# Power-law estimates for the central limit theorem for convex sets

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**Abstract**

We investigate the rate of convergence in the central limit theorem for convex sets established in [9]. We obtain bounds with a power-law dependence on the dimension. These bounds are asymptotically better than the logarithmic estimates which follow from the original proof of the central limit theorem for convex sets.

## 1 Introduction

This article is a continuation of [9]. In [9] we provided a proof for a basic conjecture in convex geometry (see [1], [4]), and showed that the uniform distribution on any high-dimensional convex body has marginals that are approximately gaussian. Here we improve some quantitative estimates regarding the degree of that approximation.

We denote by $G_{n,\ell}$ the grassmannian of all $\ell$-dimensional subspaces of $\mathbb{R}^n$, and let $\sigma_{n,\ell}$ stand for the unique rotationally invariant probability measure on $G_{n,\ell}$. The standard Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. For a subspace $E \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we write $Proj_E(x)$ for the orthogonal projection of $x$ onto $E$. A convex body in $\mathbb{R}^n$ is a compact, convex set with a non-empty interior. We write $Prob$ for probability.

**Theorem 1.1** *Let $1 \leq \ell \leq n$ be integers and let $K \subset \mathbb{R}^n$ be a convex body. Let $X$ be a random vector that is distributed uniformly in $K$, and suppose that $X$ has zero mean and identity covariance matrix. Assume that $\ell \leq cn^\kappa$.*

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*Then there exists a subset $\mathcal{E} \subseteq G_{n,\ell}$ with $\sigma_{n,\ell}(\mathcal{E}) \geq 1 - \exp(-c\sqrt{n})$ such that for any $E \in \mathcal{E}$,*

$$\sup_{A \subseteq E} \left| Prob\{Proj_E(X) \in A\} - \frac{1}{(2\pi)^{\ell/2}} \int_A \exp\left(-\frac{|x|^2}{2}\right) dx \right| \leq \frac{1}{n^\kappa},$$

*where the supremum runs over all measurable sets $A \subseteq E$. Here, $c, \kappa > 0$ are universal constants.*

Our methods in [9] have yielded bounds that depend logarithmically on the dimension; the estimates in [9] are closer to those in Milman's quantitative theory of Dvoretzky's theorem (see, e.g., [12] and references therein). In one of its formulations, Dvoretzky's theorem states that for any convex body $K \subset \mathbb{R}^n$ and $\varepsilon > 0$, there exists a subspace $E \subseteq \mathbb{R}^n$ of dimension at least $c\varepsilon^2 \log n$ with

$$(1 - \varepsilon)\mathcal{D} \subseteq Proj_E(K) \subseteq (1 + \varepsilon)\mathcal{D}.$$

Here, $\mathcal{D}$ is some Euclidean ball in the subspace $E$ that is centered at the origin, and $c > 0$ is a universal constant. The logarithmic dependence on the dimension is known to be tight in Dvoretzky's theorem (consider, e.g., an $n$-dimensional simplex). In contrast to that, we learn from Theorem 1.1 that the uniform measure on $K$, once projected to subspaces whose dimension is a power of $n$, becomes approximately gaussian.

**Corollary 1.2** *Let $1 \leq \ell \leq n$ be integers with $\ell \leq cn^\kappa$. Let $X$ be a random vector in $\mathbb{R}^n$ that is distributed uniformly in some convex body. Then there exist an $\ell$-dimensional subspace $E \subset \mathbb{R}^n$ and $r > 0$ such that*

$$\sup_{A \subseteq E} \left| Prob\{Proj_E(X) \in A\} - \frac{1}{(2\pi r)^{\ell/2}} \int_A \exp\left(-\frac{|x|^2}{2r}\right) dx \right| \leq \frac{1}{n^\kappa},$$

*where the supremum runs over all measurable sets $A \subseteq E$. Here, $c, \kappa > 0$ are universal constants.*

Thus, we observe a sharp distinction between the measure-projection and the geometric-projection of high-dimensional convex bodies. We did not expect such a distinction. Our results are also valid for random vectors with a log-concave density. Recall that a function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave when

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$. The characteristic function of any convex set is log-concave. As a matter of fact, the assumptions of Theorem 1.1 and Corollary 1.2 may be slightly weakened: It is sufficient for the density of $X$ to be log-concave; the density is not necessarily required to be proportional to the characteristic function of a convex body. A function $f : \mathbb{R}^n \to [0, \infty)$ is isotropic if it is the density of a random vector in $\mathbb{R}^n$ with zero mean and identity covariance matrix.

**Theorem 1.3** *Let $n \geq 1$ be an integer and let $X$ be a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density. Then,*

$$Prob \left\{ \left| \frac{|X|}{\sqrt{n}} - 1 \right| \geq \frac{1}{n^\kappa} \right\} \leq C \exp\left(-n^\kappa\right),$$

*where $C, \kappa > 0$ are universal constants.*

The bounds we obtain for $\kappa$ from Theorem 1.3 are not very good. Our proof of Theorem 1.3 works for, say, $\kappa = 1/15$. See Theorem 4.4 below for more precise information. Compare Theorem 1.3 with the sharp large-deviation estimate of Paouris [14], [15]: Paouris showed that under the assumptions of Theorem 1.3,

$$Prob \left\{ |X| \geq C\sqrt{n} \right\} \leq \exp\left(-\sqrt{n}\right) \tag{1}$$

for some universal constant $C > 0$. The estimate (1) is known to be essentially the best possible, unlike the results in this note which probably miss the optimal exponents.

Define

$$\gamma(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \qquad (t \in \mathbb{R}),$$

the standard gaussian density. We write $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ for the standard Euclidean sphere in $\mathbb{R}^n$. The unique rotationally-invariant probability measure on $S^{n-1}$ is denoted by $\sigma_{n-1}$. The standard scalar product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. Theorem 1.3 is the basic requirement needed in order to apply Sodin's moderate deviation estimates [17]. We arrive at the following result:

**Theorem 1.4** *Let $n \geq 1$ be an integer and let $X$ be a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density.*

*Then there exists $\Theta \subseteq S^{n-1}$ with $\sigma_{n-1}(\Theta) \geq 1 - C\exp(-\sqrt{n})$ such that for all $\theta \in \Theta$, the real-valued random variable $\langle X, \theta \rangle$ has a density $f_\theta : \mathbb{R} \to [0, \infty)$ with the following two properties:*

*(i)* $\int_{-\infty}^{\infty} |f_\theta(t) - \gamma(t)|\, dt \leq \frac{1}{n^\kappa}$,

*(ii)* *For all* $|t| \leq n^\kappa$ *we have* $\left| \frac{f_\theta(t)}{\gamma(t)} - 1 \right| \leq \frac{1}{n^\kappa}$.

*Here,* $C, \kappa > 0$ *are universal constants.*

The direction of research we pursue in [9] and here builds upon the investigations of Anttila, Ball and Perissinaki [1], Brehm and Voigt [4] and others (e.g., [3], [5], [10]). The methods of proof in this article have much in common with the technique in [9]. As in [9], the basic idea is to show that a typical multi-dimensional marginal is approximately spherically-symmetric. Since the marginal is also log-concave, then most of the mass of the marginal must be concentrated in a thin spherical shell, and hence the marginal is close to the uniform distribution on a sphere. An important difference between the argument here and the one in [9], is the use of concentration inequalities on the orthogonal group, rather than on the sphere. Even though the proof here is more technical and conceptually more complicated than the argument in [9], it may be considered more "direct" in some respects, since we avoid the use of the Fourier inversion formula.

As the reader has probably guessed, we write $c, C, c', \tilde{C}$ etc., and also $\kappa$, for various positive universal constants, whose value may change from one line to the next. The symbols $C, C', \bar{C}, \tilde{C}$ etc. denote universal constants that are assumed to be sufficiently large, while $c, c', \bar{c}, \tilde{c}$ etc. denote sufficiently small universal constants. The universal constants denoted by $\kappa$ are usually exponents; it is desirable to obtain reasonable lower bounds on $\kappa$. The natural logarithm is denoted here by $\log$, and $\mathbb{E}$ stands for expectation.

# 2 Computations with log-concave functions

This section contains certain estimates related to log-concave functions. Underlying these estimates is the usual paradigm, that log-concave densities in high-dimension are quite rigid, up to an affine transformation. We refer the reader, e.g., to [9, Section 2] and to [8, Section 2] for a quick overview of log-concave functions and for appropriate references. The following result of Fradelizi [6, Theorem 4] will be frequently used.

**Lemma 2.1** *Let $n \geq 1$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable, log-concave function. Denote $x_0 = \int_{\mathbb{R}^n} x f(x) dx / \int_{\mathbb{R}^n} f(x) dx$, the barycenter of $f$. Then,*

$$f(x_0) \geq e^{-n} \sup f.$$

The proof of our next lemma appears in [9, Corollary 5.3].

**Lemma 2.2** *Let $n \geq 2$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable, log-concave function. Denote $K = \{x \in \mathbb{R}^n ; f(x) \geq e^{-10n} \cdot \sup f\}$. Then,*

$$\int_K f(x) dx \geq \left(1 - e^{-n}\right) \int_{\mathbb{R}^n} f(x) dx.$$

The following lemma is essentially taken from [8]. However, the proof in [8] relates only to even functions. Below we describe a reduction to the even case. Another proof may be obtained by adapting the arguments from [8] to the general case. We write $Vol_n$ for the Lebesgue measure on $\mathbb{R}^n$, and $D^n = \{x \in \mathbb{R}^n ; |x| \leq 1\}$ for the centered Euclidean unit ball in $\mathbb{R}^n$.

**Lemma 2.3** *Let $n \geq 1$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Denote $K = \{x \in \mathbb{R}^n ; f(x) \geq e^{-n} f(0)\}$. Then,*

$$K \subseteq C n D^n, \tag{2}$$

*where $C > 0$ is a universal constant.*

*Proof:* We may assume that $n \geq 2$ (the case $n = 1$ follows, e.g., from [3, Lemma 3.2]). Suppose first that $f$ is an even function. Consider the logarithmic Laplace transform $\Upsilon f(x) = \log \int_{\mathbb{R}^n} e^{\langle x, y \rangle} f(y) dy$, defined for $x \in \mathbb{R}^n$. According to [8, Lemma 2.7] the set $T = \{x \in \mathbb{R}^n ; \Upsilon f(x) \leq n\}$ satisfies

$$T \subseteq C n K^\circ, \tag{3}$$

where $K^\circ = \{x \in \mathbb{R}^n ; \forall y \in K, \langle x, y \rangle \leq 1\}$ is the dual body. Since $f$ is isotropic, $\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 f(x) dx = 1$ for all $\theta \in S^{n-1}$. We use Borell's lemma (e.g., [13, Appendix III.3]) and conclude that for any $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} \exp\left(c \langle x, \theta \rangle\right) f(x) dx \leq 2.$$

Consequently, $cD^n \subseteq T$. Since $K$ is convex and centrally-symmetric, the inclusion (3) entails that $K \subseteq CnD^n$. This completes the proof for the case where $f$ is an even function.

It remains to deal with the case where $f$ is not necessarily even. The log-concavity of $f$ implies that $f(\alpha x)/f(0) \leq (f(x)/f(0))^\alpha$ for any $\alpha \geq 1, x \in \mathbb{R}^n$. Therefore,

$$K' := \{x \in \mathbb{R}^n; f(x) \geq e^{-10n} f(0)\} \subseteq 10K. \tag{4}$$

According to Lemma 2.1 and Lemma 2.2,

$$e^n f(0) Vol_n(K') \geq \sup f \cdot Vol_n(K') \geq \int_{K'} f(x)dx \geq 1/2. \tag{5}$$

From (4) and (5) we see that

$$Vol_n(K) > \frac{c^n}{f(0)}. \tag{6}$$

Denote

$$g(x) = 2^{n/2} \int_{\mathbb{R}^n} f(y) f(y + \sqrt{2}x) dy.$$

Then $g$ is an even, isotropic, log-concave density on $\mathbb{R}^n$, as follows from the Prékopa-Leindler inequality (for Prékopa-Leindler, see, e.g., the first pages of [16]). Moreover,

$$g(0) \leq 2^{n/2} \sup f \int_{\mathbb{R}^n} f(y)dy = 2^{n/2} \sup f \leq (\sqrt{2}e)^n f(0) \tag{7}$$

by Lemma 2.1. The set $K$ is convex with $0 \in K$. For any $x, y \in K/4$ we have $y + \sqrt{2}x \in K$ and hence $f(y), f(y + \sqrt{2}x) \geq e^{-n} f(0)$. Therefore, by (6) and (7), if $x \in K/4$,

$$g(x) \geq \int_{K/4} f(y) f(y + \sqrt{2}x) dy \geq Vol_n(K/4)(e^{-n} f(0))^2 \geq c^n f(0) \geq \tilde{c}^n g(0).$$

We conclude that

$$K/4 \subseteq \{x \in \mathbb{R}^n; g(x) \geq e^{-Cn} g(0)\} \subseteq C\{x \in \mathbb{R}^n; g(x) \geq e^{-n} g(0)\}, \tag{8}$$

where the last inclusion follows directly from the log-concavity of $g$. Recall that $g$ is even, isotropic and log-concave, and that (2) was already proven for functions that are even, isotropic and log-concave. We may thus assert that

$$\{x \in \mathbb{R}^n; g(x) \geq e^{-n} g(0)\} \subseteq CnD^n. \tag{9}$$

The conclusion of the lemma thereby follows from (8) and (9). $\square$

**Corollary 2.4** *Let $n \geq 1$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function. Then, for any $x \in \mathbb{R}^n$,*

$$f(x) \leq f(0) \exp\left(Cn - c|x|\right)$$

*for some universal constants $c, C > 0$.*

*Proof:* According to Lemma 2.3, for any $x \in \mathbb{R}^n$,

$$f(x) \leq e^{-n} f(0) \quad \text{if} \quad |x| \geq Cn.$$

By log-concavity, whenever $|x| \geq Cn$,

$$e^{-n} f(0) \geq f\left(\frac{Cn}{|x|} \cdot x\right) \geq f(0)^{1 - Cn/|x|} \cdot f(x)^{Cn/|x|}.$$

Equivalently,

$$f(x) \leq f(0) e^{-|x|/C} \quad \text{whenever} \quad |x| \geq Cn. \tag{10}$$

According to Lemma 2.1 we know that $f(x) \leq e^n f(0)$ for any $x \in \mathbb{R}^n$. In particular, $f(x) \leq f(0) \exp(2n - |x|/C)$ when $|x| < Cn$. Together with (10) we obtain

$$f(x) \leq f(0) e^{2n - |x|/C} \quad \text{for all} \quad x \in \mathbb{R}^n.$$

This completes the proof. $\square$

We will make use of the following elementary result.

**Lemma 2.5** *Let $f : [0, \infty) \to [0, \infty)$ be a measurable function with $0 < \int_0^\infty (1 + t) f(t) dt < \infty$. Suppose that $g : [0, \infty) \to (0, \infty)$ is a monotone decreasing function. Then,*

$$\frac{\int_0^\infty t^2 f(t) g(t) dt}{\int_0^\infty f(t) g(t) dt} \leq \frac{\int_0^\infty t^2 f(t) dt}{\int_0^\infty f(t) dt}.$$

*Proof:* Since $g$ is non-increasing, for any $t \geq 0$,

$$\frac{\int_0^t f(s) g(s) ds}{\int_t^\infty f(s) g(s) ds} \geq \frac{g(t) \int_0^t f(s) ds}{g(t) \int_t^\infty f(s) ds} = \frac{\int_0^t f(s) ds}{\int_t^\infty f(s) ds}.$$

Equivalently, for any $t \geq 0$,

$$\frac{\int_t^\infty f(s) g(s) ds}{\int_0^\infty f(s) g(s) ds} \leq \frac{\int_t^\infty f(s) ds}{\int_0^\infty f(s) ds}.$$

We conclude that

$$\frac{\int_0^\infty t^2 f(t)g(t)dt}{\int_0^\infty f(t)g(t)dt} = \frac{\int_0^\infty \int_{\sqrt{t}}^\infty f(s)g(s)dsdt}{\int_0^\infty f(t)g(t)dt} \leq \frac{\int_0^\infty \int_{\sqrt{t}}^\infty f(s)dsdt}{\int_0^\infty f(t)dt} = \frac{\int_0^\infty t^2 f(t)dt}{\int_0^\infty f(t)dt}.$$

□

The identity matrix is denoted here by $Id$. For a $k$-dimensional subspace $E \subseteq \mathbb{R}^n$ and for $v > 0$ we define $\gamma_E[v] : E \to [0, \infty)$ to be the density

$$\gamma_E[v](x) = \frac{1}{(2\pi v)^{k/2}} \exp\left(-\frac{|x|^2}{2v}\right) \qquad (x \in E).$$

That is, $\gamma_E[v]$ is the density of a gaussian random vector in $E$ with mean zero and covariance matrix that equals $vId$. A standard gaussian random vector in $E$ is a random vector whose density is $\gamma_E[1]$. We abbreviate $\gamma_n[v]$ for $\gamma_{\mathbb{R}^n}[v]$.

**Corollary 2.6** *Let $n \geq 1$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic function. Let $x \in \mathbb{R}^n, v > 0$ and denote*

$$g(y) = f(y) \exp\left(-\frac{|x-y|^2}{2v}\right) \qquad (y \in \mathbb{R}^n),$$

*an integrable function. Suppose that $X$ is a random vector in $\mathbb{R}^n$ whose density is proportional to $g$. Then,*

*(i)* $\mathbb{E}|X - x|^2 \leq n + |x|^2$,

*(ii)* $|\mathbb{E}X - x| \leq \sqrt{n} + |x|$.

*Proof:* For $r > 0$ denote

$$h(r) = r^{n-1} \int_{S^{n-1}} f(x + r\theta) d\theta$$

where the integration is with respect to the surface area measure on $S^{n-1}$. By integration in polar coordinates,

$$\int_0^\infty h(r)dr = \int_{\mathbb{R}^n} f = 1.$$

Let $Y$ be a random vector in $\mathbb{R}^n$ that is distributed according to the density $f$. We integrate in polar coordinates and obtain

$$\begin{aligned} n + |x|^2 &= \mathbb{E}|Y - x|^2 \qquad (11) \\ &= \int_{\mathbb{R}^n} |y|^2 f(y+x) dy = \int_0^\infty \int_{S^{n-1}} r^{n+1} f(x + r\theta) d\theta dr = \int_0^\infty r^2 h(r) dr. \end{aligned}$$

Similarly,

$$\mathbb{E}|X - x|^2 = \frac{\int_0^\infty r^2 h(r) \exp(-r^2/2v) dr}{\int_0^\infty h(r) \exp(-r^2/2v) dr}. \qquad (12)$$

We apply the elementary Lemma 2.5, based on (11) and (12). This proves (i). The inequality (ii) follows from (i) by Jensen's inequality. □

The crude estimates in the next two lemmas are the main results of this section. Our first lemma does not use the log-concavity assumption in an essential way.

**Lemma 2.7** *Let $n \geq 1$, let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function, and let $v > 0$. Denote $g = f * \gamma_n[v]$, the convolution of $f$ with $\gamma_n[v]$. Then, for any $x \in \mathbb{R}^n$ with $|x| \leq 10\sqrt{n}$,*

$$|\nabla \log g(x)| \leq C\sqrt{n}/v,$$

*where $C > 0$ is a universal constant.*

*Proof:* Since

$$g(x) = (2\pi v)^{-(n/2)} \int_{\mathbb{R}^n} f(y) \exp\left(-|x-y|^2/(2v)\right) dy,$$

we may differentiate under the integral sign and obtain that

$$\nabla g(x) = (2\pi v)^{-(n/2)} \int_{\mathbb{R}^n} f(y) \frac{y-x}{v} \exp\left(-|x-y|^2/(2v)\right) dy.$$

Fix $x \in \mathbb{R}^n$. Denote $g_x(y) = f(y) \exp\left(-|y-x|^2/(2v)\right)$. Then,

$$\nabla \log g(x) = \frac{\nabla g(x)}{g(x)} = v^{-1} \int_{\mathbb{R}^n} (y-x) g_x(y) dy \Big/ \int_{\mathbb{R}^n} g_x(y) dy. \qquad (13)$$

Let $X$ be a random vector in $\mathbb{R}^n$ whose density is proportional to $g_x$. Then $|\mathbb{E}X - x| \leq \sqrt{n} + |x|$, as we learn from Corollary 2.6(ii). We conclude from (13) that

$$|\nabla \log g(x)| = v^{-1} |\mathbb{E}X - x| \leq \frac{\sqrt{n} + |x|}{v} \leq 11 \frac{\sqrt{n}}{v},$$

since $|x| \leq 10\sqrt{n}$. The lemma is thus proven. □

For a subspace $E \subseteq \mathbb{R}^n$ we write $E^\perp = \{x \in \mathbb{R}^n; \forall y \in E, \langle x, y \rangle = 0\}$ for its orthogonal complement. For an integrable function $f : \mathbb{R}^n \to [0, \infty)$, a subspace $E \subseteq \mathbb{R}^n$ and a point $x \in E$ we write

$$\pi_E(f)(x) = \int_{x+E^\perp} f(y)dy.$$

That is, $\pi_E(f) : E \to [0, \infty)$ is the marginal of $f$ onto $E$.

**Lemma 2.8** *Let $n \geq 1, 0 < v \leq 1, x \in \mathbb{R}^n$ and $e \in S^{n-1}$. Let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, log-concave function and denote $g = f * \gamma_n[v]$. For $\theta \in \mathbb{R}^n$ with $|\theta - e| < 1/2$ define*

$$F(\theta) = \log \int_{-\infty}^{\infty} g(x + t\theta)dt.$$

*Assume that $|x| \leq 10\sqrt{n}$. Then,*

$$|\nabla F(e)| \leq Cn^{3/2}/v^2,$$

*where $C > 0$ is a universal constant.*

*Proof:* For $\theta \in \mathbb{R}^n$ with $|\theta - e| < 1/2$, denote

$$G(\theta) = \int_{-\infty}^{\infty} g(x+t\theta)dt = (2\pi v)^{-n/2} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{|x + t\theta - y|^2}{2v}\right) dtdy$$

$$= (2\pi v)^{-(n-1)/2} \cdot |\theta|^{-1} \cdot \int_{\mathbb{R}^n} f(y) \exp\left(-\frac{|Proj_{\theta^\perp}(x - y)|^2}{2v}\right) dy$$

where $\theta^\perp = \{x \in \mathbb{R}^n; \langle x, \theta \rangle = 0\}$. We may differentiate under the integral sign (recall that $f$ decays exponentially fast at infinity by, e.g., [8, Lemma 2.1]) and obtain

$$\nabla G(e) = \frac{1}{v(2\pi v)^{n/2}} \int_{\mathbb{R}^n} f(y) \int_{-\infty}^{\infty} (y - x - te)t \exp\left(-\frac{|x + te - y|^2}{2v}\right) dtdy.$$

We write $y - x = z + re$ where $r \in \mathbb{R}$ and $z \in e^\perp$. A direct computation reveals that

$$\int_{-\infty}^{\infty} [z + (r - t)e]t \exp\left(-\frac{|(t - r)e - z|^2}{2v}\right) dt = \sqrt{2\pi v} \exp\left(-\frac{|z|^2}{2v}\right) \cdot (rz - ve).$$

Denote $H = e^{\perp}$ and set $g_x(y) = f(y) \exp\left(-|Proj_H(x-y)|^2/(2v)\right)$ for $y \in \mathbb{R}^n$. Then,

$$|\nabla G(e)| \leq \frac{1}{v(2\pi v)^{(n-1)/2}} \int_{\mathbb{R}^n} \left(|Proj_H(y-x)| \cdot |\langle y-x, e\rangle| + v\right) g_x(y) dy.$$

According to the Cauchy-Schwartz inequality,

$$\begin{aligned} |\nabla F(e)| = \frac{|\nabla G(e)|}{G(e)} &\leq 1 + \frac{\int_{\mathbb{R}^n} |Proj_H(x-y)| \cdot |\langle x-y, e\rangle| \cdot g_x(y) dy}{v \int_{\mathbb{R}^n} g_x(y) dy} \qquad (14) \\ \leq \quad 1 + \frac{1}{v} &\left( \frac{\int_{\mathbb{R}^n} |Proj_H(x-y)|^2 g_x(y) dy}{\int_{\mathbb{R}^n} g_x(y) dy} \cdot \frac{\int_{\mathbb{R}^n} \langle x-y, e\rangle^2 g_x(y) dy}{\int_{\mathbb{R}^n} g_x(y) dy} \right)^{1/2}. \end{aligned}$$

Our derivation of the inequality (14) relies only on integrability properties of $f$. The log-concavity and isotropicity of $f$ will come into play next. Let $Y$ be a random vector in $\mathbb{R}^n$ whose density is proportional to $g_x$. Then the density of the random vector $Proj_H(Y)$ is proportional to

$$y \mapsto \pi_H(f)(y) \cdot \exp\left(-\frac{|y - Proj_H(x)|^2}{2v}\right) \qquad (y \in H).$$

The density $\pi_H(f)$ is isotropic and log-concave, by Prékopa-Leindler. We may thus apply Corollary 2.6(i) and conclude that

$$\mathbb{E}|Proj_H(Y) - Proj_H(x)|^2 \leq (n-1) + |Proj_H(x)|^2.$$

Therefore,

$$\frac{\int_{\mathbb{R}^n} |Proj_H(y-x)|^2 g_x(y) dy}{\int_{\mathbb{R}^n} g_x(y) dy} \leq 2 \max\left\{n, |x|^2\right\}. \qquad (15)$$

Next, we deal with the second factor of the product in (14). We write $H^+ = \{y \in \mathbb{R}^n; \langle y, e\rangle \geq 0\}$ and $H^- = \{y \in \mathbb{R}^n; \langle y, e\rangle \leq 0\}$. Let $g^+ : \mathbb{R}^n \to [0, \infty)$ be the function defined by

$$g^+(y) = \left(\langle y-x, e\rangle + 2|x| + 1\right)^2 g_x(y) \quad \text{for } y \in H^+$$

and $g^+(y) = 0$ for $y \notin H^+$. Observe that $g^+$ is log-concave, since both $y \mapsto (\langle y-x, e\rangle + 2|x| + 1)^2$ and $y \mapsto g_x(y)$ are log-concave on $H^+$. Additionally, $g^+$ is integrable, since $f$ decays exponentially fast at infinity. We claim that

$$g^+(y) < e^{-10n} g^+(0) \quad \text{if } |y| > \tilde{C}\left(\frac{|x|^2}{v} + |x| + n\right). \qquad (16)$$

Indeed, by Corollary 2.4, $f(y) \leq f(0)e^{Cn-c|y|}$ for all $y \in \mathbb{R}^n$. Hence, when $|y| > \tilde{C}\left(|x|^2/v + |x| + n\right)$,

$$\begin{aligned} g^+(y) &\leq (1+2|x|+|y-x|)^2 g_x(y) \leq (1+2|y|)^2 \cdot f(0)e^{Cn-c|y|} \\ &\leq \; f(0)e^{C'n-c'|y|} < f(0)\exp\left(-10n - \frac{|x|^2}{2v}\right) \leq e^{-10n}g_x(0) \leq e^{-10n}g^+(0), \end{aligned}$$

and (16) is proven. Denote $K^+ = \{y \in \mathbb{R}^n; g^+(y) \geq e^{-10n}g^+(0)\}$. Then, by (16) and by Lemma 2.2,

$$\begin{aligned} &\left[1+3|x| + \tilde{C}\left(\frac{|x|^2}{v} + |x| + n\right)\right]^2 \cdot \int_{K^+} g_x(y)dy \\ &\geq \; \int_{K^+} g^+(y)dy \geq \left(1-e^{-n}\right)\int_{\mathbb{R}^n} g^+(y)dy. \end{aligned}$$

We deduce that

$$\frac{\int_{H^+}\langle y-x,e\rangle^2 g_x(y)dy}{\int_{\mathbb{R}^n} g_x(y)dy} \leq \frac{\int_{\mathbb{R}^n} g^+(y)dy}{\int_{K^+} g_x(y)dy} \leq C\max\left\{\frac{|x|^4}{v^2}, |x|^2, n^2\right\}. \tag{17}$$

The proof that

$$\int_{H^-}\langle y-x,e\rangle^2 g_x(y)dy \leq C\max\left\{\frac{|x|^4}{v^2}, |x|^2, n^2\right\} \cdot \int_{\mathbb{R}^n} g_x(y)dy \tag{18}$$

is completely analogous. One just needs to work with the function $y \mapsto g^-(y) = \left(\langle y-x,e\rangle - 2|x| - 1\right)^2 g_x(y)$, which is log-concave on $H^-$. By adding (17) and (18) we find that

$$\frac{\int_{\mathbb{R}^n}\langle y-x,e\rangle^2 g_x(y)dy}{\int_{\mathbb{R}^n} g_x(y)dy} \leq C'\max\left\{\frac{|x|^4+1}{v^2}, n^2\right\}, \tag{19}$$

since $0 < v \leq 1$. We combine (19) with (15) and (14), and conclude that

$$|\nabla F(e)| \leq 1 + \frac{C}{v^2}\max\{\sqrt{n}, |x|\} \cdot \max\{|x|^2+1, nv\}.$$

The lemma follows, since $|x| \leq 10\sqrt{n}$ and $0 < v \leq 1$. □

## 3 Concentration of measure on the orthogonal group

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function, let $E \subseteq \mathbb{R}^n$ be a subspace and let $x \in E$. Recall that we define

$$\pi_E(f)(x) = \int_{x+E^\perp} f(y)dy.$$

We will consider the group $SO(n)$, that consists of all orthogonal transformations in $\mathbb{R}^n$ of determinant one. The group $SO(n)$ admits a canonical Riemannian metric which it inherits from the obvious embedding $SO(n) \subset \mathbb{R}^{n^2}$ (that is, a real $n \times n$-matrix has $n^2$ real numbers in it). For $U \in SO(n)$ we set

$$M_{f,E,x}(U) = \log \pi_E(f \circ U)(x) = \log \pi_{U(E)}(f)(Ux). \tag{20}$$

Clearly, for any $U_1, U_2 \in SO(n)$,

$$M_{f,E,x}(U_1U_2) = M_{f,U_2(E),U_2(x)}(U_1). \tag{21}$$

For $U_1, U_2 \in SO(n)$ we write $d(U_1, U_2)$ for the geodesic distance between $U_1$ and $U_2$ in the connected Riemannian manifold $SO(n)$. It is well-known that for any $U_1, U_2 \in SO(n)$,

$$\|U_1 - U_2\|_{HS} \leq d(U_1, U_2) \leq \frac{\pi}{2} \|U_1 - U_2\|_{HS}$$

where $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt norm, i.e., for a matrix $A = (A_{i,j})_{i,j=1,\ldots,n}$ we have $\|A\|_{HS} = \sqrt{\sum_{i,j=1}^n |A_{i,j}|^2}$.

**Lemma 3.1** *Let $2 \leq k \leq n$ be integers, let $\alpha \geq 0$, and assume that $0 < \lambda \leq 1$ is such that $k = n^\lambda$. Suppose that $f : \mathbb{R}^n \to [0, \infty)$ is an isotropic, log-concave function and define $g = f * \gamma_n[n^{-\alpha\lambda}]$. Let $E_0 \subseteq \mathbb{R}^n$ be a $k$-dimensional subspace, and let $x_0 \in E_0$ be a point with $|x_0| \leq 10\sqrt{k}$. Then, for any $U_1, U_2 \in SO(n)$,*

$$|M_{g,E_0,x_0}(U_1) - M_{g,E_0,x_0}(U_2)| \leq Cn^{\lambda(2\alpha+2)} \cdot d(U_1, U_2),$$

*where $C > 0$ is a universal constant.*

*Proof:* We abbreviate $M(U) = M_{g,E_0,x_0}(U)$ for $U \in SO(n)$. We need to show that $M$ is $Cn^{\lambda(2\alpha+2)}$-Lipshitz on $SO(n)$. By rotational-symmetry (see (21)), it is enough to show that

$$|\nabla M(Id)| \leq Cn^{\lambda(2\alpha+2)}, \tag{22}$$

where $\nabla M(Id)$ is the Riemannian gradient of $M : SO(n) \to \mathbb{R}$ at $Id$, and $|\nabla M(Id)|$ is its length. Fix an orthonormal basis $e_1, ..., e_k \in E_0$. For $v = (v_1, ..., v_k) \in (\mathbb{R}^n)^k$, let $A_v : E_0 \to \mathbb{R}^n$ stand for the unique linear operator with $A_v(e_i) = v_i$ for $i = 1, ..., k$. Define,

$$G(v_1, ..., v_k) = \log \int_{A_v x_0 + (A_v E_0)^\perp} g(x) dx$$

which is well-defined because $x_0 \in E_0$. Note that by (20), for any $U \in SO(n)$,

$$M(U) = \log \int_{Ux_0 + (UE_0)^\perp} g(x) dx = G(Ue_1, ..., Ue_k).$$

Furthermore, for any $U \in SO(n)$,

$$\sum_{i=1}^k |Ue_i - e_i|^2 \leq \|U - Id\|_{HS}^2 \leq d(U, Id)^2.$$

Therefore, to prove (22), it is sufficient to demonstrate that

$$|\nabla G(e_1, ..., e_k)| \leq Cn^{\lambda(2\alpha+2)} = Ck^{2\alpha+2}, \tag{23}$$

where $\nabla G$ is the usual gradient of the function $G : (\mathbb{R}^n)^k \to \mathbb{R}$ in the Euclidean space $(\mathbb{R}^n)^k$. For $i = 1, ..., k$ and $v \in \mathbb{R}^n$ we set $F_i(v) = G(e_1, ..., e_{i-1}, v, e_{i+1}, ..., e_k)$. Then

$$|\nabla G(e_1, .., e_k)|^2 = \sum_{i=1}^k |\nabla F_i(e_i)|^2$$

(note that $G$ is a smooth function in a neighborhood of $(e_1, ..., e_k) \in (\mathbb{R}^n)^k$, since $g$ is $C^\infty$-smooth and decays exponentially fast at infinity). Therefore, it is sufficient to prove that for any $i = 1, .., k$,

$$|\nabla F_i(e_i)| \leq Ck^{2\alpha+3/2}. \tag{24}$$

By symmetry, it is enough to focus on the case $i = 1$. Thus, we denote $F(v) = F_1(v) = G(v, e_2, ..., e_k)$. Then $F : \mathbb{R}^n \to \mathbb{R}$ is a smooth function in a neighborhood of $e_1$, and our goal is to prove that

$$|\nabla F(e_1)| \leq Ck^{2\alpha+3/2}.$$

Equivalently, fix an arbitrary $v \in \mathbb{R}^n$ with $|v| = 1$. To prove the lemma, it suffices to show that

$$\frac{d}{dt} F(e_1 + tv) \bigg|_{t=0} = \langle \nabla F(e_1), v \rangle \leq C k^{2\alpha + 3/2}. \tag{25}$$

We thus focus on proving (25). Denote $E = sp(E_0 \cup \{v\})$ where $sp$ denotes linear span, and let $\bar{g} = \pi_E(g)$. For $t \in \mathbb{R}$ and $x \in E_0$ set $A_t(x) = x + t\langle x, e_1 \rangle v$. Then for any $|t| \leq 1/2$ we have $A_t(E_0) \subseteq E$ and

$$F(e_1 + tv) = \log \pi_{A_t(E_0)}(g)(A_t x_0) = \log \pi_{A_t(E_0)}\left(\bar{g}\right)(A_t x_0).$$

We have thus reduced our $n$-dimensional problem to a $(k+1)$-dimensional problem; the function $\bar{f} := \pi_E(f)$ is isotropic and log-concave on $E$ (by Prékopa-Leindler), and

$$\bar{g} = \pi_E(g) = \pi_E\left(f * \gamma_n[n^{-\alpha\lambda}]\right) = \bar{f} * \gamma_E[k^{-\alpha}]. \tag{26}$$

We divide the proof of (25) into two cases. Suppose first that $v \in E_0$. In this case, $E = E_0 = A_t(E_0)$ for all $|t| < 1/2$. Therefore, for $|t| < 1/2$,

$$F(e_1 + tv) = \log \bar{g}(A_t x_0) = \log \bar{g}(x_0 + t\langle x_0, e_1 \rangle v).$$

The subspace $E$ is $k$-dimensional. We may apply Lemma 2.7 for $k, \bar{f}, \bar{g}, x_0$ and $v = k^{-\alpha}$ because of (26). By the conclusion of that lemma,

$$\frac{d}{dt} F(e_1 + tv) \bigg|_{t=0} = \langle \nabla \log \bar{g}(x_0), v \rangle \cdot \langle x_0, e_1 \rangle \leq C k^{\alpha + 1/2} |x_0| \leq C' k^{\alpha+1},$$

since $|v| = 1$ and $|x_0| \leq 10\sqrt{k}$. Thus (25) is proven for the case where $v \in E_0$.

From this point on and until the end of the proof, we suppose that $v \notin E_0$ and our aim is to prove (25). Then $\dim(E) = k + 1$, and for $|t| < 1/2$,

$$F(e_1 + tv) = \log \pi_{A_t(E_0)}(\bar{g})(A_t x_0) = \log \int_{-\infty}^{\infty} \bar{g}\left(A_t x_0 + r\theta_t\right) dr,$$

where $\theta_t$ is a unit vector that is orthogonal to the hyperplane $A_t(E_0)$ in $E$. There exist two such unit vectors, and we may select any of them. For concreteness, we choose

$$\theta_t = \frac{t|Proj_{E_0^\perp} v|^2 e_1 - (1 + t\langle v, e_1 \rangle) Proj_{E_0^\perp} v}{|\, t|Proj_{E_0^\perp} v|^2 e_1 - (1 + t\langle v, e_1 \rangle) Proj_{E_0^\perp} v \,|}.$$

An elementary geometric argument (or, alternately, a tedious computation) shows that $d\theta_t/dt|_{t=0}$ is a vector whose length is at most one, since $|v| \leq 1$.

For $s, t \in \mathbb{R}$ with $|s|, |t| \leq 1/2$, we define

$$\bar{F}(s,t) = \log \int_{-\infty}^{\infty} \bar{g}\left(A_t x_0 + r\theta_s\right) dr,$$

which is a smooth function in $s$ and $t$. Then $\bar{F}(t,t) = F(e_1 + tv)$. Recall that $\bar{g}$ is the convolution of an isotropic, log-concave function with $\gamma_E[k^{-\alpha}]$, according to (26). Lemma 2.8 implies that

$$\left. \frac{\partial \bar{F}(s,t)}{\partial s} \right|_{s=t=0} \leq C(k+1)^{3/2} k^{2\alpha} \cdot \left| \left. \partial\theta_s/\partial s \right|_{s=0} \right| \leq C' k^{2\alpha+3/2}.$$

Next, we estimate $\partial \bar{F}/\partial t$. Note that for any $x \in \mathbb{R}^n$,

$$\log \int_{-\infty}^{\infty} \bar{g}(x + r\theta_0) dr = \log \int_{-\infty}^{\infty} \bar{g}(Proj_{E_0}(x) + r\theta_0) dr = \log \pi_{E_0}(\bar{g})(Proj_{E_0}(x)).$$

Therefore, $\bar{F}(0,t) = \log \pi_{E_0}(\bar{g})(x_0 + t\langle x_0, e_1\rangle Proj_{E_0}(v))$ for any $|t| < 1/2$. The function $\pi_{E_0}(\bar{g})$ is the convolution of an isotropic, log-concave function with $\gamma_{E_0}[k^{-\alpha}]$. We appeal to Lemma 2.7 and conclude that

$$\left. \frac{\partial \bar{F}(s,t)}{\partial t} \right|_{t=s=0} \leq C k^{\alpha+1/2} \cdot |\langle x_0, e_1\rangle| \cdot |Proj_{E_0}(v)| \leq C' k^{\alpha+1},$$

since $|v| \leq 1$ and $|x_0| \leq 10\sqrt{k}$. We have thus shown that

$$\left. \frac{dF(e_1+tv)}{dt} \right|_{t=0} = \left. \frac{\partial \bar{F}(s,t)}{\partial s} \right|_{t=s=0} + \left. \frac{\partial \bar{F}(s,t)}{\partial t} \right|_{t=s=0} \leq C k^{2\alpha+3/2}.$$

This completes the proof of (25) in the case where $v \notin E_0$. Thus (25) holds in all cases. The lemma is proven. $\square$

The group $SO(n)$ admits a unique Haar probability measure $\mu_n$, which is invariant under both left and right translations. Our next proposition is a concentration of measure inequality on the orthogonal group from Gromov and Milman [7], see also [13, Section 6 and Appendix V]. This measure-concentration inequality is deduced in [7] from a very general isoperimetric inequality due to Gromov, which requires only lower bounds on the Ricci curvature of the manifold in question. In the specific case of the orthogonal group, a more elementary proof of Proposition 3.2 may be obtained by using two-point symmetrization (see [2]).

**Proposition 3.2** *Let $n \geq 1, \varepsilon > 0, L > 0$ and let $f : SO(n) \to \mathbb{R}$ be such that*

$$f(U) - f(V) \leq Ld(U,V)$$

*for all $U, V \in SO(n)$. Denote $M = \int_{SO(n)} f(U) d\mu_n(U)$. Then,*

$$\mu_n \{U \in SO(n); |f(U) - M| \geq \varepsilon\} \leq C \exp\left(-cn\varepsilon^2/L^2\right),$$

*where $C, c > 0$ are universal constants.*

Milman's principle from [11] states, very roughly, that Lipshitz functions on certain high-dimensional mathematical structures are approximately constant when restricted to typical sub-structures. Behind this principle there usually stands a concentration of measure inequality, such as Proposition 3.2. Our next lemma is yet another manifestation of Milman's principle, whose proof is rather similar to the original argument in [11].

Recall that $G_{n,k}$ stands for the grassmannian of all $k$-dimensional subspaces in $\mathbb{R}^n$. There is a unique rotationally-invariant probability measure on $G_{n,k}$, denoted by $\sigma_{n,k}$. Whenever we say that $E$ is a random $k$-dimensional subspace in $\mathbb{R}^n$, and whenever we say that $U$ is a random rotation in $SO(n)$, we relate to the probability measures $\sigma_{n,k}$ and $\mu_n$, respectively. For a subspace $E \in G_{n,k}$, let $S(E) = \{x \in E; |x| = 1\}$ stand for the unit sphere in $E$, and let $\sigma_E$ denote the unique rotationally-invariant probability measure on $S(E)$.

**Lemma 3.3** *Let $2 \leq k \leq n$ be integers, let $0 \leq \alpha \leq 10^5, -10 \leq \eta \leq 10^5, 0 < u < 1$, and assume that*

$$0 < \lambda \leq \min\{1/(4\alpha + 2\eta + 5.01), u/(4\alpha + 2\eta + 4)\} \tag{27}$$

*is such that $k = n^\lambda$. Suppose that $f : \mathbb{R}^n \to [0, \infty)$ is an isotropic, log-concave function and define $g = f * \gamma_n[n^{-\lambda\alpha}]$.*

*Let $E \in G_{n,k}$ be a random subspace. Then, with probability greater than $1 - Ce^{-cn^{1-u}}$,*

$$|\log \pi_E(g)(x_1) - \log \pi_E(g)(x_2)| \leq \frac{1}{k^\eta}$$

*for all $x_1, x_2 \in E$ with $|x_1| = |x_2| \leq 10\sqrt{k}$. Here, $c, C > 0$ are universal constants.*

*Proof:* Fix a $k$-dimensional subspace $E_0 \subseteq \mathbb{R}^n$. By rotational-invariance, for any $r > 0$ and a unit vector $v \in E_0$,

$$\int_{SO(n)} M_{g,E_0,rv}(U) d\mu_n(U) = \int_{SO(n)} \log \pi_{U(E_0)}(g)(rUv) d\mu_n(U) \tag{28}$$
$$= \int_{G_{n,k}} \int_{S(E)} \log \pi_E(g)(r\theta) d\sigma_E(\theta) d\sigma_{n,k}(E).$$

Consequently, for $r > 0$ we may define

$$M(r) = \int_{SO(n)} M_{g,E_0,rv}(U) d\mu_n(U), \tag{29}$$

where $v \in E_0$ is an arbitrary unit vector, and the definition does not depend on the choice of the unit vector $v \in E_0$.

According to Lemma 3.1, for any $x \in E_0$ with $|x| \leq 10\sqrt{k}$, the function

$$U \mapsto M_{g,E_0,x}(U)$$

is $Ck^{2\alpha+2}$-Lipshitz on $SO(n)$. Therefore, by Proposition 3.2 and by (29), for any $x \in E_0$ with $|x| \leq 10\sqrt{k}$ and for any $\varepsilon > 0$,

$$\mu_n \left\{ U \in SO(n); |M_{g,E_0,x}(U) - M(|x|)| > \varepsilon \right\} \leq C \exp\left( -c\frac{\varepsilon^2 n}{k^{4\alpha+4}} \right). \tag{30}$$

Let $\varepsilon, \delta > 0$ be some small numbers to be specified later. Let $\mathcal{N} \subset 10\sqrt{k}D^n \cap E_0$ be an $\varepsilon$-net for $10\sqrt{k}D^n \cap E_0$ of at most $(C\sqrt{k}/\varepsilon)^k$ elements (see, e.g. [16, Lemma 4.10]). That is, for any $x \in 10\sqrt{k}D^n \cap E_0$ there exists $y \in \mathcal{N}$ with $|x - y| \leq \varepsilon$. Suppose $U \in SO(n)$ is such that

$$|\, M_{g,E_0,x}(U) - M(|x|) \,| \leq \delta \quad \text{for all } x \in \mathcal{N}. \tag{31}$$

Denote $E = U(E_0)$, and let $\mathcal{N}' = \{Ux; x \in \mathcal{N}\}$. Then (31) and the definition (20) imply that

$$|\log \pi_E(g)(x) - M(|x|) \,| \leq \delta \quad \text{for all } x \in \mathcal{N}'. \tag{32}$$

The function $\pi_E(f)$ is isotropic and log-concave, and $\pi_E(g) = \pi_E(f) * \gamma_E[k^{-\alpha}]$. We may thus apply Lemma 2.7 and conclude that the function $x \mapsto \log \pi_E(f)(x)$ is $Ck^{\alpha+1/2}$-Lipshitz on $10\sqrt{k}D^n \cap E$. Therefore, from (28) and (29) we deduce that $r \mapsto M(r)$ is $Ck^{\alpha+1/2}$-Lipshitz on the interval $(0, 10\sqrt{k})$. Since $\mathcal{N}'$ is an $\varepsilon$-net for $10\sqrt{k}D^n \cap E$, we infer from (32) that

$$| \log \pi_E(g)(x) - M(|x|) | \leq \delta + 2\varepsilon C k^{\alpha+1/2} \quad \text{for all } x \in 10\sqrt{k}D^n \cap E.$$

To summarize, if $U \in SO(n)$ is such that (31) holds, then for all $x, x' \in E = U(E_0)$,

$$| \log \pi_E(g)(x) - \log \pi_E(g)(x') | \leq 2\delta + 4\varepsilon C k^{\alpha+1/2} \quad \text{when } |x| = |x'| \leq 10\sqrt{k}. \tag{33}$$

Recall the estimate (30). Recall that $\mathcal{N}$ is a subset of $10\sqrt{k}D^n \cap E_0$ of cardinality at most $(C'\sqrt{k}/\varepsilon)^k$. Therefore, the probability of a random rotation $U \in SO(n)$ to satisfy (31) is greater than

$$1 - (C'\sqrt{k}/\varepsilon)^k \cdot C \exp\left(-c'\delta^2 k^{-4\alpha-4} n\right).$$

Set $\delta = k^{-\eta}/10$ and $\varepsilon = \delta k^{-\alpha-1/2}/C$ where $C$ is the constant from (33). Since

$$k \leq \min\left\{ n^{1/(4\alpha+2\eta+5.01)}, n^{u/(4\alpha+2\eta+4)} \right\}$$

by (27), then $\delta^2 k^{-4\alpha-4} n > c n^{1-u}$ and also

$$1 - (C'\sqrt{k}/\varepsilon)^k \cdot \exp\left(-c'\delta^2 k^{-4\alpha-4} n\right) \geq 1 - \bar{C} e^{-\bar{c} n^{1-u}}.$$

We conclude that if $U$ is a random rotation in $SO(n)$, then (31) holds with probability greater than $1 - \bar{C}\exp(-\bar{c}n^{1-u})$. Whenever $U \in SO(n)$ satisfies (31), the subspace $E = U(E_0)$ necessarily satisfies (33). Hence, with probability greater than $1 - \bar{C}\exp(-\bar{c}n^{1-u})$ of selecting $U \in SO(n)$, for any $x, x' \in U(E_0)$,

$$\left| \log \pi_{U(E_0)}(g)(x) - \log \pi_{U(E_0)}(g)(x') \right| \leq \frac{1}{k^\eta} \quad \text{when } |x| = |x'| \leq 10\sqrt{k}.$$

Note that the subspace $U(E_0)$ is distributed uniformly on $G_{n,k}$. The proof is complete. $\square$

# 4 Almost-radial marginals

Suppose that $f : \mathbb{R}^n \to [0, \infty)$ is an isotropic, log-concave function. Lemma 3.3 states that a typical $n^c$-dimensional marginal of $f$ is approximately radial, after convolving with a gaussian. In this section we will show – mostly by referring to

[9] – that a large portion of the mass of this typical marginal is located in a very thin spherical shell.

Let $f : [0, \infty) \to [0, \infty)$ be a log-concave function with $0 < \int_0^\infty f < \infty$ that is continuous on $[0, \infty)$ and $C^2$-smooth on $(0, \infty)$. As in [9], for $p > 1$, we denote by $t_p(f)$ the unique $t > 0$ for which $f(t) > 0$ and also

$$(\log f)'(t) = \frac{f'(t)}{f(t)} = -\frac{p-1}{t}.$$

The quantity $t_p(f)$ is well-defined according to [9, Lemma 4.3]. The following lemma asserts that most of the mass of $t \mapsto t^{p-1} f(t)$ is located in a small neighborhood of $t_p(f)$. We refer the reader to [9, Lemma 4.5] for the proof.

**Lemma 4.1** *Let $p \geq 2$, and let $f : [0, \infty) \to [0, \infty)$ be a continuous, log-concave function, $C^2$-smooth on $(0, \infty)$, with $0 < \int_0^\infty f < \infty$. Then for all $0 \leq \varepsilon \leq 1$,*

$$\int_{t_p(f)(1-\varepsilon)}^{t_p(f)(1+\varepsilon)} t^{p-1} f(t) dt \geq \left(1 - Ce^{-c\varepsilon^2 p}\right) \int_0^\infty t^{p-1} f(t) dt,$$

*where $C, c > 0$ are universal constants.*

Next, we analyze log-concave densities that are almost-radial in the sense of Lemma 3.3.

**Lemma 4.2** *Let $n \geq C$ and let $f : \mathbb{R}^n \to [0, \infty)$ be a $C^2$-smooth, log-concave probability density. Let $X$ be a random vector in $\mathbb{R}^n$ whose density is $f$, and assume that $\mathbb{E}X = 0$ and $n \leq \mathbb{E}|X|^2 \leq 2n$. Suppose that $\delta > 0$ is such that*

$$|\log f(x_1) - \log f(x_2)| \leq \delta n \tag{34}$$

*for all $x_1, x_2 \in \mathbb{R}^n$ with $|x_1| = |x_2| \leq 10\sqrt{n}$. Denote $r = \sqrt{\mathbb{E}|X|^2}$. Then for all $\varepsilon > 0$ with $C\sqrt{\delta} \leq \varepsilon \leq 1$,*

$$Prob \left\{ \left| \frac{|X|}{r} - 1 \right| > \varepsilon \right\} \leq Ce^{-c\varepsilon^2 n}.$$

*Here, $C, c > 0$ are universal constants.*

*Proof:* We may assume that $\delta < 10^{-3}$; otherwise, there is no $\varepsilon > 0$ with $C\sqrt{\delta} \leq \varepsilon \leq 1$ for a sufficiently large universal constant $C$. For $\theta \in S^{n-1}$ and $r \geq 0$ we denote $f_\theta(r) = f(r\theta)$. Since $\int f = 1$ then $f$ decays exponentially fast at infinity (e.g., [8, Lemma 2.1]). Consequently, $t_n(f_\theta)$ is well-defined for all $\theta \in S^{n-1}$. Let $M > 0$ be such that

$$\int_{\{x\in\mathbb{R}^n; |x|\leq M\}} f(x)dx = \frac{2}{3}\int_{\mathbb{R}^n} f(x)dx = \frac{2}{3}. \tag{35}$$

Since $\int_{\mathbb{R}^n}|x|^2 f(x)dx \geq n$, then Borell's lemma (e.g. [13, Appendix III.3]) implies that $M \geq \sqrt{n}/10$. Additionally, since $\int_{\mathbb{R}^n}|x|^2 f(x)dx \leq 2n$, then necessarily $M \leq 3\sqrt{n}$, by Markov's inequality. We integrate (35) in polar coordinates and obtain

$$\int_{S^{n-1}}\int_0^M f_\theta(t)t^{n-1}dtd\theta = \frac{2}{3}\int_{S^{n-1}}\int_0^\infty f_\theta(t)t^{n-1}dtd\theta. \tag{36}$$

We claim that there exists $\theta_0 \in S^{n-1}$ with

$$\frac{1}{20} \leq \frac{t_n(f_{\theta_0})}{\sqrt{n}} \leq 6. \tag{37}$$

Otherwise, by continuity, either $t_n(\theta) > 6\sqrt{n}$ for all $\theta \in S^{n-1}$ or else $t_n(\theta) < \sqrt{n}/20$ for all $\theta \in S^{n-1}$. In the first case, for all $\theta \in S^{n-1}$ we have $t_n(\theta) > 6\sqrt{n} \geq 2M$, and by Lemma 4.1,

$$\forall \theta \in S^{n-1}, \quad \int_0^M f_\theta(t)t^{n-1}dt \leq \int_0^{t_n(\theta)/2} f_\theta(t)t^{n-1}dt < \frac{2}{3}\int_0^\infty f_\theta(t)t^{n-1}dt,$$

provided that $n > C$, in contradiction to (36). Similarly, in the second case, for all $\theta \in S^{n-1}$, we have $2t_n(\theta) < \sqrt{n}/10 \leq M$ and by Lemma 4.1,

$$\forall \theta \in S^{n-1}, \quad \int_0^M f_\theta(t)t^{n-1}dt \geq \int_0^{2t_n(\theta)} f_\theta(t)t^{n-1}dt > \frac{2}{3}\int_0^\infty f_\theta(t)t^{n-1}dt,$$

in contradiction to (36). We have thus proven that there exists $\theta_0 \in S^{n-1}$ such that (37) holds. Fix such $\theta_0 \in S^{n-1}$. Denote $\varphi_0(t) = \log f_{\theta_0}(t)$ for $t \geq 0$ (where $\log 0 = -\infty$) and $r_0 = t_n(f_{\theta_0})$. Then,

$$\sqrt{n}/20 \leq r_0 \leq 6\sqrt{n}. \tag{38}$$

Fix $\theta \in S^{n-1}$ and denote $r = t_n(f_\theta)$ and $\varphi(t) = \log f_\theta(t)$ for $t \geq 0$. Then $\varphi'(r) = -(n-1)/r$. We will prove that

$$r_0/r \leq 1 + 60\sqrt{\delta}. \tag{39}$$

Indeed, assume the contrary. Then $r_0 > r(1 + 60\sqrt{\delta})$. Since $r_0 \geq \sqrt{n}/20$ and $\sqrt{\delta} < 1/30$, then necessarily

$$r_0 - r > \sqrt{\delta n}. \tag{40}$$

Recall that $\varphi$ and $\varphi_0$ are concave functions, hence their derivatives are non-increasing. Therefore for all $t \in [r, r_0]$,

$$\begin{aligned}\varphi'(t) \leq \varphi'(r) = -\frac{n-1}{r} &< -\frac{n-1}{r_0}\left(1 + 60\sqrt{\delta}\right) \\ \leq \quad -\frac{n-1}{r_0} - 6\sqrt{\delta n} &= \varphi_0'(r_0) - 6\sqrt{\delta n} \leq \varphi_0'(t) - 6\sqrt{\delta n},\end{aligned} \tag{41}$$

where we used the fact that $(n-1)/r_0 \geq (n-1)/(6\sqrt{n}) \geq \sqrt{n}/10$ by (38). Note that $r < r_0 \leq 6\sqrt{n}$, and hence (34) implies that $|\varphi_0(t) - \varphi(t)| \leq \delta n$ for all $t \in [r, r_0]$. However, by (41) and (40),

$$[\varphi_0(r_0) - \varphi(r_0)] - [\varphi_0(r) - \varphi(r)] = \int_r^{r_0} \varphi_0'(t) - \varphi'(t)dt > 6\sqrt{\delta n}(r_0 - r) > 6\delta n,$$

in contradiction to (34). Thus (39) is proven. Next we will demonstrate that

$$r/r_0 \leq 1 + 200\sqrt{\delta}. \tag{42}$$

The proof of (42) is very similar to the proof of (39). Assume on the contrary that (42) does not hold. Then $r > r_0 + 10\sqrt{\delta n}$. Denote $\bar{r} = r_0 + 10\sqrt{\delta n}$. Since $\varphi'$ is non-increasing, then $\varphi'(\bar{r}) \geq \varphi'(r) = -(n-1)/r > -(n-1)/\bar{r}$. Hence, for $t \in [r_0, \bar{r}]$,

$$\varphi_0'(t) \leq \varphi_0'(r_0) = -\frac{n-1}{r_0} < -\frac{n-1}{\bar{r}} - \frac{\sqrt{\delta n}}{5} < \varphi'(\bar{r}) - \frac{\sqrt{\delta n}}{5} \leq \varphi'(t) - \frac{\sqrt{\delta n}}{5},$$

according to (38). As before, this entails

$$[\varphi(\bar{r}) - \varphi_0(\bar{r})] - [\varphi(r_0) - \varphi_0(r_0)] > (\sqrt{\delta n}/5) \cdot (\bar{r} - r_0) = 2\delta n,$$

in contradiction to (34). To summarize, in (39) and (42) we proved that for all $\theta \in S^{n-1}$,

$$\left(1 - C\sqrt{\delta}\right) r_0 \leq t_n(f_\theta) \leq \left(1 + C\sqrt{\delta}\right) r_0. \tag{43}$$

We may assume that $10C\sqrt{\delta} < 1$, where $C$ is the constant from (43). Let $\varepsilon > 0$ satisfy $10C\sqrt{\delta} < \varepsilon < 1$. According to (43), for any $\theta \in S^{n-1}$,

$$r_0(1-\varepsilon) \leq t_n(f_\theta)(1-\varepsilon/2) \quad \text{and} \quad r_0(1+\varepsilon) \geq t_n(f_\theta)(1+\varepsilon/2).$$

Integration in polar coordinates yields

$$\begin{aligned} Prob\left\{\left|\frac{|X|}{r_0}-1\right| \leq \varepsilon\right\} &= \int_{S^{n-1}}\int_{r_0(1-\varepsilon)}^{r_0(1+\varepsilon)} f(t\theta)t^{n-1}dtd\theta \\ \geq & \int_{S^{n-1}}\int_{t_n(f_\theta)(1-\varepsilon/2)}^{t_n(f_\theta)(1+\varepsilon/2)} f_\theta(t)t^{n-1}dtd\theta \\ \geq & \left(1-C'e^{-c'\varepsilon^2 n}\right)\int_{S^{n-1}}\int_0^\infty f_\theta(t)t^{n-1}dtd\theta = 1-C'e^{-c'\varepsilon^2 n}, \end{aligned}$$

where we used Lemma 4.1. Therefore, when $\tilde{C}\sqrt{\delta} < \varepsilon < 1$,

$$Prob\left\{\left|\frac{|X|^2}{r_0^2}-1\right| \geq \varepsilon\right\} \leq Ce^{-c\varepsilon^2 n}. \tag{44}$$

As in the proof of Lemma 4.6 in [9], we use (44) and Borell's lemma to obtain

$$\begin{aligned} \left|\frac{\mathbb{E}|X|^2}{r_0^2}-1\right| &\leq \mathbb{E}\left|\frac{|X|^2}{r_0^2}-1\right| = \int_0^\infty Prob\left\{\left|\frac{|X|^2}{r_0^2}-1\right| \geq t\right\}dt \qquad (45) \\ \leq & \tilde{C}\sqrt{\delta} + \int_{\tilde{C}\sqrt{\delta}}^1 Ce^{-ct^2 n}dt + C'\int_1^\infty \min\{e^{-cn}, e^{-c'\sqrt{t}}\}dt \leq \bar{C}\sqrt{\delta+\frac{1}{n}}. \end{aligned}$$

Recall that we denote $r = \sqrt{\mathbb{E}|X|^2}$. From (44) and (45) we conclude that

$$Prob\left\{\left|\frac{|X|^2}{r^2}-1\right| \geq \varepsilon\right\} \leq Ce^{-c\varepsilon^2 n}$$

for all $\varepsilon > 0$ with $\tilde{C}\sqrt{\delta} < \varepsilon < 1$. This completes the proof. □

**Lemma 4.3** *Let $n \geq 2$ be an integer, let*

$$\max\left\{\frac{1}{10}, \frac{1}{\log n}\right\} \leq \lambda \leq \frac{1}{3.01} - \frac{C'}{\log n}, \tag{46}$$

*and assume that $X$ is a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density. Let $Y$ be a standard gaussian random vector in $\mathbb{R}^n$, independent of $X$. Then*

$$Prob\left\{\left|\frac{|X+Y|}{\sqrt{2n}}-1\right| > Cn^{(3.01\lambda-1)/4}\right\} \leq C\exp\left(-cn^{(5.01\lambda-1)/2}\right).$$

*Here, $c, C, C' > 0$ are universal constants.*

*Proof:* We may clearly assume that $n \geq \tilde{C}$. Denote by $f$ the density of the random vector $X$. Then $f : \mathbb{R}^n \to [0, \infty)$ is isotropic and log-concave. Define $g = f * \gamma_n[1]$. Let $k$ be the maximal integer such that $k \leq n^\lambda$. Then $k \geq \max\{2, n^\lambda/2\}$ because of (46). Define

$$\eta = \frac{1}{2\lambda} - 2.505.$$

Then $-1 \leq \eta \leq 10$. We apply Lemma 3.3 for $\alpha = 0$, for $\eta$ as was just defined and for $u = 1 - 1.01\lambda$. Note that

$$\lambda \leq \min\{1/(2\eta + 5.01), u/(2\eta + 4)\}.$$

Thus the appeal to Lemma 3.3 is legitimate. By the conclusion of Lemma 3.3, there exists $\mathcal{E} \subseteq G_{n,k}$ with

$$\sigma_{n,k}(\mathcal{E}) > 1 - Ce^{-cn^{1-u}} \geq 1 - Ce^{-cn^\lambda}$$

and with the following property: For any $E \in \mathcal{E}$ and $x_1, x_2 \in E$,

$$|\log \pi_E(g)(x_1) - \log \pi_E(g)(x_2)| \leq k^{-\eta} \quad \text{when } |x_1| = |x_2| \leq 10\sqrt{k}.$$

Equivalently, denote $\delta = k^{-\eta-1}$. For an appropriate choice of a large universal constant $C'$ in (46), we have that $C\sqrt{\delta} \leq 1$ where $C$ is the constant from Lemma 4.2. Then for any $E \in \mathcal{E}$ and $x_1, x_2 \in E$,

$$|\log \pi_E(g)(x_1) - \log \pi_E(g)(x_2)| \leq \delta k \quad \text{when } |x_1| = |x_2| \leq 10\sqrt{k}. \tag{47}$$

Fix a subspace $E \in \mathcal{E}$. The function $\pi_E(g)$ is $C^2$-smooth (it is a convolution with a gaussian), log-concave (by Prékopa-Leindler), and it satisfies (47). The random vector $Proj_E(X + Y)$ is distributed according to the density $\pi_E(g) = \pi_E(f) * \gamma_E[1]$ in the subspace $E$. Furthermore,

$$\mathbb{E} Proj_E(X + Y) = 0 \qquad \text{and} \qquad \mathbb{E}|Proj_E(X + Y)|^2 = 2k.$$

We have thus verified the assumptions of Lemma 4.2 for the function $\pi_E(g)$, the random vector $Proj_E(X + Y)$ and the number $\delta$. By the conclusion of Lemma 4.2 (the case $\varepsilon = C\sqrt{\delta} \leq 1$),

$$Prob\left\{ \left| \frac{|Proj_E(X + Y)|}{\sqrt{2k}} - 1 \right| > C\sqrt{\delta} \right\} \leq C' e^{-c'\delta k}. \tag{48}$$

The rest of the argument is similar to the proof of Theorem 1.4 in [9]; see the derivation involving formulas (49), (50) in [9, Section 4]. We have proven that (48) holds for all $E \in \mathcal{E}$. Recall that $\sigma_{n,k}(\mathcal{E}) > 1 - C\exp(-cn^{\lambda})$. Let $E \in G_{n,k}$ be a random subspace, independent of $X$ and $Y$. Then

$$Prob\left\{\left|\frac{|Proj_E(X+Y)|}{\sqrt{2k}} - 1\right| > C\sqrt{\delta}\right\} \leq C'e^{-c'\delta k} + Ce^{-cn^{\lambda}}. \qquad (49)$$

However, according to the Johnson-Lindenstrauss dimension reduction lemma (see, e.g. [9, Lemma 4.8]),

$$Prob\left\{\left|\frac{|Proj_E(X+Y)|}{|X+Y|} - \sqrt{\frac{k}{n}}\right| \geq \sqrt{\delta}\cdot\sqrt{\frac{k}{n}}\right\} \geq 1 - Ce^{-c\delta k}. \qquad (50)$$

From (49) and (50) we obtain

$$Prob\left\{\left|\frac{|X+Y|}{\sqrt{2n}} - 1\right| > C\sqrt{\delta}\right\} < Ce^{-c\delta k} + Ce^{-cn^{\lambda}} < C'e^{-c\delta k}, \qquad (51)$$

since $\delta < 1$ and $k \leq n^{\lambda}$. Recall that $n^{\lambda}/2 \leq k \leq n^{\lambda}$ and that $\delta = k^{(3.01-1/\lambda)/2}$. The lemma follows from (51). $\square$

**Theorem 4.4** *Let $n \geq 1$ be an integer and let $X$ be a random vector with an isotropic, log-concave density in $\mathbb{R}^n$. Then for all $0 \leq \varepsilon \leq 1$,*

$$Prob\left\{\left|\frac{|X|}{\sqrt{n}} - 1\right| \geq \varepsilon\right\} \leq C\exp\left(-c\varepsilon^{3.33}n^{0.33}\right), \qquad (52)$$

*where $c, C > 0$ are universal constants.*

*Proof:* We may assume that $n \geq C$. Let $Y$ be a standard gaussian random vector in $\mathbb{R}^n$, independent of $X$. Let $n^{-0.1} \leq \varepsilon \leq \bar{c}$, for a sufficiently small universal constant $\bar{c} > 0$. Then the real number $\lambda$ defined by the equation $\varepsilon = n^{(3.01\lambda-1)/4}$ satisfies (46). Consequently, by Lemma 4.3,

$$Prob\left\{\left|\frac{|X+Y|}{\sqrt{2n}} - 1\right| > C\varepsilon\right\} \leq C\exp\left(-cn^{(5.01\lambda-1)/2}\right) \leq C'\exp\left(-c'\varepsilon^{3.33}n^{0.33}\right).$$

By adjusting the constants, we conclude that for all $0 < \varepsilon < 1$,

$$Prob\left\{\left|\frac{|X+Y|}{\sqrt{2n}} - 1\right| > \varepsilon\right\} \leq \tilde{C}\exp\left(-\tilde{c}\varepsilon^{3.33}n^{0.33}\right). \qquad (53)$$

The random vector $X$ has an isotropic, log-concave density. The standard gaussian random vector $Y$ is independent of $X$. The simple argument that leads from (53) to (52) was described in great detail in the proof of Proposition 4.1 in [9]; see the derivation involving formulas (40),...,(45) in [9, Section 4]. We will not repeat that argument here. This completes the proof. □

*Proof of Theorem 1.3:* Substitute $\varepsilon = n^{-1/14}$ in Theorem 4.4. □

*Remark:* The exponents $3.33$ and $0.33$ in Theorem 4.4 are not optimal. They may be replaced by constants arbitrarily close to $10/3$ and $1/3$, respectively, at the expense of increasing $C$ and decreasing $c$ in Theorem 4.4, as is easily seen from our proof. We conjecture that slightly better exponents may be obtained by optimizing our argument; for example, the transition from (23) to (24) seems inefficient, and it also makes sense to try and play with the function $M(U) = \langle \nabla \log \pi_{U(E_0)} g(Ux_0), Ux_0 \rangle$ in place of the definition (20).

# 5 Tying up loose ends

Next we discuss the proof of Theorem 1.1. As in the previous section, we rely heavily on results from [9]. For two random vectors $X$ and $Y$ attaining values in some measurable space $\Omega$, we write

$$d_{TV}(X, Y) = 2 \sup_{A \subseteq \Omega} |\, Prob\{X \in A\} \,-\, Prob\{Y \in A\}\,|,$$

for the total-variation distance between $X$ and $Y$, where the supremum runs over all measurable subsets $A \subseteq \Omega$. The following lemma is no more than an adaptation of [9, Proposition 5.7].

**Lemma 5.1** *Let $2 \leq \ell \leq n$ be integers and assume that $X$ is a random vector in $\mathbb{R}^n$ with an isotropic, log-concave density. Suppose that*

$$\ell \leq n^{\kappa}.$$

*Then, there exists a subset $\mathcal{E} \subseteq G_{n,\ell}$ with $\sigma_{n,\ell}(\mathcal{E}) > 1 - Ce^{-\sqrt{n}}$ such that for all $E \in \mathcal{E}$ there exists a random vector $Y$ in $E$ for which the following hold:*

*(i)* $d_{TV}(Proj_E(X), Y) \leq C/\ell^{10}$.

*(ii)* $Y$ *has a spherically-symmetric distribution. That is,*

$$Prob\{Y \in A\} = Prob\{Y \in U(A)\}$$

*for any measurable subset* $A \subseteq E$ *and any* $U \in SO(n)$ *with* $U(E) = E$*.*

*(iii)* $Prob\left\{\left| |Y| - \sqrt{\ell} \right| \geq \varepsilon\sqrt{\ell}\right\} \leq Ce^{-c\varepsilon^2\ell}$ *for all* $0 \leq \varepsilon \leq 1$*.*

*Here,* $c, C, \kappa > 0$ *are universal constants.*

*Proof:* We may clearly assume that $n \geq \tilde{C}$. We begin by verifying the requirements of Lemma 3.3. Let $\alpha = 3000, \eta = 10$ and $u = 1/3$. Our universal constant $\kappa$ will be defined by

$$\kappa = \min\left\{1/(4\alpha + 2\eta + 5.01), u/(4\alpha + 2\eta + 4)\right\}.$$

Recall that $\ell \leq n^\kappa$. Let $f : \mathbb{R}^n \to [0, \infty)$ stand for the density of $X$, and denote $g = f * \gamma_n[\ell^{-\alpha}]$. The requirements of Lemma 3.3 thus hold true. By the conclusion of that lemma, there exists $\mathcal{E} \subseteq G_{n,\ell}$ with $\sigma_{n,\ell}(\mathcal{E}) > 1 - C\exp(-\sqrt{n})$ such that for $E \in \mathcal{E}$,

$$\pi_E(g)(r\theta_1) \leq \left(1 + \frac{2}{\ell^{10}}\right) \cdot \pi_E(g)(r\theta_2) \tag{54}$$

for all $\theta_1, \theta_2 \in S(E), 0 \leq r \leq 10\sqrt{\ell}$. Fix $E \in \mathcal{E}$. We need to construct a random vector $Y$ in $E$ that satisfies (i), (ii) and (iii). Consider first the random vector $X'$ in the subspace $E$ whose density is $\pi_E(g) = \pi_E(f) * \gamma_E[\ell^{-\alpha}]$. The function $\pi_E(f)$ is isotropic, log-concave and it is the density of $Proj_E(X)$. According to [9, Lemma 5.1],

$$d_{TV}\left(Proj_E(X), X'\right) \leq \frac{C}{\ell^{10}}. \tag{55}$$

The density $\pi_E(g)$ is $C^2$-smooth (it is a convolution with a gaussian) and log-concave (by Prékopa-Leindler). Additionally,

$$\ell \leq \mathbb{E}|X'|^2 = \ell + \ell^{1-\alpha} \leq 2\ell. \tag{56}$$

We may thus apply Lemma 4.2, based on (54), for $\delta = 2/\ell^{10}$. According to the conclusion of Lemma 4.2,

$$Prob\left\{\left|\frac{|X'|}{\sqrt{\ell + \ell^{1-\alpha}}} - 1\right| > \varepsilon\right\} \leq Ce^{-c\varepsilon^2\ell} \quad \text{for all } 0 \leq \varepsilon \leq 1. \tag{57}$$

Since $\alpha > 1/2$ then $\sqrt{\ell + \ell^{1-\alpha}}$ is sufficiently close to $\sqrt{\ell}$, and from (57) we obtain

$$Prob\left\{\left|\frac{|X'|}{\sqrt{\ell}} - 1\right| > \varepsilon\right\} \le C' e^{-c'\varepsilon^2 \ell} \quad \text{for all } 0 \le \varepsilon \le 1. \tag{58}$$

Define $Y$ to be the random vector in the subspace $E$ whose density is

$$\tilde{g}(x) = \int_{S(E)} \pi_E(g)(|x|\theta) d\sigma_E(\theta). \tag{59}$$

Then $\tilde{g}$ is the spherical average of $g$, hence $Y$ is spherically-symmetric in $E$ and (ii) holds. Additionally, since $|X'|$ and $|Y|$ have the same distribution, then (iii) holds in view of (58). All that remains is to prove (i). According to (54) and (59), for any $x \in E$ with $|x| \le 10\sqrt{\ell}$,

$$|\tilde{g}(x) - \pi_E(g)(x)| \le \frac{C}{\ell^{10}} \cdot \tilde{g}(x). \tag{60}$$

From (58), (iii) and (60),

$$\begin{aligned} d_{TV}(Y, X') &= \int_{\mathbb{R}^n} |\tilde{g}(x) - \pi_E(g)(x)| dx \\ &\le 4Prob\{|X'| \ge 2\sqrt{\ell}\} + \int_{|x| \le 2\sqrt{\ell}} |\pi_E(g)(x) - \tilde{g}(x)| dx \\ &\le Ce^{-c\ell} + \frac{C'}{\ell^{10}} \int_{|x| \le 2\sqrt{\ell}} \tilde{g}(x) dx \le \frac{\tilde{C}}{\ell^{10}}. \end{aligned} \tag{61}$$

Now (i) follows from (61) and (55). This completes the proof. □

*Proof of Theorem 1.1:* We will actually prove the theorem under the weaker assumption that the density of $X$ is log-concave. The deduction of Theorem 1.1 from Lemma 5.1 is very similar to the argument in the proof of Theorem 5.9 in [9]. We supply a few details. We may assume that $n \ge \tilde{C}$, since otherwise there is no $\ell \ge 1$ with $\ell \le cn^\kappa$. It is sufficient to prove the theorem for $\ell = \lfloor n^{\kappa/2} \rfloor + 2$ where $\kappa$ is the constant from Lemma 5.1. Let $\mathcal{E} \subseteq G_{n,\ell^2}$ be the subset from the conclusion of Lemma 5.1 for $\ell^2$. Then

$$\sigma_{n,\ell^2}(\mathcal{E}) \ge 1 - C\exp(-\sqrt{n}). \tag{62}$$

Fix $E \in \mathcal{E}$, and let $Y$ be the random vector in $E$ whose existence is guaranteed by Lemma 5.1. Let $F \subset E$ be any $\ell$-dimensional subspace. We may apply [9, Lemma 5.8], based on properties (ii) and (iii) of $Y$ from Lemma 5.1. We conclude that

$$d_{TV}\left(Proj_F(Y), Z_F\right) \leq C\sqrt{\frac{\ell}{\ell^2}} \leq \frac{C'}{\sqrt{\ell}}, \tag{63}$$

where $Z_F$ is a standard gaussian random vector in the subspace $F$. Recall that $d_{TV}(Y, Proj_E(X)) < C/\ell^{10}$, by property (i) from Lemma 5.1. Therefore, from (63),

$$d_{TV}\left(Proj_F(X), Z_F\right) \leq \frac{C'}{\sqrt{\ell}} + \frac{C}{\ell^{10}} \leq \frac{\bar{C}}{n^{\kappa/4}}. \tag{64}$$

Recall that $E \in \mathcal{E}$ and $F \subset E$ were arbitrary. Denote

$$\mathcal{F} = \left\{F \in G_{n,\ell}; \exists E \in \mathcal{E},\ F \subset E\right\}.$$

We have proved that (64) holds for all $F \in \mathcal{F}$. From (62) we deduce that $\sigma_{n,\ell}(\mathcal{F}) \geq \sigma_{n,\ell^2}(\mathcal{E}) \geq 1 - \exp(-c\sqrt{n})$. The theorem is thus proven. $\square$

The remaining statements that were announced in Section 1 above follow, in a rather straightforward manner, from the theorems obtained so far in this note and from results found in the literature. The argument that leads from Theorem 1.1 to Corollary 1.2 is elementary and well-known (in the context of Dvoretzky's theorem, we learned it from G. Schechtman). It is based on the observation that any $(2n+1)$-dimensional ellipsoid $\mathcal{E} \subset \mathbb{R}^{2n+1}$ has a projection onto some $n$-dimensional subspace $F \subset \mathbb{R}^{2n+1}$ such that $Proj_F(\mathcal{E})$ is a multiple of the standard Euclidean ball in the subspace $F$. We omit the standard linear algebra details.

Regarding the proof of Theorem 1.4: We may assume that $n \geq C$. Note that the desired conclusion (i) is equivalent to the case $\ell = 1$ in Theorem 1.1, since the total-variation distance between two random variables equals the $L^1$ distance between their densities. In order to prove (ii), we use Sodin's results [17]. We may apply [17, Theorem 2] with $\alpha = 0.33$ and $\beta = 3.33$, in view of Theorem 4.4. According to the conclusion of [17, Theorem 2], for any $t \in \mathbb{R}$ with $|t| < cn^{1/24}$,

$$\left| \int_{S^{n-1}} [f_\theta(t)/\gamma(t)] d\sigma_{n-1}(\theta) - 1 \right| \leq \frac{C}{n^{1/24}}. \tag{65}$$

Next, we would like to use a ready-made concentration of measure phenomenon argument, such as [17, Theorem 5]. However, the results in the relevant section in [17] are proven under the additional assumption that $X$ is a symmetric random vector (i.e., $X$ and $-X$ have the same distribution). Sodin's argument formally relies on the fact that in the symmetric case, $f_\theta(t)$ is non-increasing for $t > 0$, as an even, log-concave density. In the case where $X$ is symmetric, we may directly apply [17, Theorem 5] for $\varepsilon = Cn^{-1/24}$ and $T = cn^{1/24}$, because of (65). We deduce that there exists $\Theta \subseteq S^{n-1}$ with $\sigma_{n-1}(\Theta) \geq 1 - C\exp(-\sqrt{n})$ such that for any $\theta \in \Theta$,

$$|f_\theta(t)/\gamma(t) - 1| \leq \frac{C'}{n^{1/24}} \quad \text{when } |t| < cn^{1/24}.$$

This completes the proof of Theorem 1.4 for the case where the density of $X$ is an even function.

We claim that Theorem 1.4(ii) is true as stated, without the additional assumption that the random vector $X$ is symmetric; it is possible to modify Sodin's argument (specifically, Proposition 13 in [17]) for the general, log-concave case. We will not carry out the details here, and they are left to the reader as a (rather interesting) exercise. An alternative route to establish Theorem 1.4(ii) from (65), in the general, non-even case, may be very roughly summarized as follows: Observe that after a convolution with a small gaussian, estimates such as (54) directly lead us to the desired result. Then, show that the convolution of a log-concave function with a small gaussian has only a minor effect in the moderate-deviation scale.

*Remark:* It is also possible to improve the quantitative bound for $\varepsilon_n$ from [9, Theorem 1.2]. The most straightforward adaptation of the proof of Theorem 1.2 in [9], using the new Theorem 4.4, leads to the estimate $\varepsilon_n \leq C/n^\kappa$ for some universal constants $C, \kappa > 0$.

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