|x|^2/2}} - 1 \right| \leqslant \frac{C}{\sqrt{n}}. \tag{35}$$

(We omit the details of the simple computation. An almost identical computation is done, for example, in [13, Lemma 1]. Note that in addition to the computation there, we have to use, e.g., Stirling's formula to estimate the constants ε_n.) Using the above fact (35), we see that it suffices to prove the following inequality:

$$\left| \frac{\int_0^\infty \psi_{n,\ell,r}(|x|) g(r)\, dr}{\psi_{n,\ell,\sqrt{n}}(|x|)} - 1 \right| < Cn^{-\frac{1}{60}} \tag{36}$$

for all $x \in \mathbb{R}^\ell$ with $|x| < 2n^{1/40}$. To that end, fix $x_0 \in \mathbb{R}^\ell$ with $|x_0| < 2n^{1/40}$, define

$$A = \left[\sqrt{n}\left(1 - n^{-\frac{1}{15}}\right), \sqrt{n}\left(1 + n^{-\frac{1}{15}}\right)\right], \qquad B = [0, \infty) \setminus A,$$

and write

$$\int_0^\infty \psi_{n,\ell,r}\left(|x_0|\right) g(r)\, dr = \int_A \psi_{n,\ell,r}\left(|x_0|\right) g(r)\, dr + \int_B \psi_{n,\ell,r}\left(|x_0|\right) g(r)\, dr. \tag{37}$$

We estimate the two terms separately. For the second term, we have,

$$\begin{aligned} \int_B \psi_{n,\ell,r}\left(|x_0|\right) g(r)\, dr &= \Gamma_{n,\ell} \int_B \frac{1}{r^\ell} \left(1 - \frac{|x_0|^2}{r^2}\right)^{\frac{n-\ell-2}{2}} 1_{[-r,r]}\left(|x_0|\right) g(r)\, dr \\ &< \Gamma_{n,\ell} \int_B \frac{1}{r^\ell} g(r)\, dr < \Gamma_{n,\ell} C e^{-cn^{1/15}}, \end{aligned} \tag{38}$$

where the last inequality follows from (27). Therefore,

$$\begin{aligned} \frac{\int_B \psi_{n,\ell,r}(|x_0|) g(r)\, dr}{\psi_{n,\ell,\sqrt{n}}(|x_0|)} &< \frac{Ce^{-cn^{1/15}}}{\left(\frac{1}{\sqrt{n}}\right)^\ell \left(1 - \frac{|x_0|^2}{n}\right)^{\frac{n-l-2}{2}}} \\ &< Ce^{-cn^{1/15} + |x_0|^2 + \frac{1}{2}\ell \log n} < Ce^{-n^{1/20}}. \end{aligned} \tag{39}$$

To estimate the first term on the right-hand side of (37), we will show that the following inequality holds:

$$\left|\frac{\int_A \psi_{n,\ell,r}(|x_0|)g(r)\,dr}{\psi_{n,\ell,\sqrt{n}}(|x_0|)} - 1\right| < Cn^{-1/60} \tag{40}$$

for some constant $C > 0$. For $r > 0$ such that $\frac{|x_0|^2}{r^2} < \frac{1}{2}$, we have,

$$\left|\frac{d}{dr}\log\psi_{n,\ell,r}\big(|x_0|\big)\right| = \left|-\frac{\ell}{r} + (n-\ell-2)\frac{|x_0|^2}{r^3}\frac{1}{\big(1-\frac{|x_0|^2}{r^2}\big)}\right| < \frac{\ell}{r} + 2n\frac{|x_0|^2}{r^3}. \tag{41}$$

Recalling that $|x_0| < 2n^{1/40}$ and $\ell \leqslant n^{1/20}$, the above estimate gives, for all $r \in [\frac{1}{2}\sqrt{n}, \frac{3}{2}\sqrt{n}\,]$,

$$\left|\frac{d}{dr}\log\psi_{n,\ell,r}\big(|x_0|\big)\right| < 2n^{\frac{1}{20}-\frac{1}{2}} + 16n^{1+\frac{1}{20}-\frac{3}{2}} < Cn^{-\frac{9}{20}} \tag{42}$$

which gives, for $r \in [\frac{1}{2}\sqrt{n}, \frac{3}{2}\sqrt{n}\,]$,

$$\left|\frac{\psi_{n,\ell,r}(|x_0|)}{\psi_{n,\ell,\sqrt{n}}(|x_0|)} - 1\right| < Cn^{-\frac{9}{20}}|r-\sqrt{n}|. \tag{43}$$

Recall that for $r \in A$ we have $|r-\sqrt{n}| \leqslant n^{13/30}$. Hence the last estimate yields,

$$\left|\frac{\int_A \psi_{n,\ell,r}(|x_0|)g(r)\,dr}{\psi_{n,\ell,\sqrt{n}}(|x_0|)\int_A g(r)\,dr} - 1\right| < Cn^{-\frac{9}{20}}n^{\frac{13}{30}} = Cn^{-\frac{1}{60}}. \tag{44}$$

Combining the last inequality with (23), we get

$$\left|\frac{\int_A \psi_{n,\ell,r}(|x_0|)g(r)\,dr}{\psi_{n,\ell,\sqrt{n}}(|x_0|)} - 1\right| < \tilde{C}e^{-cn^{\frac{1}{15}}} + Cn^{-\frac{1}{60}} < C'n^{-\frac{1}{60}}. \tag{45}$$

From (39) and (45) we deduce (36), and the lemma is proved. □

Recall the definitions (9) and (16) of $\tilde{M}(|x|)$ and $\tilde{M}_1(|x|)$; the only difference is the normalization of $X+Y$. By an easy scaling argument, we deduce from (33) and Lemma 6 that when $n \geqslant C$,

$$\left|\frac{\tilde{M}(|x|)}{\gamma_\ell[1+n^{-\lambda\alpha}](x)} - 1\right| < C_1 n^{-\frac{1}{60}} \tag{46}$$

for all $x \in \mathbb{R}^\ell$ with $|x| < n^{1/40}$, for $C_1 > 0$ a universal constant. By substituting (10) and (46) into Lemma 2, we conclude the following.

Proposition 7. *Let $1 \leqslant \ell \leqslant n$ be integers. Let $0 < \alpha < 10^5$ and denote $\lambda = \frac{1}{5\alpha+20}$. Assume that $\ell \leqslant n^\lambda$. Suppose that $f : \mathbb{R}^n \to [0, \infty)$ is a log-concave function that is the density of an isotropic random vector. Define $g = f * \gamma_n[n^{-\lambda\alpha}]$, the convolution of f and $\gamma_n[n^{-\lambda\alpha}]$. Let $E \in G_{n,\ell}$ be a random subspace. Then, with probability greater than $1 - Ce^{-cn^{1/10}}$ of selecting E, we have*

$$\left| \frac{\pi_E(g)(x)}{\gamma_\ell[1 + n^{-\lambda\alpha}](x)} - 1 \right| \leqslant Cn^{-\lambda} \tag{47}$$

for all $x \in E$ with $|x| < n^{\lambda/2}$, where $C > 0$ is a universal constant.

We did not have to explicitly assume that $n \geqslant C$ in Proposition 7, since otherwise the proposition is vacuously true. In the next section we will show that the above estimate still holds without taking the convolution, though perhaps with slightly worse constants.

3. Deconvolving the Gaussian

Our goal in this section is to establish the following principle. Suppose that X is a random vector with a log-concave density, and that Y is an independent, Gaussian random vector whose covariance matrix is small enough with respect to that of X. Then, in the case where $X + Y$ is approximately Gaussian, the density of X is also approximately Gaussian, in a rather large domain. We begin with a lower bound for the density of X.

Note that the notation n in this section corresponds to the dimension of the subspace, that was denoted by ℓ in the previous section.

Lemma 8. *Let $n \geqslant 1$ be a dimension, and let $\alpha, \beta, \varepsilon, R > 0$. Suppose that X is an isotropic random vector in $\mathbb{R}^n$ with a log-concave density, and that Y is an independent Gaussian random vector in $\mathbb{R}^n$ with mean zero and covariance matrix $\alpha\,\mathrm{Id}$. Denote by f_X and f_{X+Y} the respective densities. Suppose that,*

$$f_{X+Y}(x) \geqslant (1 - \varepsilon)\gamma_n[1 + \alpha](x) \tag{48}$$

for all $|x| \leqslant R$. Assume that $\alpha \leqslant c_0 n^{-8}$ and that

$$100(2n)^{\max\{3\beta, 3/2\}}\alpha^{1/4} < \varepsilon < \frac{1}{100}. \tag{49}$$

Then,

$$f_X(x) \geqslant (1 - 6\varepsilon)\gamma_n[1](x) \tag{50}$$

for all $x \in \mathbb{R}^n$ with $|x| \leqslant \min\{R - 1, (2n)^\beta\}$. Here, $0 < c_0 < 1$ is a universal constant.

Proof. Suppose first that f_X is positive everywhere in $\mathbb{R}^n$, and that $\log f_X$ is strictly concave. Fix $x_0 \in \mathbb{R}^n$ with $|x_0| \leqslant \min\{R - 1, (2n)^\beta\}$. Assume that $\varepsilon_0 > 0$ is such that

$$f_X(x_0) < (1 - \varepsilon_0)\gamma_n[1](x_0). \tag{51}$$

To prove the lemma (for the case where $\log f_X$ is strictly concave) it suffices to show that

$$\varepsilon_0 \leqslant 6\varepsilon. \tag{52}$$

Consider the level set $L = \{x \in \mathbb{R}^n;\ f_X(x) \geqslant f_X(x_0)\}$. Then L is convex and bounded, as f_X is log-concave and integrable (here we used the fact that $f_X(x_0) > 0$). Let H be an affine hyperplane that supports L at its boundary point x_0, and denote by D the open ball of radius $\alpha^{1/4}$ tangent to H at x_0, that is disjoint from the level set L. By definition, $f_X(x) < f_X(x_0)$ for $x \in D$. Denote the center of D by x_1. Then, $|x_1 - x_0| \leqslant \alpha^{1/4}$ with $|x_0| \leqslant (2n)^\beta$, and a straightforward computation yields

$$\big||x_1|^2 - |x_0|^2\big| \leqslant \big(2(2n)^\beta + \alpha^{1/4}\big)\alpha^{1/4} \leqslant \frac{\varepsilon}{2}, \tag{53}$$

where we used (49). Note that $|x_1| \leqslant |x_0| + \alpha^{1/4} \leqslant R$. Apply the last inequality and (48) to obtain,

$$f_{X+Y}(x_1) \geqslant (1-\varepsilon)\gamma_n[1+\alpha](x_0)e^{\frac{|x_0|^2-|x_1|^2}{2(1+\alpha)}} > (1-2\varepsilon)\gamma_n[1+\alpha](x_0). \tag{54}$$

By definition,

$$\begin{aligned} f_{X+Y}(x_1) &= \int_{\mathbb{R}^n} f_X(x)\gamma_n[\alpha](x_1-x)\,dx \\ &= \int_{x\in D} f_X(x)\gamma_n[\alpha](x_1-x)\,dx + \int_{x\notin D} f_X(x)\gamma_n[\alpha](x_1-x)\,dx. \end{aligned} \tag{55}$$

We will estimate both integrals. First, recall that $f_X(x) < f_X(x_0)$ for $x \in D$ and use (51) to deduce

$$\int_{x\in D} f_X(x)\gamma_n[\alpha](x_1-x)\,dx < f_X(x_0) < (1-\varepsilon_0)\gamma_n[1](x_0). \tag{56}$$

For the integral outside D, a rather rough estimate would suffice. We may write,

$$\int_{x\notin D} f_X(x)\gamma_n[\alpha](x_1-x)\,dx < \mathbb{P}\left(|G_n| \geqslant \frac{1}{\alpha^{1/4}}\right)\sup_{\mathbb{R}^n} f_X \tag{57}$$

where $G_n \sim \gamma_n[1]$ is a standard Gaussian random vector. To bound the right-hand side term, we shall use a standard tail bound for the norm of a Gaussian random vector,

$$\mathbb{P}\big(|G_n| > t\sqrt{n}\big) < Ce^{-ct^2}, \tag{58}$$

and the following crude bound for the isotropic constant of f_X (see, e.g., [11, Theorem 5.14(e)]),

$$\sup_{\mathbb{R}^n} f_X < e^{\frac{1}{2}n\log n + 6n} < e^{Cn\log n}. \tag{59}$$

Consequently,

$$\int_{x\notin D} f_X(x)\gamma_n[\alpha](x_1-x)\,dx < Ce^{-cn^{-1}\alpha^{-1/2}}e^{Cn\log n} < e^{-\alpha^{-1/3}}, \tag{60}$$

for an appropriate choice of a sufficiently small universal constant $c_0 > 0$ (so that all other constants are absorbed). Combining (55), (56) and (60) gives

$$f_{X+Y}(x_1) < \left(1-\varepsilon_0+\frac{e^{-\alpha^{-1/3}}}{\gamma_n[1](x_0)}\right)\gamma_n[1](x_0). \tag{61}$$

Using the fact that $n+(2n)^{2\beta} < \frac{\alpha^{-1/3}}{2}$, which follows easily from our assumptions, we have

$$\frac{e^{-\alpha^{-1/3}}}{\gamma_n[1](x_0)} = e^{\frac{|x_0|^2}{2}+\frac{n}{2}\log(2\pi)-\alpha^{-1/3}} < e^{-\frac{1}{2}\alpha^{-1/3}} \leqslant 2\alpha^{1/3} < \frac{\varepsilon}{2} < \frac{\varepsilon_0}{2} \tag{62}$$

(for the last inequality, note that if $\varepsilon_0 < 6\varepsilon$ then (10) holds and we have nothing to prove. So we can assume that $\varepsilon_0 > \varepsilon$). From (61) and (62) we obtain the bound

$$f_{X+Y}(x_1) < \left(1-\frac{\varepsilon_0}{2}\right)\gamma_n[1](x_0). \tag{63}$$

Combining (54) and (63) we get,

$$(1-2\varepsilon)\gamma_n[1+\alpha](x_0) < \left(1-\frac{\varepsilon_0}{2}\right)\gamma_n[1](x_0). \tag{64}$$

A calculation yields,

$$\frac{\gamma_n[1](x_0)}{\gamma_n[1+\alpha](x_0)} \leqslant \frac{\gamma_n[1](0)}{\gamma_n[1+\alpha](0)} = (1+\alpha)^{\frac{n}{2}} < 1+\varepsilon. \tag{65}$$

From the above two inequalities, we finally deduce,

$$\frac{1-\varepsilon_0/2}{1-2\varepsilon} > \frac{1}{1+\varepsilon} > 1-\varepsilon \quad\Rightarrow\quad \varepsilon_0 < 6\varepsilon, \tag{66}$$

which proves (10). The lemma is proved, under the additional assumption that $\log f_X$ is strictly concave. The general case follows by a standard approximation argument. □

After proving a lower bound, we move to the upper bound. We will show that if we add to the requirements of the previous lemma an assumption that the density of f_{X+Y} is bounded from above, then we can provide an upper bound for f_X.

Lemma 9. *Let $n, X, Y, \alpha, \beta, \varepsilon, R, c_0$ be defined as in Lemma 8, and suppose that all of the conditions of Lemma 8 are satisfied. Suppose that in addition, we have the following upper bound for f_{X+Y}:*

$$f_{X+Y}(x) < (1+\varepsilon)\gamma_n[1+\alpha](x) \tag{67}$$

for all $|x| < R$. Then we have:

$$f_X(x) < (1+8\varepsilon)\gamma_n[1](x) \tag{68}$$

for all x with $|x| < \min\{(2n)^\beta, R\} - 3$.

Proof. Denote $F(x) = -\log f_X(x)$. Again we use the upper bound for the supremum of the density (59),

$$F(x) > 6n - \frac{1}{2}n\log n > -n\log n, \quad \forall x \in \mathbb{R}^n. \tag{69}$$

Use the conclusion of Lemma 8 to deduce that for $|x| < \min\{(2n)^\beta, R\} - 1$ the following holds:

$$F(x) < -\log\left(\frac{1}{2}\gamma_n[1](x)\right) < \log 2 + \frac{n}{2}\log(2\pi) + (2n)^{2\beta} < 3(2n)^{\max\{2\beta, \frac{3}{2}\}}. \tag{70}$$

Next we will show that for $x, y \in A = \{x \in \mathbb{R}^n;\ |x| < \min\{(2n)^\beta, R\} - 2\}$, the following Lipschitz condition holds:

$$\left|F(x) - F(y)\right| \leqslant 5(2n)^{\max\{2\beta, 3/2\}}|x-y|. \tag{71}$$

To that end, denote $a = 5(2n)^{\max\{2\beta, \frac{3}{2}\}}$ and suppose by contradiction that $x, y \in A$ are such that

$$F(y) - F(x) > a|y-x|. \tag{72}$$

Since $F(y) - F(x) < a$ (as implied by (69) and (70)), we have $|y-x| < 1$ and for the point

$$y_1 := x + \frac{y-x}{|y-x|},$$

we have, using the convexity of F,

$$F(y_1) - F(x) \geqslant \frac{F(y) - F(x)}{|y-x|} > a.$$

Note that $|y_1| \leqslant |x| + 1 < \min\{(2n)^\beta, R\} - 1$, thus we obtain a contradiction of (69) and (70). This proves (71).

Therefore, given two points $x, x_0 \in A$ such that $|x_0 - x| < \alpha^{1/4}$, (71) implies,

$$\left|F(x_0) - F(x)\right| < 5\alpha^{1/4}(2n)^{\max\{2\beta, 3/2\}} < \varepsilon/20. \tag{73}$$

Recall that $F = -\log f_X$, hence the above translates to

$$\left|f_X(x_0) - f_X(x)\right| < 2\left(e^{\varepsilon/20} - 1\right) f_X(x_0) < \frac{\varepsilon}{4} f_X(x_0). \tag{74}$$

Now, suppose $x_0 \in \mathbb{R}^n$ and $0 < \varepsilon_0 < 1$ are such that

$$f_X(x_0) > (1 + \varepsilon_0)\gamma_n[1](x_0), \tag{75}$$

with $|x_0| < \min\{R, (2n)^\beta\} - 3$. Again, to prove the lemma it suffices to show that in fact $\varepsilon_0 < 8\varepsilon$. Let D be a ball of radius $\alpha^{1/4}$ around x_0.

Since we can assume that $\varepsilon_0 > \varepsilon$ (otherwise, there is nothing to prove), we deduce from (74) and (75) that for all $x \in D$,

$$f_X(x) > \left(1 - \frac{\varepsilon_0}{4}\right)(1 + \varepsilon_0)\gamma_n[1](x_0) > \left(1 + \frac{\varepsilon_0}{2}\right)\gamma_n[1](x_0). \tag{76}$$

Thus,

$$\begin{aligned} f_{X+Y}(x_0) &= \int_{\mathbb{R}^n} f_X(x)\gamma_n[\alpha](x_0 - x)\,dx > \int_{x \in D} f_X(x)\gamma_n[\alpha](x_0 - x)\,dx \\ &> \left(1 + \frac{\varepsilon_0}{2}\right)\gamma_n[1](x_0) \cdot \left(1 - \mathbb{P}\left(|G_n| > \frac{1}{\alpha^{1/4}}\right)\right) > \left(1 + \frac{\varepsilon_0}{3}\right)\gamma_n[1](x_0), \end{aligned} \tag{77}$$

where in the last inequality we used the estimate (58) and the assumption $\varepsilon_0 > \varepsilon$. Now, a computation yields,

$$\frac{\gamma_n[1+\alpha](x_0)}{\gamma_n[1](x_0)} < e^{\frac{1}{2}(|x_0|^2 - \frac{|x_0|^2}{1+\alpha})} = e^{\frac{1}{2}|x_0|^2 \frac{\alpha}{1+\alpha}} < e^{(2n)^{2\beta}\alpha} < 1 + \varepsilon. \tag{78}$$

We thus obtain, combining (67) and (77) and using (78), that

$$\frac{1 + \varepsilon_0/3}{1 + \varepsilon} < \frac{\gamma_n[1+\alpha](x_0)}{\gamma_n[1](x_0)} < 1 + \varepsilon,$$

so $\varepsilon_0 < 8\varepsilon$, and the proof of the lemma is complete. □

The combination of the two lemmas above gives us the desired estimate for the density of X, as proclaimed in the beginning of this section.

4. Proof of the main theorem

Proof of Theorem 1. We may clearly assume that n exceeds some positive universal constant (otherwise, take $\mathcal{E} = \emptyset$). Let $1 \leqslant \ell \leqslant n^{1/100}$ be an integer, and let $\delta \geqslant 0$ be such that $\ell = n^\delta$. Set $\alpha = 10$ and $\lambda = \frac{1}{5\alpha+20} = \frac{1}{70}$. Let Y be a Gaussian random vector in $\mathbb{R}^n$ with mean zero and covariance matrix $n^{-\alpha\lambda} Id$, independent of X. We first apply Proposition 7 for the random vector

$X+Y$ with parameters ℓ and α (noting that $\ell \leqslant n^{1/100} \leqslant n^{\lambda}$). According to the conclusion of that proposition, if E is a random subspace of dimension ℓ, then

$$\left| \frac{\pi_E(f_{X+Y})(x)}{\gamma_n[1+n^{-\alpha\lambda}](x)} - 1 \right| \leqslant Cn^{-1/100}, \tag{79}$$

for all $x \in E$ with $|x| < n^{\frac{1}{200}}$, with probability greater than $1 - Ce^{-cn^{1/10}}$ of selecting E.

Next, we apply Lemmas 8 and 9 in the ℓ-dimensional subspace E, with the parameters $\alpha = n^{-10\lambda} \leqslant n^{-1/20}\ell^{-8}$, $\beta = \frac{1}{600(\delta + 1/\log_2 n)}$, $R = n^{1/200}$, $\varepsilon = Cn^{-1/100}$ where C is the constant from (79). It is straightforward to verify that the requirements of these two lemmas hold, since n may be assumed to exceed a given universal constant. According to the conclusions of Lemmas 8 and 9, for any $x \in E$ with $|x| < n^{\frac{1}{700}}$,

$$\left| \frac{\pi_E(f_X)(x)}{\gamma_n[1](x)} - 1 \right| \leqslant C' n^{-1/100}.$$

This completes the proof. □

Remark. The numerical values of the exponents c_1, c_2, c_3, c_4 provided by our proof of Theorem 1 are far from optimal. The theorem is tight only in the sense that the power-law dependencies on n cannot be improved to, say, exponential dependence. The only constant among c_1, c_2, c_3, c_4 for which the best value is essentially known to us is c_2. It is clear from the proof that c_2 can be made arbitrarily close to 1 at the expense of decreasing the other constants. Note also that necessarily $c_4 \leqslant 1/4$, as is shown by the example where X is distributed uniformly in a Euclidean ball (see [13, Section 4.1]).

5. An additional Gaussian deconvolution

In this section we improve an estimate from [7,8] which is related to Gaussian convolution. This improvement can be used to obtain slightly better bounds on certain exponents related to the central limit theorem for convex bodies. The following proposition was conjectured by Meckes [12].

Proposition 10. *Let $n \geqslant 1$ and $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave density. Suppose that $\varepsilon > 0$ and denote $g_\varepsilon = f * \gamma_n[\varepsilon^2]$, the convolution of f with $\gamma_n[\varepsilon^2]$. Then,*

$$\|g_\varepsilon - f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left| g_\varepsilon(x) - f(x) \right| dx \leqslant Cn\varepsilon,$$

where $C > 0$ is a universal constant.

Proposition 10 improves upon Lemma 5.1 in [7] and the results of Section 3 in [12], and it admits a simpler proof. It is straightforward to adapt the argument in [8], and to use Proposition 10 in place of the inferior Lemma 5.1 of [7]. This leads to slightly better estimates. For instance, we conclude that whenever X is a random vector with a log-concave density in $\mathbb{R}^n$, one may find a subspace $E \subset \mathbb{R}^n$ of dimension, say, $cn^{1/15}$ such that $\mathrm{Proj}_E(X)$ is approximately Gaussian, in the total variation sense. The exponent $1/15$ is probably far from optimal, yet it is better than previous bounds.

Meckes has observed that Proposition 10 would follow from the next lemma.

Lemma 11. *Let $f:\mathbb{R}^n \to [0,\infty)$ be a C^∞-smooth, isotropic, log-concave density. Then,*

$$\int_{\mathbb{R}^n} \left|\nabla f(x)\right| dx \leqslant C' n,$$

where $C' > 0$ is a universal constant.

To see that Lemma 11 leads to Proposition 10, one only needs to apply an inequality from Ledoux [10]. In the notation of Proposition 10, it is proven in [10] that when f is C^∞-smooth,

$$\|g_\varepsilon - f\|_{L^1(\mathbb{R}^n)} \leqslant \sqrt{2}\varepsilon \int_{\mathbb{R}^n} \left|\nabla f(x)\right| dx. \tag{80}$$

Thus, Proposition 10 follows from Lemma 11 in virtue of (80), by approximating f with a C^∞-smooth function. Proposition 10 and Lemma 11 are tight, for small ε, up to the value of the constants C, C'. This is shown, e.g., by the example of f being close to the isotropic, log-concave function that is proportional to the characteristic function of the cube $[-\sqrt{3}, \sqrt{3}]^n$.

Proof of Lemma 11. The case $n = 1$ is covered, e.g., in [12]. We assume from now on that $n \geqslant 2$. Our method builds on the main idea of the proof of Lemma 2.3 in [9]. Fix $x \in \mathbb{R}^n$. We claim that

$$\left|\nabla f(x)\right| \leqslant C_1 n f(x) - C_2 \nabla f(x) \cdot x, \tag{81}$$

for some universal constants $C_1, C_2 > 0$. Suppose first that $f(x) = 0$. Since $f \geqslant 0$ and f is C^∞-smooth, then necessarily $\nabla f(x) = 0$. Therefore (81) is trivial in this case. It remains to prove (81) for the case where $f(x) > 0$. Denote $F = -\log f$. Then $F:\mathbb{R}^n \to (-\infty, \infty]$ is convex. Additionally, F is finite and C^∞-smooth in a neighborhood of x. The graph of the convex function F lies entirely above the supporting hyperplane to F at x. That is,

$$F(x) + \nabla F(x) \cdot (y - x) \leqslant F(y) \quad \text{for all } y \in \mathbb{R}^n.$$

Consequently, for any $y \in \mathbb{R}^n$,

$$\nabla F(x) \cdot y \leqslant \left[F(y) - \inf F\right] + \nabla F(x) \cdot x.$$

By taking the supremum over all $y \in \mathbb{R}^n$ with $|y| \leqslant \frac{1}{10}$, we see that

$$\frac{|\nabla F(x)|}{10} \leqslant \nabla F(x) \cdot x + \sup_{|y| \leqslant 1/10} F(y) - \inf F. \tag{82}$$

Denote

$$K = \{x \in \mathbb{R}^n;\ f(x) \geqslant e^{-10n} \sup f\}.$$

Then K is clearly convex. Additionally, $\int_K f(x)\,dx \geqslant 1 - e^{-5n/4} \geqslant 9/10$, by Corollary 5.3 in [7] (we actually use the formulation from Lemma 2.2 in [8]). According to Lemma 5.4 from [7] we have the inclusion $\{y \in \mathbb{R}^n;\ |y| \leqslant \frac{1}{10}\} \subseteq K$. Therefore,

$$\sup_{|y|\leqslant 1/10} F(y) - \inf F \leqslant \sup_{y\in K} F(y) - \inf F \leqslant [10n + \inf F] - \inf F = 10n.$$

Hence (82) implies that for any $x \in \mathbb{R}^n$,

$$\left|\nabla F(x)\right| \leqslant 10(\nabla F(x)\cdot x) + 100n. \tag{83}$$

Since $\nabla f(x) = -f(x)\nabla F(x)$, then (81) follows from (83). This completes the proof of (81). Next, we integrate by parts and see that

$$-\int_{\mathbb{R}^n} \nabla f(x)\cdot x\,dx = -\sum_{i=1}^n \int_{\mathbb{R}^n} x_i \partial^i f\,dx_1 \ldots dx_n = \sum_{i=1}^n \int_{\mathbb{R}^n} f(x)\,dx = n.$$

The boundary terms vanish, since $|x| f(x) \to 0$ as $|x| \to \infty$ (see, e.g., [9, Lemma 2.1]). According to (81),

$$\int_{\mathbb{R}^n} \left|\nabla f(x)\right| dx \leqslant C_1 n \int_{\mathbb{R}^n} f(x)\,dx - C_2 \int_{\mathbb{R}^n} \nabla f(x)\cdot x\,dx = (C_1 + C_2)n. \qquad \square$$

Acknowledgments

The first named author would like to express his sincere gratitude to his supervisor, Prof. Vitali Milman who introduced him to the subject, guided him and encouraged him to write this note. We would also like to thank Sasha Sodin and Prof. Vitali Milman for reviewing a preliminary version of this note.

References

[1] D. Alonso-Gutierrez, J. Bastero, J. Bernues, G. Paouris, High-dimensional random sections of isotropic convex bodies, preprint.
[2] M. Anttila, K. Ball, I. Perissinaki, The central limit problem for convex bodies, Trans. Amer. Math. Soc. 355 (12) (2003) 4723–4735.
[3] J. Bastero, J. Bernués, Asymptotic behavior of averages of k-dimensional marginals of measures on $\mathbb{R}^n$, preprint, available at http://www.unizar.es/galdeano/preprints/2005/preprint34.pdf.
[4] U. Brehm, J. Voigt, Asymptotics of cross sections for convex bodies, Beiträge Algebra Geom. 41 (2) (2000) 437–454.
[5] P. Diaconis, D. Freedman, A dozen de Finetti-style results in search of a theory, Ann. Inst. H. Poincaré Probab. Statist. 23 (2) (1987) 397–423.
[6] B. Klartag, On convex perturbations with a bounded isotropic constant, Geom. Funct. Anal. 16 (6) (2006) 1274–1290.
[7] B. Klartag, A central limit theorem for convex sets, Invent. Math. 168 (2007) 91–131.

[8] B. Klartag, Power-law estimates for the central limit theorem for convex sets, J. Funct. Anal. 245 (2007) 284–310.
[9] B. Klartag, Uniform almost sub-Gaussian estimates for linear functionals on convex sets, Algebra i Analiz (St. Petersburg Math. J.) 19 (1) (2007) 109–148.
[10] M. Ledoux, Spectral gap, logarithmic Sobolev constant, and geometric bounds, in: Surveys Differ. Geom., vol. IX, Int. Press, Somerville, MA, 2004, pp. 219–240.
[11] L. Lovász, S. Vempala, The geometry of logconcave functions and sampling algorithms, Random Structures Algorithms 30 (3) (2007) 307–358.
[12] M.W. Meckes, Gaussian marginals of convex bodies with symmetries, available under http://arxiv.org/abs/math/0606073.
[13] S. Sodin, Tail-sensitive Gaussian asymptotics for marginals of concentrated measures in high dimension, in: Geometric Aspects of Functional Analysis, Israel Seminar, in: Lecture Notes in Math., vol. 1910, Springer, 2007, pp. 271–295.