Uniform almost sub-gaussian estimates for linear functionals on convex sets

B. Klartag*

School of Mathematics
Institute for Advanced Study
Einstein Drive
Princeton, NJ 08540, USA

Abstract

A well-known consequence of the Brunn-Minkowski inequality, is that the distribution of a linear functional on a convex set has a sub-exponential tail. That is, for any dimension n, a convex set $K \subset \mathbb{R}^n$ of volume one, and a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$, we have

$$Vol_n \left(\left\{ x \in K ; |\varphi(x)| > t \|\varphi\|_{L_1(K)} \right\} \right) \leq e^{-ct} \quad \text{for all } t > 1,$$

where $\|\varphi\|_{L_1(K)} = \int_K |\varphi(x)| dx$ and $c > 0$ is a universal constant. In this note we prove that for any dimension n and a convex set $K \subset \mathbb{R}^n$ of volume one, there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that

$$Vol_n \left(\left\{ x \in K ; |\varphi(x)| > t \|\varphi\|_{L_1(K)} \right\} \right) \leq e^{-c \frac{t^2}{\log^5 (t+1)}} \quad \text{for all } t > 1,$$

where $c > 0$ is a universal constant.

1 Introduction

For two subsets $A, B \subset \mathbb{R}^n$ and $\lambda > 0$ we denote $\lambda A = \{\lambda x ; x \in A\}$ and $A + B = \{x + y ; x \in A, y \in B\}$. The latter operation is the well-known Minkowski sum of sets. The classical Brunn-Minkowski inequality (see e.g., [35]) states that for any dimension n, for any Borel sets $A, B \subset \mathbb{R}^n$ and for any $0 < \lambda < 1$,

$$Vol_n (\lambda A + (1 - \lambda) B) \geq Vol_n(A)^{\lambda} Vol_n(B)^{1-\lambda}, \tag{1}$$

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where Vol_n is the standard Lebesgue measure on $\mathbb{R}^n$. A convex body is a compact, convex set with a non-empty interior. Suppose that $K \subset \mathbb{R}^n$ is a convex body of volume one, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a non-zero linear functional. Let $M > 0$ be such that the set $T = \{x \in \mathbb{R}^n; |\varphi(x)| \leq M\}$ satisfies $Vol_n(K \cap T) = \frac{2}{3}$. Clearly $M \leq 3\|\varphi\|_{L_1(K)} = 3\int_K |\varphi(x)|dx$. An elegant argument by Borell [6] states that the Brunn-Minkowski inequality (1), together with the easily-verified inclusion,

$$\frac{2}{t+1}(K \setminus tT) + \frac{t-1}{t+1}(K \cap T) \subset K \setminus T \quad \text{for all } t \geq 1,$$

imply that

$$Vol_n\left(\{x \in K; |\varphi(x)| > tM\}\right) \leq \frac{1}{3}\frac{1}{2^{\frac{t-1}{2}}} \quad \text{for all } t \geq 1. \tag{2}$$

Inequality (2) is a dimension-free sub-exponential estimate for the tail of the distribution of an arbitrary linear functional on an arbitrary convex body. Recall the definition of the Orlicz norm; Let $\psi : [0,\infty) \to [0,\infty)$ be a convex, non-decreasing function that vanishes at the origin, and let (Ω, μ) be a probability space. For a measurable function $g : \Omega \to \mathbb{R}$, we denote

$$\|g\|_{L_\psi(\mu)} = \inf\left\{\lambda > 0; \int_\Omega \psi\left(\frac{|g(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

For a convex set $K \subset \mathbb{R}^n$ of volume one, we write $\|\cdot\|_{L_\psi(K)}$ to denote $\|\cdot\|_{L_\psi(\lambda_K)}$ where λ_K is the restriction of the Lebesgue measure on $\mathbb{R}^n$ to K. We will mainly consider the Young functions $\psi_1(t) = e^t - 1, \psi_2(t) = e^{t^2} - 1$ and their variants. Inequality (2) now translates as follows: For a convex body $K \subset \mathbb{R}^n$ of volume one and a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\|\varphi\|_{L_{\psi_1}(K)} \leq C\|\varphi\|_{L_1(K)}, \tag{3}$$

where $C > 0$ is some universal constant. We would like to emphasize that the constant C in (3) is a universal constant, independent of φ, K and the dimension n.

Let us consider an example. Suppose $K \subset \mathbb{R}^n$ is a simplex of volume one, with the origin lying on one of its facets. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a non-zero linear functional, that is non-negative on K, and vanishes on the facet of K that contains the origin. We assume that φ is normalized so that $\|\varphi\|_{L_1(K)} = 1$. Suppose that X is a random vector that is distributed uniformly over K. Then the random variable $\varphi(X)$ has a density that is proportional to the function

$$t \mapsto \begin{cases} \left(1 - \frac{t}{n+1}\right)^{n-1} & 0 \leq t \leq n+1 \\ 0 & \text{otherwise} \end{cases}$$

Since $(1 - t/(n+1))^{n-1} \approx e^{-t}$ for large n, then the distribution of the random variable $\varphi(X)$ is very close to being an exact exponential, when the dimension n is large. Note that there is nothing special about the simplex; Any cone over a convex base exhibits such approximately-exponential behavior (see Figure 1).

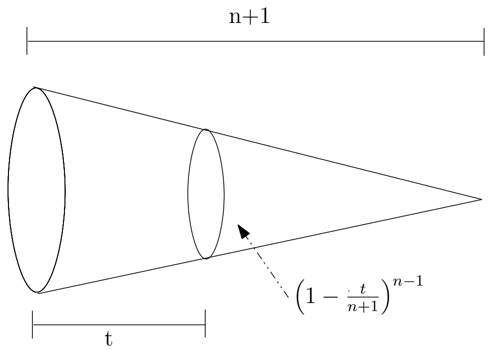

Figure 1

We conclude that (3) is sharp, in the following strong sense: Suppose that $\psi : [0,\infty) \to [0,\infty)$ is a convex, non-decreasing function that vanishes at the origin. Suppose also that $C > 0$ has the property that for any dimension n, a convex body $K \subset \mathbb{R}^n$ of volume one, and a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ we have $\|\varphi\|_{L_\psi(K)} \le C\|\varphi\|_{L_1(K)}$. Then, necessarily there exist $C', C'' > 0$ such that $\psi(t) \le C'\psi_1(C''t)$ for all $t \ge 0$.

Estimate (3) is the best possible general estimate. However, for some specific classes of convex bodies, there exist much better estimates than (3). Consider, for instance, an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ of volume one, centered at the origin. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a non-zero linear functional. Note that the random variable $\varphi(X)$, where X is distributed uniformly over $\mathcal{E}$, has a density that is proportional to

$$t \mapsto \left(1 - \frac{at^2}{n}\right)_+^{\frac{n-1}{2}} \approx \exp\left(-\frac{at^2}{2}\right),$$

for some $a > 0$. This distribution is very close to the gaussian distribution. We thus conclude that for any linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\|\varphi\|_{L_{\psi_2}(\mathcal{E})} \le C\|\varphi\|_{L_1(\mathcal{E})} \tag{4}$$

where $C > 0$ is a universal constant. All non-zero linear functionals are distributed approximately according to the gaussian law, and hence for each non-zero linear functional φ, the estimate (4) is essentially sharp. The example of the ellipsoid shows that for some convex bodies, a non-zero linear functional cannot satisfy a dimension-free estimate stronger than ψ_2. The purpose of this note is to prove the following theorem.

Theorem 1.1 *Let $n \ge 1$ be an integer, and let $K \subset \mathbb{R}^n$ be a convex body of volume one. Then there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $t \ge 1$,*

$$Vol_n\left(\{x \in K; |\varphi(x)| > t\|\varphi\|_{L_1(K)}\}\right) < e^{-c\frac{t^2}{\log^5(t+1)}} \tag{5}$$

where $c > 0$ is a universal constant. Equivalently, denote $\psi_{2^-}(t) = \exp\left(\frac{t^2}{\log^5(t+5)}\right) - 1$. Then the linear functional φ satisfies

$$\|\varphi\|_{L_{\psi_{2^-}}(K)} \le C\|\varphi\|_{L_1(K)}$$

where $C > 0$ is a universal constant.

Therefore, a convex body cannot display "cone-type" behavior in all directions; There always exists a direction in which better, almost sub-gaussian behavior is observed. Theorem 1.1 is stated only for convex bodies of volume one, with the generalization to general convex bodies being straightforward. An estimate that is slightly weaker than (5) actually holds for "most" linear functionals on $\mathbb{R}^n$ in some sense (see precise formulation in Corollary 5.3 below).

Aside from the logarithmic factors, Theorem 1.1 is sharp, as shown by the above discussion. Up to the logarithmic factors, Theorem 1.1 provides an affirmative answer to a question attributed to V. Milman in [2] (see also [28, 29]):

Question 1.2 (V. Milman) *Does there exist a constant $C > 0$ for which the following holds: For any integer $n \geq 1$ and a convex body $K \subset \mathbb{R}^n$ of volume one, there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\|\varphi\|_{L_{\psi_2}(K)} \leq C\|\varphi\|_{L_1(K)}$?*

Let us briefly review the literature related to Question 1.2. In high-dimensional convex geometry, the significance of dimension-free ψ_2-estimates is already apparent in Bourgain's bound for the isotropic constant of general convex bodies [8, 9]. In particular, in [9] the isotropic constant of a convex body $K \subset \mathbb{R}^n$ is shown to be bounded by a function solely of

$$\sup_{\varphi \neq 0} \frac{\|\varphi\|_{L_{\psi_2}(K)}}{\|\varphi\|_{L_1(K)}},$$

where the supremum runs over all non-zero linear functionals $\varphi : \mathbb{R}^n \to \mathbb{R}$.

An affirmative answer to Question 1.2 was obtained for some particular classes of convex bodies. Bobkov and Nazarov [4, 5] have provided a positive answer to Question 1.2 in the case where K is assumed to be an unconditional convex body. Paouris [28, 29] has given an affirmative answer to Question 1.2 in the case where K is a zonoid, or when K is a convex body with a "small diameter". See [2] for more information on the case of K being the unit ball of l_p^n, for $2 \leq p \leq \infty$.

In the context of an arbitrary convex body, no general, dimension-free estimates were obtained, beyond the ψ_1-estimate (3). Note that it is easy to establish some dimension-dependent estimates for the tail distribution of linear functionals on convex bodies. For instance, it follows from the Brunn-Minkowski inequality that $\|\varphi\|_{L_\infty(K)} \leq Cn\|\varphi\|_{L_1(K)}$ for any n-dimensional convex body of volume one and a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$, where $C > 0$ is a universal constant.

Remarkable progress pertaining to the understanding of mass distribution in high dimensional convex bodies was recently obtained by Paouris [30]. Among the consequences of Paouris theorem [30], is a version of Theorem 1.1, without the logarithmic factors, but with t restricted to the range $[1, n^{1/4}]$. This version follows immediately by combining Paouris theorem [30] with the methods of [29]. Some of our techniques here are related to and influenced by the approach taken by Paouris.

Theorem 1.1 allows generalizations slightly beyond the context of uniform measures on convex sets. Recall that a function $f : \mathbb{R}^n \to [0,\infty)$ is a logarithmically concave function, log-concave in short, if

$$f\left(\lambda x + (1-\lambda)y\right) \geq f(x)^{\lambda} f(y)^{1-\lambda}, \quad \text{for all } x, y \in \mathbb{R}^n,\ 0 < \lambda < 1.$$

A basic sub-class of log-concave functions consists of the s-concave functions for $s > 0$. A function $f : \mathbb{R}^n \to [0,\infty)$ is s-concave, for some $s > 0$, if

$$f^{\frac{1}{s}}\left(\lambda x + (1-\lambda)y\right) \geq \lambda f^{\frac{1}{s}}(x) + (1-\lambda) f^{\frac{1}{s}}(y),$$

for all $0 < \lambda < 1$ and for all $x, y \in \mathbb{R}^n$ with $f(x), f(y) > 0$. The characteristic function of a convex set is an s-concave function, for any $s > 0$. Theorem 1.1 does not hold when we replace uniform measures on convex sets with arbitrary log-concave densities (consider, e.g., the one-dimensional log-concave density $t \mapsto e^{-|t|}$). However, for functions that are s-concave the following theorem holds.

Theorem 1.3 *Let $\alpha > 0$, let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0,\infty)$ be an αn-concave function with $\int f = 1$. Denote by μ the measure whose density is f. Then there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $t \geq 1$,*

$$\mu\left(\{x \in \mathbb{R}^n; \varphi(x) > t\|\varphi\|_{L_1(\mu)}\}\right) < e^{-c_\alpha \frac{t^2}{\log^5(t+1)}}$$

where $c_\alpha > 0$ is a constant depending solely on α.

Our main technical tool here is the logarithmic Laplace transform, as in our previous work [20]. The logarithmic Laplace transform is introduced and discussed in Section 2, and then applied in Section 3 to the study of log-concave functions with a bounded isotropic constant. Section 3 contains the main technical steps of the proof. Theorem 1.1 and Theorem 1.3 are proved in Section 4. In Section 5 we address questions regarding the behavior of a "typical" linear functional. Throughout this text, unless mentioned otherwise, we use the symbols $c, c', \tilde{c}, \hat{c}, c_1, c_2, C, C', \tilde{C}, \hat{C}$ etc. to denote various positive universal constants, whose value is not necessarily the same in different appearances.

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2 Logarithmic Laplace transform

In this section we develop a number of estimates that are related to the logarithmic Laplace transform of log-concave functions. We denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the standard Euclidean norm and scalar product in $\mathbb{R}^n$, respectively. We also write $D^n = \{x \in$

$\mathbb{R}^n; |x| \leq 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$, the unit Euclidean ball and the unit sphere in $\mathbb{R}^n$, respectively. When the dimension is clear from the context, we may write D in place of D^n. Let us begin with a very standard lemma.

Lemma 2.1 *Let $n \geq 1$ and let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $0 < \int f < \infty$. Then there exist $A, B > 0$ such that for all $x \in \mathbb{R}^n$,*

$$f(x) \leq Ae^{-B|x|}.$$

Proof: Since $\int f > 0$, then there exists $0 < \varepsilon < 1$ such that $K = \{x \in \mathbb{R}^n; f(x) > \varepsilon\}$ satisfies $Vol_n(K) > 0$. The set K is convex, because f is log-concave. Therefore K has a non-empty interior, since it is a convex set of positive volume. By translating f, we may assume that $rD^n \subset K$, for some $r > 0$. Denote $T = \{x \in \mathbb{R}^n; f(x) > \frac{\varepsilon}{2}\}$. The set T is convex, $Vol_n(T) \geq Vol_n(K) > 0$, and also $Vol_n(T) < \infty$ since $\int f < \infty$. Consequently, T is bounded, as a convex set of a finite, positive volume. Thus for some $R > 0$ we have $T \subset \frac{R}{2} D^n$. Pick $x \notin RD^n$. Then

$$R\frac{x}{|x|} = \frac{|x| - R}{|x| - r} \cdot r\frac{x}{|x|} + \frac{R - r}{|x| - r} \cdot x \tag{6}$$

is a convex combination. Note that $R\frac{x}{|x|} \notin T$ and that $r\frac{x}{|x|} \in K$. Using (6) and the log-concavity of f, we deduce that

$$\frac{\varepsilon}{2} \geq f\left(R\frac{x}{|x|}\right) \geq f\left(r\frac{x}{|x|}\right)^{\frac{|x|-R}{|x|-r}} \cdot f(x)^{\frac{R-r}{|x|-r}} \geq \varepsilon^{\frac{|x|-R}{|x|-r}} f(x)^{\frac{R-r}{|x|-r}}.$$

In particular,

$$f(x) \leq \varepsilon \left(\frac{1}{2}\right)^{\frac{|x|-r}{R-r}} \leq Ae^{-B|x|} \quad \text{for any } x \notin RD^n$$

for some $A, B > 0$ depending on f. Note also that f is bounded in RD^n; Otherwise the log-concavity implies that $f = \infty$ on K, in contradiction to our assumption that $\int f < \infty$. We conclude that an estimate of the form $f(x) \leq A'e^{-B'|x|}$, for some numbers $A', B' > 0$, holds in the entire $\mathbb{R}^n$. $\square$

For a function $F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ we write $Dom(F) = \{x \in \mathbb{R}^n; F(x) < \infty\}$. Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $0 < \int f < \infty$. We define $\Upsilon f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by setting

$$\Upsilon f(x) = \log \int_{\mathbb{R}^n} e^{\langle x, y \rangle} f(y) \frac{dy}{\int f}. \tag{7}$$

In this note "log" stands for the natural logarithm. According to Lemma 2.1, the function Υf is finite in some open neighborhood of the origin. Furthermore, if $x \in Dom(\Upsilon f)$ then $\tilde{f}(y) = e^{\langle x, y \rangle} f(y)$ is a log-concave function that has a positive, finite integral. Lemma 2.1 implies that $\tilde{f}$ decays exponentially, and hence we may differentiate under the integral sign any finite number of times. We conclude that $Dom(\Upsilon f)$ is open, and that Υf is C^∞-smooth in $Dom(\Upsilon f)$.

A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is strictly-convex if for any $x_1, x_2 \in Dom(F)$ with $x_1 \neq x_2$, we have

$$F\left(\frac{x_1 + x_2}{2}\right) < \frac{F(x_1) + F(x_2)}{2}.$$

The function Υf as defined in (7) is strictly-convex. Indeed, for any distinct points $x_1, x_2 \in Dom(\Upsilon f)$,

$$\int_{\mathbb{R}^n} e^{\langle \frac{x_1+x_2}{2}, y\rangle} f(y) \frac{dy}{\int f} < \sqrt{\int_{\mathbb{R}^n} e^{\langle x_1, y\rangle} f(y) \frac{dy}{\int f} \cdot \int_{\mathbb{R}^n} e^{\langle x_2, y\rangle} f(y) \frac{dy}{\int f}}$$

by the Cauchy-Schwartz inequality. The following lemma is standard. Its straightforward proof appears, e.g., in [20, Lemma 3.1].

Lemma 2.2 *Let $n \geq 1$, and let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $0 < \int f < \infty$. Then, for any $x \in Dom(\Upsilon f)$,*

$$\nabla(\Upsilon f)(x) = \int_{\mathbb{R}^n} y \, d\mu_x(y),$$

where μ_x is the probability measure on $\mathbb{R}^n$ whose density is proportional to $y \mapsto e^{\langle x, y\rangle} f(y)$. Additionally,

$$Hess(\Upsilon f)(x) = Cov(\mu_x) = \int_{\mathbb{R}^n} y \otimes y \, d\mu_x(y) - \left[\int_{\mathbb{R}^n} y \, d\mu_x(y)\right] \otimes \left[\int_{\mathbb{R}^n} y \, d\mu_x(y)\right],$$

the covariance matrix of μ_x. Here $Hess$ stands for Hessian, and $x \otimes x$ stands for the matrix whose entries are $(x_i x_j)_{i,j=1,\dots,n}$.

A set $K \subset \mathbb{R}^n$ is centrally-symmetric when $K = -K$. Recall that for a centrally-symmetric convex body $K \subset \mathbb{R}^n$, the set

$$K^\circ = \{x \in \mathbb{R}^n ; \forall y \in K, \ \langle x, y\rangle \leq 1\}$$

is the polar body of K. For a convex body $K \subset \mathbb{R}^n$, we denote

$$v.rad.(K) = \left(\frac{Vol_n(K)}{Vol_n(D^n)}\right)^{\frac{1}{n}},$$

the radius of the Euclidean ball that has the same volume as K. Recall that $c_1 < \sqrt{n} Vol_n(D^n)^{\frac{1}{n}} < c_2$ for some universal constants $c_1, c_2 > 0$ (see, e.g., the first pages of [31]). Therefore $v.rad.(K)$ has the order of magnitude of $\sqrt{n} Vol_n(K)^{1/n}$.

The following lemma may be interesting in its own right. It is related to the use of $\nabla(\Upsilon f)$ as a "transportation map", an idea developed already in [20]. It may be beneficial, when reading this lemma, to have in mind the simplest example of the function $F(x) = A|x|^2/2$.

Lemma 2.3 *Let $n \geq 1$ and let $F : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a C^2-smooth, strictly-convex, even function with $F(0) = 0$. Fix $t > 0$ and set $K = \{x \in \mathbb{R}^n; F(x) \leq t\}$. Let $A > 0,\ 0 < \alpha \leq 1/2$ and suppose that*

$$\det Hess(F)(x) \geq A^n \quad \textit{for } x \in \alpha K. \tag{8}$$

Then,

$$v.rad.(K) \leq \sqrt{\frac{2}{\alpha A}} \cdot \sqrt{t}.$$

Proof: Fix $x \in \alpha K$. Since F is convex, the graph of F lies above the supporting hyperplane to F at x. In particular,

$$F(x) + \langle \nabla F(x), 2x - x \rangle \leq F(2x) \leq t, \tag{9}$$

since $2x \in 2\alpha K \subset K$. Note that F is non-negative, as a convex, even function that vanishes at the origin. In particular $F(x) \geq 0$, and we conclude from (9) that

$$\langle \nabla F(x), x \rangle \leq t. \tag{10}$$

Next, fix $y \in K$. Using (10) and the convexity of F we obtain

$$t \geq F(y) \geq F(x) + \langle \nabla F(x), y - x \rangle \geq F(x) + \langle \nabla F(x), y \rangle - t. \tag{11}$$

Since $F(x) \geq 0$, we deduce from (11) that

$$\langle \nabla F(x), y \rangle \leq 2t \quad \text{for all } x \in \alpha K, y \in K. \tag{12}$$

Denote $K' = \{\nabla F(x); x \in \alpha K\}$. Then (12) is interpreted as

$$K' \subset 2tK^\circ. \tag{13}$$

The function F is strictly-convex on K, hence its gradient is a one-to-one map. By substituting $x = \nabla F(y)$ we obtain

$$Vol_n(K') = \int_{\{\nabla F(y); y \in \alpha K\}} dx = \int_{\alpha K} \det Hess(F)(y) dy \geq Vol_n(\alpha K) A^n, \tag{14}$$

where we have used (8). By combining (13) and (14) we conclude that

$$Vol_n(K^\circ) \geq \left(\frac{\alpha A}{2t}\right)^n Vol_n(K). \tag{15}$$

The set K is centrally-symmetric and convex. According to the Santaló inequality (see [34] or [1, 24]), we know that $v.rad.(K)v.rad.(K^\circ) \leq 1$. Thus (15) implies that

$$v.rad.(K) \leq \frac{\sqrt{2t}}{\sqrt{\alpha A}}.$$

The lemma is proven. $\square$

Remark. It is also possible to "reverse" the inequalities in Lemma 2.3. Suppose that (8) is replaced by the requirement that $\det HessF(x) \leq A^n$ for all $x \in K = \{y \in \mathbb{R}^n; F(y) \leq t\}$. By slightly modifying the above argument, and in particular by applying Bourgain-Milman's inequality [10] in place of the Santaló inequality, one may prove that, in this case, $v.rad.(K) \geq c\sqrt{t}/\sqrt{A}$, for some universal constant $c > 0$.

Next, we review some of the basic, non-trivial properties of log-concave functions. Suppose $f : \mathbb{R}^n \to [0, \infty)$ is a log-concave function with $0 < \int f < \infty$. According to the Prékopa-Leindler inequality (see [32, 22, 33] or, e.g., the first pages of [31]),

$$\sqrt{\int_{\lambda A + (1-\lambda)B} f(x)dx} \geq \left(\int_A f(x)dx \right)^{\lambda} \left(\int_B f(x)dx \right)^{1-\lambda} \tag{16}$$

for any Borel sets $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$. Denote by μ the measure whose density is f, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a linear functional. As was explained in the introduction for the case of convex bodies, Borell's lemma [6] states that

$$\|\varphi\|_{L_{\psi_1}(\mu)} \leq C\|\varphi\|_{L_1(\mu)}, \tag{17}$$

where $C > 0$ is a universal constant. We conclude from Borell's lemma, in particular, that $\|\varphi\|_{L_2(\mu)} \leq C\|\varphi\|_{L_1(\mu)}$. Furthermore, let $M > 0$ be such that

$$\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| \leq M\}\right) = \frac{2}{3}.$$

Then we have $c\|\varphi\|_{L_1(\mu)} \leq M \leq 3\|\varphi\|_{L_1(\mu)}$. Another corollary of (16) is that the marginal function $(x_1, .., x_k) \mapsto \int f(x_1, .., x_n)dx_{k+1}...dx_n$ is a log-concave function. Consequently, the convolution of two log-concave functions is again a log-concave function, a fact that was already known to Lekkerkerker [23] in one dimension, and to Davidovič, Korenbljum and Hacet [12] in the general case. Next, suppose that $f : \mathbb{R}^n \to [0, \infty)$ is not only log-concave, but also s-concave for some $s > 0$. Then (16) may be strengthened, as follows:

$$\left(\int_{\alpha A + \beta B} f(x)dx \right)^{\frac{1}{n+s}} \geq \alpha \left(\int_A f(x)dx \right)^{\frac{1}{n+s}} + \beta \left(\int_B f(x)dx \right)^{\frac{1}{n+s}} \tag{18}$$

for any Borel sets $A, B \subset \mathbb{R}^n$ and $\alpha, \beta > 0$. Inequality (18) in n dimensions was proven by Dinghas [13, 14], by Borell [7], and by Brascamp and Lieb [11]. It follows directly from the Brunn-Minkowski inequality (1), see [19]. The one-dimensional case of (18) is due to Henstock and Macbeath [16]. The same line of reasoning also leads to the conclusion that the convolution of an s_1-concave function on $\mathbb{R}^n$ with an s_2-concave function on $\mathbb{R}^n$ is an $(s_1 + s_2 + n)$-concave function (see, e.g., [7]).

The next lemma is standard, and is contained, e.g., in [17, Lemma 2.3]. Rather than referring the reader to the proof in [17], here we will present a simpler proof that we learned from Mark Rudelson.

Lemma 2.4 *Let $n \geq 1$, let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body, and let $x \in \mathbb{R}^n, t > 0$. Suppose that $\sup_{y \in K} \langle x, y \rangle \geq t$. Denote*

$$A = \left\{ y \in K; \langle x, y \rangle \geq \frac{t}{2} \right\}.$$

Then,

$$Vol_n(A) \geq 2^{-n-1} Vol_n(K).$$

Proof: Denote $K^+ = \{y \in K; \langle x, y \rangle \geq 0\}$. Let $y_0 \in K$ be a point such that $\langle x, y_0 \rangle \geq t$. Since $y_0 \in K$ and $K^+ \subset K$, by convexity,

$$\frac{1}{2} y_0 + \frac{1}{2} K^+ \subset K.$$

For any $y \in (y_0 + K^+)/2$ we have $\langle x, y \rangle \geq t/2$. Therefore $(y_0 + K^+)/2 \subset A$. Hence,

$$Vol_n(A) \geq Vol_n \left(\frac{y_0 + K^+}{2} \right) = 2^{-n} Vol_n(K^+) = 2^{-n-1} Vol_n(K).$$

$\square$

Later on, we will make use of the following standard estimate for the gamma function: Begin with the routinely-verified fact, that for any $n > 0, A > 1$ and $t > An$, we have $t^n e^{-t} \leq (An)^n e^{-An} \exp\left(-(1 - \frac{1}{A})(t - An)\right)$. Consequently,

$$\int_{An}^{\infty} t^n e^{-t} dt \leq (An)^n e^{-An} \frac{A}{A-1}, \quad \text{for any } A > 1. \tag{19}$$

Our next lemma is an application of Laplace's asymptotic method, and is a direct extension of [21, Lemma 2.1].

Lemma 2.5 *Let $g : [0, \infty) \to [0, \infty]$ be a non-decreasing, continuous, convex function such that $g(0) = 0$ and $g \not\equiv 0$. Suppose that g is continuously differentiable on $Dom(g)$. Let $n \geq 1$ be an integer, and let $t_0 > 0$ be a point such that*

$$e^{-g(t_0)} t_0^n = \sup_{t > 0} e^{-g(t)} t^n. \tag{20}$$

Then t_0 exists and is unique. Furthermore, t_0 satisfies:

(i) $g(t_0) \leq n$.

(ii) $\int_{5t_0}^{\infty} e^{-g(t)} t^n dt \leq e^{-n} \int_0^{\infty} e^{-g(t)} t^n dt$.

(iii) *Suppose $a \in \mathbb{R}$ satisfies $|at_0| \leq \frac{n}{20}$. Then,*

$$\int_0^{\infty} e^{at - g(t)} t^n dt \leq e^n \int_0^{\infty} e^{-g(t)} t^n dt.$$

Proof: The function g is a non-constant, convex function. Hence, $g(t)$ tends to infinity as $t \to \infty$. Consequently, the function $\varphi(t) = -g(t) + n \log t$ tends to $-\infty$ when $t \to 0$ or $t \to \infty$. Since φ is continuous, its finite maximum is attained. Furthermore, φ is strictly concave, hence the maximum is attained at a unique point $t_0 \in Dom(g)$. Thus a point $t_0 > 0$ that satisfies (20) exists and is unique. Since $\varphi'(t_0) = 0$, we have

$$g'(t_0)t_0 = n. \tag{21}$$

By convexity, g' is non-decreasing, hence $g'(t) \leq \frac{n}{t_0}$ for any $t < t_0$. Therefore,

$$g(t_0) = g(0) + \int_0^{t_0} g'(t)dt \leq \int_0^{t_0} \frac{n}{t_0} dt = n.$$

This proves (i). Next, we prove (ii). Recall that g is non-decreasing, hence,

$$\int_0^\infty e^{-g(t)} t^n dt \geq e^{-g(t_0)} \int_0^{t_0} t^n dt = \frac{1}{n+1} e^{-g(t_0)} t_0^{n+1}. \tag{22}$$

The convexity of g and (21) imply that for any $t > 0$,

$$g(t) \geq g(t_0) + \frac{n}{t_0}\left(t - t_0\right). \tag{23}$$

Therefore,

$$\int_{5t_0}^\infty e^{-g(t)} t^n dt \leq e^{n-g(t_0)} \int_{5t_0}^\infty e^{-\frac{n}{t_0}t} t^n dt = \frac{e^n}{n^{n+1}} e^{-g(t_0)} t_0^{n+1} \int_{5n}^\infty e^{-t} t^n dt. \tag{24}$$

From (24), (22) and the case where $A = 5$ in (19), we conclude that

$$\int_{5t_0}^\infty e^{-g(t)} t^n dt \leq \frac{5}{4} \frac{e^n}{n^{n+1}} \left(\frac{5n}{e^5}\right)^n e^{-g(t_0)} t_0^{n+1} \leq e^{-n} \int_0^\infty e^{-g(t)} t^n dt,$$

as $n \geq 1$. This completes the proof of (ii). Now we proceed to prove (iii). Denote $\psi(t) = g(t) - at$, a convex function. According to (21), and since $at_0 \leq \frac{n}{20}$, we know that

$$\psi'(t_0) \geq \frac{19n}{20t_0}.$$

The convexity of ψ implies that $\psi(t) \geq \psi(t_0) + \frac{19n}{20t_0}(t - t_0)$ for $t > t_0$. Thus,

$$\begin{aligned}
&\int_{5t_0}^\infty e^{-\psi(t)} t^n dt \\
\leq\ & e^{\frac{19n}{20} - \psi(t_0)} \int_{5t_0}^\infty e^{-\frac{19n}{20t_0}t} t^n dt \leq e^{-g(t_0)} e^{\frac{19n}{20} + at_0} \left(\frac{20t_0}{19n}\right)^{n+1} \int_{4n}^\infty e^{-t} t^n dt \\
\leq\ & \frac{t_0^{n+1} e^{-g(t_0)}}{n} \frac{e^n}{n^n} \left(\frac{20}{19}\right)^{n+1} \cdot \frac{4}{3} \left(\frac{4n}{e^4}\right)^n \leq \frac{t_0^{n+1} e^{-g(t_0)}}{n+1} \frac{e^n}{2} \leq \frac{e^n}{2} \int_0^\infty e^{-g(t)} t^n dt
\end{aligned}$$

for any $n \geq 1$, where we used (22) as well as (19). Consequently,

$$\int_{5t_0}^{\infty} e^{at-g(t)} t^n \leq \frac{e^n}{2} \int_0^{\infty} e^{-g(t)} t^n dt. \tag{25}$$

In addition, since $5|at_0| \leq \frac{1}{4}$,

$$\int_0^{5t_0} e^{at-g(t)} t^n \leq e^{5|at_0|} \int_0^{5t_0} e^{-g(t)} t^n dt \leq \frac{e^n}{2} \int_0^{\infty} e^{-g(t)} t^n dt. \tag{26}$$

By adding (25) to (26) we obtain the third part of the lemma, and the lemma is proven. $\square$

Let $f : \mathbb{R}^n \to [0, \infty)$ be an even, log-concave function. We define

$$K(f) = \left\{ x \in \mathbb{R}^n ; f(x) \geq e^{-n} f(0) \right\}.$$

The set $K(f)$ is convex and centrally-symmetric, and it has a non-empty interior when $\int f > 0$. The next lemma is, again, very similar to the methods in [21].

Lemma 2.6 *Let $n \geq 2$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, continuously differentiable, log-concave function with $0 < \int f < \infty$. Then,*

(i) $\int_{5K(f)} f(y)dy \geq \left(1 - e^{-(n-1)}\right) \int_{\mathbb{R}^n} f(y)dy$.

(ii) Suppose that $x \in \mathbb{R}^n$ satisfies $\sup_{y \in K(f)} \langle x, y \rangle \leq \frac{n}{20}$. Then,

$$\int_{\mathbb{R}^n} e^{\langle x, y \rangle} f(y)dy \leq e^n \int_{\mathbb{R}^n} f(y)dy.$$

Proof: Since f is even, log-concave and $\int f > 0$, then $f(0) > 0$. We divide f by $f(0)$, and assume from now on that $f(0) = 1$. For a unit vector $\theta \in S^{n-1}$, define $g_\theta(t) = -\log f(t\theta)$. Since f is even, log-concave and $f(0) = 1$, we conclude that $g_\theta : [0, \infty) \to [0, \infty]$ is non-decreasing, convex and $g_\theta(0) = 0$. The function g_θ is also continuous on $[0, \infty)$ and continuously differentiable on $Dom(g_\theta)$. By Lemma 2.1, $g_\theta \not\equiv 0$. Thus the function g_θ and the integer $n-1$ satisfy the requirements of Lemma 2.5. Let $t_0(\theta)$ be the point t_0 from Lemma 2.5, corresponding to g_θ and $n-1$. According to Lemma 2.5(i), for any $\theta \in S^{n-1}$,

$$g_\theta(t_0(\theta)) \leq n - 1 \leq n.$$

Consequently,

$$t_0(\theta)\theta \in K(f) \quad \text{for all } \theta \in S^{n-1}. \tag{27}$$

For a set $K \subset \mathbb{R}^n$, we denote by 1_K the characteristic function of K. We integrate in polar coordinates, and use (27) and Lemma 2.5(ii), to obtain

$$\begin{aligned}
\int_{5K(f)} f(y)dy &= \int_{S^{n-1}} \int_0^{\infty} 1_{5K(f)}(t\theta) f(t\theta) t^{n-1} dt d\theta \\
&\geq \int_{S^{n-1}} \int_0^{5t_0(\theta)} t^{n-1} e^{-g_\theta(t)} dt d\theta \geq \left(1 - e^{-(n-1)}\right) \int_{S^{n-1}} \int_0^{\infty} t^{n-1} e^{-g_\theta(t)} dt d\theta \\
&= \left(1 - e^{-(n-1)}\right) \int_{\mathbb{R}^n} f(y)dy.
\end{aligned}$$

This proves (i). We move to (ii). Our assumption in (ii) and (27) implies that

$$|\langle x, \theta \rangle t_0(\theta)| \le \frac{n}{20} \quad \text{for any } \theta \in S^{n-1} \tag{28}$$

(we also used the fact that $K(f)$ is centrally-symmetric). Using (28) and Lemma 2.5(iii), we obtain that for any $\theta \in S^{n-1}$,

$$\int_0^\infty e^{\langle x, \theta \rangle t} t^{n-1} e^{-g_\theta(t)} dt \le e^{n-1} \int_0^\infty t^{n-1} e^{-g_\theta(t)} dt.$$

By integrating the last inequality in polar coordinates, we conclude that

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{\langle x, y \rangle} f(y) dy &= \int_{S^{n-1}} \int_0^\infty e^{\langle x, t\theta \rangle} f(t\theta) t^{n-1} dt d\theta \\
&= \int_{S^{n-1}} \int_0^\infty e^{\langle x, \theta \rangle t} t^{n-1} e^{-g_\theta(t)} dt d\theta \le e^{n-1} \int_{S^{n-1}} \int_0^\infty t^{n-1} e^{-g_\theta(t)} dt d\theta \\
&\le e^n \int_{\mathbb{R}^n} f(x) dx.
\end{aligned}$$

This establishes (ii). The proof of the lemma is complete. □

For convenience, our notation treats both f and Υf as functions defined on $\mathbb{R}^n$. It is important to keep in mind that Υf is defined in principle on the dual space. In particular, for any linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to [0, \infty)$,

$$\Upsilon (f \circ T) = (\Upsilon f) \circ (T^*)^{-1}, \tag{29}$$

where $(f \circ T)(x) = f(T(x))$. Also note that $K(f \circ T) = T^{-1} K(f)$. In addition, we clearly have, for any $a > 0$,

$$\Upsilon(af) = \Upsilon f, \quad K(af) = K(f). \tag{30}$$

Lemma 2.7 *Let $n \ge 2$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, continuously differentiable, log-concave function with $0 < \int f < \infty$.*

(i) For any $x \in \mathbb{R}^n$,

$$\Upsilon f(x) \le n \quad \Rightarrow \quad \sup_{y \in K(f)} \langle x, y \rangle \le Cn.$$

(ii) For any $x \in \mathbb{R}^n$,

$$\sup_{y \in K(f)} \langle x, y \rangle \le cn \quad \Rightarrow \quad \Upsilon f(x) \le n.$$

(iii) For any $x \in \mathbb{R}^n$,

$$\Upsilon f(Cx) \le n \quad \Rightarrow \quad \sup_{y \in \mathbb{R}^n} e^{\langle x, y \rangle} f(y) \le e^n f(0).$$

Here $c, C > 0$ are universal constants.

Proof: According to (30), we may multiply f by a positive constant, and assume that $f(0) = 1$. Furthermore, for $\delta > 0$ denote $\tau_\delta(x) = \delta x$. By (29) and the discussion around it, we may substitute f with $f \circ \tau_\delta$ for an appropriate $\delta > 0$, and assume that $\int f = 1$.

We begin by proving (i). Let $x \in \mathbb{R}^n$ be such that

$$\sup_{y \in K(f)} \langle x, y \rangle \geq C_1 n \tag{31}$$

for a universal constant C_1 to be chosen later. In order to prove (i), it is sufficient to show that $\Upsilon f(x) > n$. That is, it is enough to prove that

$$\int_{\mathbb{R}^n} e^{\langle x,y \rangle} f(y) dy > e^n. \tag{32}$$

According to (31) and to Lemma 2.4, the set

$$A = K(f) \cap \left\{ y \in \mathbb{R}^n; \langle x, y \rangle \geq \frac{C_1}{2} n \right\} \tag{33}$$

satisfies $Vol_n(A) \geq 2^{-n-1} Vol_n(K(f))$. The function f is even, log-concave and $f(0) = 1$. Hence $f(y) = \sqrt{f(y)f(-y)} \leq f(0) = 1$ for all $y \in \mathbb{R}^n$, and $\sup f = 1$. According to Lemma 2.6(i),

$$5^n Vol_n(K(f)) \geq \int_{5K(f)} f(x) dx \geq 1 - e^{-(n-1)} \geq e^{-n}.$$

We conclude that

$$Vol_n(A) \geq e^{-c'n}. \tag{34}$$

Note that

$$\int_{\mathbb{R}^n} e^{\langle x,y \rangle} f(y) dy \geq \int_A e^{\langle x,y \rangle} f(y) dy \geq Vol_n(A) \cdot \min_{y \in A} e^{\langle x,y \rangle} \cdot \min_{y \in A} f(y).$$

By using (33), (34) and the definition of $K(f)$, we deduce that

$$\int_{\mathbb{R}^n} e^{\langle x,y \rangle} f(y) dy \geq \exp\left[\left(\frac{C_1}{2} - c' - 1 \right) n \right] > e^n$$

for a large enough universal constant $C_1 > 0$. Thus (32) is proven. This completes the proof of (i). To prove (ii), suppose that $x \in \mathbb{R}^n$ is such that

$$\sup_{y \in K(f)} \langle x, y \rangle \leq \frac{n}{20}. \tag{35}$$

According to Lemma 2.6(ii), the inequality (35) entails that

$$\int_{\mathbb{R}^n} e^{\langle x,y \rangle} f(y) dy \leq e^n.$$

Thus (35) implies that $\Upsilon f(x) \leq n$, and (ii) is proven. It remains to establish (iii). We will assume that $x \in \mathbb{R}^n$ satisfies

$$\Upsilon f(C_2 x) \leq n \tag{36}$$

for a large enough constant $C_2 > 0$ to be specified later on. According to part (i) of the present lemma, which was already proven, we have

$$\sup_{y \in K(f)} \langle x, y \rangle \leq \frac{C}{C_2} n \leq n. \tag{37}$$

where we assume, as we may, that the constant C_2 is larger than the constant C from part (i). The function f is continuous. Therefore, whenever $y \in \partial K(f)$ we have that $f(y) = e^{-n} f(0) = e^{-n}$. Denote $g(y) = e^{\langle x,y \rangle} f(y)$, a continuous log-concave function. From (37) we obtain that for all $y \in \partial K(f)$,

$$g(y) = e^{\langle x,y \rangle} f(y) = e^{\langle x,y \rangle} e^{-n} \leq 1. \tag{38}$$

Let $y \notin K(f)$. Then $f(y) < e^{-n}$. Since f is continuous and $f(0) = 1$, there exists $0 < \theta < 1$ such that $f(\theta y) = e^{-n}$. Suppose that $0 < \theta < 1$ is the maximal possible number such that $f(\theta y) = e^{-n}$. Then

$$\theta y \in \partial K(f).$$

According to (38) we know that $g(\theta y) \leq 1$, and the log-concavity of g implies that

$$1 \geq g(\theta y) \geq g(y)^\theta g(0)^{1-\theta} = g(y)^\theta.$$

Consequently, $g(y) \leq 1$ for all $y \notin K(f)$. Since $0 \in K(f)$ and $g(0) = 1$,

$$\sup_{y \in \mathbb{R}^n} e^{\langle x,y \rangle} f(y) = \sup_{y \in \mathbb{R}^n} g(y) = \sup_{y \in K(f)} g(y) \leq \sup_{y \in K(f)} e^{\langle x,y \rangle} \cdot \sup_{y \in K(f)} f(y).$$

By combining this with (37) we conclude that

$$\sup_{y \in \mathbb{R}^n} e^{\langle x,y \rangle} f(y) \leq e^n \sup_{y \in K(f)} f(y) = e^n.$$

This completes the proof of (iii), and the lemma is thus proven. $\square$

Lemma 2.8 *Let $n \geq 2$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, log-concave function with $\int f = 1$. Let $K = \{x \in \mathbb{R}^n; \Upsilon f(x) \leq n\}$. Then,*

$$c\sqrt{n} f(0)^{\frac{1}{n}} \leq v.rad.(K) \leq C\sqrt{n} f(0)^{\frac{1}{n}} \tag{39}$$

where $c, C > 0$ are universal constants.

Proof: Suppose first that f is continuously differentiable. By the definition of $K(f)$,

$$e^{-n}f(0)\cdot Vol_n(K(f)) \leq \int_{K(f)} f(x)dx \leq \int_{\mathbb{R}^n} f(x)dx = 1. \tag{40}$$

In addition, from Lemma 2.6(i),

$$f(0)Vol_n(5K(f)) \geq \int_{5K(f)} f(x)dx \geq \left(1 - e^{-(n-1)}\right) \int f \geq e^{-n}. \tag{41}$$

Since $Vol_n(D)^{-\frac{1}{n}}$ has the order of magnitude of $\sqrt{n}$, we conclude from (40) and (41) that

$$c\frac{\sqrt{n}}{f(0)^{\frac{1}{n}}} < v.rad.(K(f)) < C\frac{\sqrt{n}}{f(0)^{\frac{1}{n}}}. \tag{42}$$

Now, by (i) and (ii) of Lemma 2.7,

$$\tilde{c}n[K(f)]^\circ \subset K \subset \tilde{C}n[K(f)]^\circ. \tag{43}$$

According to Santaló's inequality (see [34] or [1, 24]) and Bourgain-Milman's inequality [10], we conclude from (43) that

$$\hat{c}\frac{n}{v.rad.(K(f))} \leq v.rad.(K) \leq \hat{C}\frac{n}{v.rad.(K(f))}. \tag{44}$$

By comparing (44) with (42), we conclude the desired inequality (39), in the case where f is continuously differentiable. This completes the proof for the case of f being a smooth function.

We deal with the general case using a standard approximation argument. Denote $g_\varepsilon(x) = (2\pi\varepsilon)^{-n/2}\exp\left(-|x|^2/2\varepsilon\right)$. Then g_ε is log-concave. Since f is also log-concave, so is the convolution $f_\varepsilon = f * g_\varepsilon$. The function f_ε is C^∞, even, log-concave and $\int f_\varepsilon = 1$. Thus we have already proven the conclusion of the lemma for f_ε. Since $f_\varepsilon \to f$ in the w^*-topology when $\varepsilon \to 0$, the conclusions of the lemma also hold for f. The proof is complete. □

3 Bounded isotropic constant

Let $f : \mathbb{R}^n \to [0,\infty)$ be a log-concave function with $0 < \int f < \infty$. Recall that the covariance matrix of f is the matrix $Cov(f) = (Cov_{i,j}(f))_{i,j=1,\dots,n}$, whose entries are

$$Cov_{i,j}(f) = \int_{\mathbb{R}^n} x_i x_j f(x)\frac{dx}{\int f} - \int_{\mathbb{R}^n} x_i f(x)\frac{dx}{\int f} \cdot \int_{\mathbb{R}^n} x_j f(x)\frac{dx}{\int f}$$

where, as usual, $x = (x_1, ..., x_n)$ are coordinates in $\mathbb{R}^n$. The covariance matrix is well-defined according to Lemma 2.1. We say that f is isotropic, or that f is in isotropic position, if the barycenter of f lies at the origin and $Cov(f)$ is a scalar matrix. Note that for any log-concave function $f : \mathbb{R}^n \to [0, \infty)$ with $0 < \int f < \infty$, there exists a volume-preserving affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $f \circ T$ is isotropic (see, e.g., [26]). The isotropic constant of f is defined as

$$L_f = \left(\frac{\sup_{x \in \mathbb{R}^n} f(x)}{\int_{\mathbb{R}^n} f(x) dx} \right)^{\frac{1}{n}} (\det Cov(f))^{\frac{1}{2n}} . \tag{45}$$

We refer the reader, e.g., to [26] for a thorough review of the isotropic position and the isotropic constant of convex bodies. Clearly $L_f = L_{\tilde{f}}$ if $\tilde{f}(x) = f(x - x_0)$ for some $x_0 \in \mathbb{R}^n$, and $L_f = L_{af}$ for any $a > 0$. Note too that $L_f = L_{f \circ T}$ for any linear transformation $T \in GL_n(\mathbb{R})$. The following lemma is well-known (it is almost identical, e.g., to [26, Lemma 4.1]). Since our definitions are not entirely standard, we reproduce its short proof below.

Lemma 3.1 *Let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $0 < \int f < \infty$. Then,*

$$L_f > c$$

where $c > 0$ is a universal constant.

Proof: By translating f if necessary, we may assume that the barycenter of f is at the origin. We may further replace f with $af \circ T$, for suitable $a > 0$ and $T \in GL_n(\mathbb{R})$, and suppose that $\sup f = \int f = 1$ and that f is isotropic. Consequently, $Cov(f) = L_f^2 Id$ and

$$nL_f^2 = \int_{\mathbb{R}^n} |x|^2 f(x) dx = \int_0^\infty \int_{\mathbb{R}^n \setminus \sqrt{t}D} f(x) dx dt = \int_0^\infty \left[1 - \int_{\sqrt{t}D} f(x) dx \right] dt$$

where we used the fact that $\int f = 1$, as well as the identity $\int g d\mu = \int_0^\infty \mu\{x; g(x) \geq t\} dt$, valid for any measure μ and a non-negative measurable function g. Denote $\kappa_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} = Vol_n(D^n)$, and recall that $c < \sqrt{n} \kappa_n^{\frac{1}{n}} < C$. Since $\sup f = 1$, then

$$nL_f^2 \geq \int_0^\infty \max\{1 - Vol_n(\sqrt{t}D), 0\} dt = \int_0^{\kappa_n^{-2/n}} \left(1 - t^{n/2} \kappa_n \right) dt = \frac{n}{n+2} \kappa_n^{-\frac{2}{n}} > cn.$$

Thus $L_f > c'$, and the proof is complete. $\square$

Note that the log-concavity assumption was barely used in the proof of Lemma 3.1. For $0 < \varepsilon \leq 1$ and a log-concave function $f : \mathbb{R}^n \to [0, \infty)$ with $0 < \int f < 1$, we define

$$\hat{T}_\varepsilon(f) = \left\{ x \in \mathbb{R}^n ; \Upsilon f \left(\frac{2^{i/2}}{i^{2+\varepsilon}} x \right) \leq 2^i, \quad \text{for } i = 1, ..., \lfloor \log_2 n \rfloor \right\} . \tag{46}$$

The set $\hat{T}_\varepsilon(f)$ is convex. Our main goal in the next few pages is to show that $\hat{T}_\varepsilon(f)$ is sufficiently largely. The following lemma serves to motivate the definition of $\hat{T}_\varepsilon(f)$, and to demonstrate the usefulness of the desired lower bound for the volume of $\hat{T}_\varepsilon(f)$.

Lemma 3.2 *Let $0 < \varepsilon \leq 1, M > 0$, let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, log-concave function with $\int f = 1$. Denote by μ the measure whose density is f. Suppose that $\theta/M \in \hat{T}_\varepsilon(f)$. Then, for any $1 \leq t \leq \sqrt{n} \log^{2+\varepsilon/2} n$,*

$$\mu\left(\{x \in \mathbb{R}^n; |\langle x, \theta\rangle| > C_\varepsilon t M\}\right) < e^{-c_\varepsilon \frac{t^2}{\log^{4+\varepsilon}(t+1)}},$$

where $C_\varepsilon, c_\varepsilon > 0$ are constants depending only on ε.

Proof: In this proof, c, C, c' etc. denote constants that depend solely on ε. Let $2 \leq t \leq \sqrt{n}$ and let $1 \leq i \leq \lfloor \log_2 n \rfloor$ be the integer such that $2^i \leq t^2 < 2^{i+1}$. Since $\theta/M \in \hat{T}_\varepsilon(f)$,

$$\int_{\mathbb{R}^n} \exp\left(\left\langle \frac{t}{CM(\log t)^{2+\varepsilon}} \theta, y \right\rangle\right) d\mu(y) \leq \int_{\mathbb{R}^n} \exp\left(\left\langle \frac{\theta}{M} \cdot \frac{2^{i/2}}{i^{2+\varepsilon}}, y \right\rangle\right) d\mu(y) \leq e^{2^i} \leq e^{t^2},$$

for an appropriate choice of the constant $C > 0$. An application of the Chebychev-Markov inequality yields that for any $2 \leq t \leq \sqrt{n}$,

$$\mu\left(\left\{x \in \mathbb{R}^n; \langle \theta, x\rangle > 2CMt \log^{2+\varepsilon} t\right\}\right) \leq \frac{e^{t^2}}{e^{2t^2}} = e^{-t^2}. \tag{47}$$

Since f is even, then $\hat{T}_\varepsilon(f)$ is centrally-symmetric and $-\theta/M \in \hat{T}_\varepsilon(f)$. By repeating the above argument for $-\theta$, and substituting $s = t \log^{2+\varepsilon} t$ into (47), we see that for any $1 \leq s \leq \sqrt{n} \log^{2+\varepsilon} n$,

$$\mu\left(\{x \in \mathbb{R}^n; |\langle \theta, x\rangle| > CMs\}\right) \leq e^{-c' \frac{s^2}{\log^{4+2\varepsilon}(s+1)}}.$$

The lemma is thus proven. $\square$

Lemma 3.3 *Let $n \geq 2$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, log-concave function with $\int f = 1$. Let $0 < t \leq n$, and denote $K = \{x \in \mathbb{R}^n; \Upsilon f(x) \leq t\}$. Then,*

$$v.rad.(K^\circ) > \frac{c}{\sqrt{t} f(0)^{\frac{1}{n}}}$$

where $c > 0$ is a universal constant.

Proof: By reasoning as in the proof of Lemma 2.8, we may clearly assume that f is continuously differentiable. The function Υf is even, strictly-convex, C^∞-smooth, and $\Upsilon f(0) = 0$. Thus Υf satisfies the requirements of Lemma 2.3. Let C_1 denote the constant from Lemma 2.7(iii). Fix $x \in \mathbb{R}^n$ such that $\max\{C_1, 2\} x \in K$. Then,

$$\Upsilon(C_1 x) \leq t \leq n.$$

Denote $f_x(y) = e^{\langle x, y\rangle} f(y)$, a log-concave function. According to Lemma 2.7(iii),

$$\sup_{y \in \mathbb{R}^n} f_x(y) = \sup_{y \in \mathbb{R}^n} e^{\langle x, y\rangle} f(y) \leq e^n f(0). \tag{48}$$

The convex, even function Υf vanishes at the origin. Hence Υf is a non-negative function, and

$$\int_{\mathbb{R}^n} f_x(y)dy = \int_{\mathbb{R}^n} e^{\langle x,y\rangle} f(y)dy = \exp \Upsilon f(x) \geq 1. \tag{49}$$

Consequently, by (45), (48), (49) and Lemma 3.1,

$$\det Cov(f_x) = L_{f_x}^{2n} \left(\frac{\int_{\mathbb{R}^n} f_x(y)dy}{\sup_{y\in\mathbb{R}^n} f_x(y)} \right)^2 \geq \frac{c^{2n}}{f(0)^2}. \tag{50}$$

According to Lemma 2.2, we know that

$$\det Hess(\Upsilon f)(x) = \det Cov(f_x).$$

We deduce from (50) that

$$\det Hess(\Upsilon f)(x) > \left(\frac{\tilde{c}}{f(0)^{2/n}} \right)^n \quad \text{if } \max\{C_1, 2\}x \in K.$$

Thus, we may apply Lemma 2.3. According to the conclusion of that lemma,

$$v.rad.(K) \leq C\sqrt{t} f(0)^{\frac{1}{n}}.$$

By Bourgain-Milman's inequality [10], we have $v.rad.(K^\circ) \geq \left(C'\sqrt{t}f(0)^{1/n}\right)^{-1}$, and the lemma is proven. $\square$

For a log-concave function $f : \mathbb{R}^n \to [0, \infty)$ with $0 < \int f < \infty$ and for a subspace $E \subset \mathbb{R}^n$, we define $\pi_E(f) : E \to [0, \infty)$ to be the marginal function

$$\pi_E(f)(x) = \int_{x+E^\perp} f(y)dy.$$

The Prékopa-Leindler inequality (16) implies that $\pi_E(f)$ is also a log-concave function. The following lemma is an immediate consequence of the definitions.

Lemma 3.4 *Let $n \geq 1$ be an integer. Suppose $f : \mathbb{R}^n \to [0, \infty)$ is an even, isotropic, log-concave function with $f(0) = \int f = 1$. Let $1 \leq k \leq n$ be an integer, and let $E \subset \mathbb{R}^n$ be a subspace with* $\dim(E) = k$. *Then,*

$$(\pi_E(f)(0))^{\frac{1}{k}} = \left(\int_{E^\perp} f(x)dx \right)^{\frac{1}{k}} = \frac{L_{\pi_E(f)}}{L_f},$$

where $E^\perp$ is the orthogonal complement to E in $\mathbb{R}^n$.

Proof: The function f is isotropic, hence

$$Cov(f) = \left(\frac{\int f}{f(0)} \right)^{\frac{2}{n}} L_f^2 Id = L_f^2 Id,$$

where we used the fact that $1 = f(0) = \sup f$ since f is even and log-concave. Consequently, also

$$Cov(\pi_E(f)) = L_f^2 Id. \tag{51}$$

Since $\pi_E(f)$ is even and log-concave, then $\sup \pi_E(f) = \pi_E(f)(0) = \int_{E^\perp} f$. Using (51) and the definition (45) we see that

$$L_{\pi_E(f)} = \left(\frac{\pi_E(f)(0)}{\int_E \pi_E(f)}\right)^{\frac{1}{k}} L_f = (\pi_E(f)(0))^{\frac{1}{k}} L_f = \left(\int_{E^\perp} f(x)dx\right)^{\frac{1}{k}} L_f$$

as $\int_E \pi_E(f) = \int_{\mathbb{R}^n} f = 1$. $\square$

For a subspace $E \subset \mathbb{R}^n$ we denote by $Proj_E : \mathbb{R}^n \to E$ the orthogonal projection operator onto E in $\mathbb{R}^n$.

Lemma 3.5 *Let $n \geq 2$ and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, isotropic, log-concave function with $f(0) = \int f = 1$. Let $2 \leq k \leq n$ be an integer, and denote $K = \{x \in \mathbb{R}^n; \Upsilon f(x) \leq k\}$. Then, for any subspace $F \subset \mathbb{R}^n$ with $\dim(F) = \ell \geq k$,*

$$v.rad.(Proj_F(K^\circ)) = \left(\frac{Vol_\ell(Proj_F(K^\circ))}{Vol_\ell(D^\ell)}\right)^{\frac{1}{\ell}} \leq C\frac{L_f}{\sqrt{k}},$$

where $C > 0$ is a universal constant.

Proof: Fix a subspace $E \subset \mathbb{R}^n$ of dimension k, and denote $g = \pi_E(f)$. From Lemma 3.4, we know that $g(0)^{\frac{1}{k}} = \frac{L_g}{L_f}$. Note that $L_g > c$ according to Lemma 3.1. Consequently,

$$g(0)^{\frac{1}{k}} > \frac{c}{L_f}. \tag{52}$$

Set $K_E = \{x \in E; \Upsilon g(x) \leq k\}$. Note that

$$\Upsilon f(x) = \Upsilon \pi_E(f)(x) = \Upsilon g(x) \quad \text{for } x \in E.$$

Therefore, $K_E = K \cap E$. Recall that $\dim(E) = k$. Lemma 2.8 implies that

$$v.rad.(K \cap E) = v.rad.(K_E) > c'\sqrt{k} g(0)^{\frac{1}{k}}. \tag{53}$$

From (52) and (53) we conclude that for any subspace $E \subset \mathbb{R}^n$ of dimension k,

$$v.rad.(K \cap E) > \tilde{c}\frac{\sqrt{k}}{L_f}. \tag{54}$$

The subspace $F \subset \mathbb{R}^n$ satisfies $\dim(F) = \ell \geq k$. Note that (54) holds for all the subspaces $E \subset F$ with $\dim(E) = k$. Next we call upon Corollary 3.1 from [17]. By the conclusion of that corollary,

$$v.rad.(K \cap F) > c\frac{\sqrt{k}}{L_f}. \tag{55}$$

Since $(K \cap F)^\circ = Proj_F(K^\circ)$, Santaló's inequality and (55) imply that

$$v.rad.(Proj_F(K^\circ)) < C \frac{L_f}{\sqrt{k}}.$$

The lemma is thus proven. $\square$

For $K, T \subset \mathbb{R}^n$, the covering number of K by T is defined as

$$N(K,T) = \min \{\#(A); K \subset A + T\},$$

where $\#(A)$ denotes the cardinality of the set A, and $+$ here is the Minkowski sum. We will frequently use the following elementary properties of covering numbers. Obviously, $N(A,C) \leq N(A,B)N(B,C)$ for any $A, B, C \subset \mathbb{R}^n$. Also, $Vol_n(K) \leq N(K,T)Vol_n(T)$. In addition, for $K_1, ..., K_m, T_1, ..., T_m \subset \mathbb{R}^n$,

$$N(K_1 + ... + K_m, T_1 + ... + T_m) \leq \prod_{i=1}^m N(K_i, T_i). \tag{56}$$

Finally, $N(u(K), u(T)) \leq N(K,T)$ whenever u is a linear map.

Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body, and let $\alpha > 0$. An ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ is called a "Milman ellipsoid of order α for K with constants a, b" if for any $t > 1$,

$$\max\{N(K, at\mathcal{E}), N(\mathcal{E}, atK), N(K^\circ, at\mathcal{E}^\circ), N(\mathcal{E}^\circ, atK^\circ)\} \leq e^{b\frac{n}{t^\alpha}}. \tag{57}$$

We refer the reader to, e.g. [25] for background on the Milman ellipsoid. A fundamental theorem of Pisier [31, Section 7] states that given a centrally-symmetric convex body $K \subset \mathbb{R}^n$ and $0 < \alpha < 2$, there exists a Milman ellipsoid of order α for K, with constants that are not larger than $\frac{C}{2-\alpha}$, where $C > 0$ is a universal constant.

Next, we apply Lemma 3.5 in order to show that a Milman ellipsoid for the convex set $\{x \in \mathbb{R}^n; \Upsilon f(x) \leq k\}$ is not too far from a Euclidean ball.

Lemma 3.6 *Let $A > 0, 0 < \varepsilon \leq 1$ and let $n \geq 4$ be an integer. Let $f : \mathbb{R}^n \to [0, \infty)$ be an even, isotropic, log-concave function with $f(0) = \int f = 1$. Suppose that $L_f < A$. Let $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$ be an integer, and denote $K = \{x \in \mathbb{R}^n; \Upsilon f(x) \leq k\}$. Then for any $1 \leq t \leq c_{A,\varepsilon}\sqrt{\frac{n}{k}}$,*

$$N\left(\sqrt{k}K^\circ, C_{A,\varepsilon} tD\right) < \exp\left(C_{A,\varepsilon} \cdot \frac{n}{t^{1-\varepsilon}}\right).$$

Here, $c_{A,\varepsilon}, C_{A,\varepsilon} > 0$ are constants depending only on A and ε.

Proof: In this proof we denote by $c, C, c', \tilde{c}$ etc. constants depending only on A and ε. Denote $T = \sqrt{k}K^\circ$. The set T is convex and centrally-symmetric. According to Lemma 3.3,

$$v.rad.(T) > c. \tag{58}$$

Let $\alpha = 2 - \varepsilon \geq 1$. Let $\mathcal{E}$ be a Milman ellipsoid of order α for T with constants $C, C' > 0$. That is, for any $t \geq 1$,

$$\max\{N(T, Ct\mathcal{E}), N(\mathcal{E}, CtT)\} \leq \exp\left(C' \frac{n}{t^\alpha}\right). \tag{59}$$

Denote by $\lambda_1 \geq ... \geq \lambda_n > 0$ the lengths of the axes of the n-dimensional ellipsoid $\mathcal{E}$. From (58) and from the case $t = 1$ in (59), we conclude that

$$c < v.rad.(T) \leq N(T, C\mathcal{E})^{\frac{1}{n}} v.rad.(C\mathcal{E}) \leq \hat{C} v.rad.(\mathcal{E}). \tag{60}$$

Note that $v.rad.(\mathcal{E}) = (\prod_{i=1}^n \lambda_i)^{\frac{1}{n}}$. Hence, (60) implies that

$$\left(\prod_{i=1}^n \lambda_i\right)^{\frac{1}{n}} > c. \tag{61}$$

Next, set $m = \lfloor \frac{n}{2} \rfloor$, and let F denote the subspace spanned by the m longest axes of the ellipsoid $\mathcal{E}$. Projecting (59) to the subspace F, and substituting $t = 1$, we get

$$N\left(Proj_F(\mathcal{E}), C Proj_F(T)\right) \leq e^{C'n}.$$

Hence $v.rad.(Proj_F(\mathcal{E})) \leq \hat{C} v.rad.(Proj_F(T))$. We combine this estimate with Lemma 3.5, to obtain

$$\left(\prod_{i=1}^m \lambda_i\right)^{\frac{1}{m}} = v.rad.(Proj_F(\mathcal{E})) \leq C v.rad.(Proj_F(\sqrt{k}K^\circ)) < C' L_f < \tilde{C}. \tag{62}$$

(recall that $L_f < A$). From (61) and (62) we have

$$\prod_{i=m+1}^n \lambda_i = \left(\prod_{i=1}^n \lambda_i\right) \cdot \left(\prod_{i=1}^m \lambda_i\right)^{-1} > c^n \cdot \bar{c}^m > \tilde{c}^{n-m}.$$

Since λ_i is a non-increasing sequence, then

$$\lambda_{\lfloor \frac{n}{2} \rfloor} \geq \lambda_{m+1} \geq \left(\prod_{i=m+1}^n \lambda_i\right)^{\frac{1}{n-m}} > \tilde{c}. \tag{63}$$

Next, let ℓ be an integer with $k \leq \ell \leq \frac{n}{2}$. Let F now denote the ℓ-dimensional subspace spanned by the ℓ longest axes of the ellipsoid $\mathcal{E}$. Projecting (59) to F, we get that for any $t > 1$,

$$N\left(Proj_F(\mathcal{E}), Ct Proj_F(\sqrt{k}K^\circ)\right) < \exp\left(C' \frac{n}{t^\alpha}\right). \tag{64}$$

Setting $t = \left(\frac{n}{\ell}\right)^{\frac{1}{\alpha}}$ in (64), we deduce that

$$\left(\prod_{i=1}^\ell \lambda_i\right)^{\frac{1}{\ell}} = v.rad.(Proj_F(\mathcal{E})) < C \left(\frac{n}{\ell}\right)^{\frac{1}{\alpha}} v.rad.(Proj_F(\sqrt{k}K^\circ)). \tag{65}$$

Lemma 3.5, combined with (65), gives

$$\left(\prod_{i=1}^{\ell}\lambda_i\right)^{\frac{1}{\ell}} < CA\left(\frac{n}{\ell}\right)^{\frac{1}{\alpha}} < \tilde{C}\left(\frac{n}{\ell}\right)^{\frac{1}{\alpha}}. \tag{66}$$

Since λ_i is a non-increasing sequence, then (66) implies that

$$\lambda_\ell \leq \left(\prod_{i=1}^{\ell}\lambda_i\right)^{\frac{1}{\ell}} < C\left(\frac{n}{\ell}\right)^{\frac{1}{\alpha}}.$$

Hence,

$$\prod_{i=1}^{n}\max\left\{\frac{1}{C}\left(\frac{\ell}{n}\right)^{\frac{1}{\alpha}}\lambda_i, 1\right\} = \prod_{i=1}^{\ell}\max\left\{\frac{1}{C}\left(\frac{\ell}{n}\right)^{\frac{1}{\alpha}}\lambda_i, 1\right\}. \tag{67}$$

We use (67) together with (63) and (66), to obtain

$$\begin{aligned}&\prod_{i=1}^{n}\max\left\{\frac{1}{C}\left(\frac{\ell}{n}\right)^{\frac{1}{\alpha}}\lambda_i, 1\right\} \qquad (68)\\ &\leq \prod_{i=1}^{\ell}\max\left\{\frac{1}{C}\left(\frac{\ell}{n}\right)^{\frac{1}{\alpha}}\lambda_i, \frac{1}{\tilde{c}}\lambda_i\right\} \leq \left(\frac{1}{\min\{C,\tilde{c}\}}\right)^{\ell}\prod_{i=1}^{\ell}\lambda_i \leq \left(C'\left(\frac{n}{\ell}\right)^{\frac{1}{\alpha}}\right)^{\ell}.\end{aligned}$$

(recall that $\ell \leq \lfloor\frac{n}{2}\rfloor$.) Next, we will use standard estimates for the number of Euclidean balls needed to cover an ellipsoid, see e.g. Remark 5.15 in [31]. Recall that $\lambda_1, ..., \lambda_n$ are the lengths of the axes of the ellipsoid $\mathcal{E}$. Then for any $r > 0$, we have $N(\mathcal{E}, 4rD^n) \leq \prod_{i=1}^{n}\max\{\lambda_i/r, 1\}$. From (68) we thus conclude that for any $k \leq \ell \leq \lfloor\frac{n}{2}\rfloor$,

$$N\left(\mathcal{E}, \tilde{C}\left(\frac{n}{\ell}\right)^{\frac{1}{\alpha}}D^n\right) < \left(C'\left(\frac{n}{\ell}\right)^{\frac{1}{\alpha}}\right)^{\ell}. \tag{69}$$

Next, suppose $t > C$ is such that $t^\alpha \log t \leq \frac{n}{k}$. Using (59) followed by the case $\ell = \frac{n}{t^\alpha \log t}$ in (69), we get

$$N(T, Ct^2\log tD) \leq N(T, C't\mathcal{E})N(C't\mathcal{E}, Ct^2\log^{\frac{1}{\alpha}} tD) \leq e^{c\frac{n}{t^\alpha}}\cdot\left(\tilde{C}t\log^{\frac{1}{\alpha}} t\right)^{\frac{n}{t^\alpha\log t}} < e^{\hat{C}\frac{n}{t^\alpha}}. \tag{70}$$

(recall that $\alpha = 2 - \varepsilon \geq 1$). Now (70) implies that for any $\bar{C} \leq t \leq c\sqrt{\frac{n}{k}}$,

$$N(T, CtD) \leq \exp\left(C'n\frac{\log t}{t^{1-\frac{\varepsilon}{2}}}\right) < \exp\left(\tilde{C}\frac{n}{t^{1-\varepsilon}}\right) \tag{71}$$

(note that $c\sqrt{\frac{n}{k}} \leq C\left(\frac{n}{k}\right)^{\frac{1}{\alpha}}\left(\log\frac{n}{k}\right)^{-1}$). By selecting appropriate constants $C, \tilde{C}$ in (71), we see that (71) holds also for $1 \leq t \leq \bar{C}$. The proof is thus complete. $\square$

Recall the definition (46) of the set $\hat{T}_\varepsilon(f)$.

Lemma 3.7 *Let $0 < \varepsilon \leq 1, A > 0$, let $n \geq 4$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, log-concave function with $\int f = 1$. Suppose that $L_f < A$. Then,*

$$v.rad.(\hat{T}_\varepsilon(f)) > c_{A,\varepsilon} f(0)^{\frac{1}{n}}.$$

Here, $c_{A,\varepsilon} > 0$ is a constant depending only on A and ε.

Proof: In this proof, c, C, c' etc. denote constants that depend solely on A and ε. First, suppose that $f(0) = 1$ and that f is isotropic. For $t > 0$ denote

$$K_t = \{x \in \mathbb{R}^n; \Upsilon f(x) \leq t\}.$$

Dualizing (46), we get

$$\left(\hat{T}_\varepsilon(f)\right)^\circ = conv\left(\bigcup_{i=1}^{\lfloor \log_2 n \rfloor} \frac{2^{i/2}}{i^{2+\varepsilon}} K_{2^i}^\circ\right),$$

where *conv* denotes convex hull. According to the Bourgain-Milman inequality [10], it is sufficient to prove that

$$v.rad.\left(conv \bigcup_{i=1}^{\lfloor \log_2 n \rfloor} \frac{2^{i/2}}{i^{2+\varepsilon}} K_{2^i}^\circ\right) < C.$$

For any centrally-symmetric convex bodies $\Omega_1, .., \Omega_k \subset \mathbb{R}^n$, we know that $conv(\Omega_1 \cup ... \cup \Omega_k) \subset \Omega_1 + ... + \Omega_k$. Consequently, it is enough to prove that

$$v.rad.\left(\sum_{i=1}^{\lfloor \log_2 n \rfloor} \frac{2^{i/2}}{i^{2+\varepsilon}} K_{2^i}^\circ\right) < C. \tag{72}$$

we define c_1 to be the largest possible number, such that the following two requirements hold:

$$i^{1+\varepsilon/2} \leq c_{A,\varepsilon/6}\sqrt{\frac{n}{2^i}}, \quad \text{whenever} \quad 1 \leq 2^i \leq \frac{c_1 n}{\log^4 n}, \tag{73}$$

$$\text{and} \qquad c_1 \leq \min\{c_{A,1/3}^2, 1\}, \tag{74}$$

where $c_{A,\varepsilon/6}$ and $c_{A,1/3}$ are the constant from Lemma 3.6. Then c_1 is a constant depending only on A and ε. We recursively define the functions $\log^{(k)} t = \max\left\{\left(\log\log^{(k-1)} t\right)^4, c_1\right\}$, starting from $\log^{(0)} t = t$. Next, we divide the Minkowski sum in (72) into parts. We set

$$S_1 = \sum_{i=1}^{\lfloor \log_2 \frac{c_1 n}{\log^4 n} \rfloor} \frac{2^{i/2}}{i^{2+\varepsilon}} K_{2^i}^\circ = \sum_{i=1}^{\lfloor \log_2 \frac{c_1 n}{\log^{(1)} n} \rfloor} \frac{2^{i/2}}{i^{2+\varepsilon}} K_{2^i}^\circ, \tag{75}$$

and for any $\mu \geq 2$ such that $\log^{(\mu-1)} n > c_1$, we define

$$S_\mu = \sum_{i=\lfloor \log_2 \frac{c_1 n}{\log^{(\mu-1)} n} \rfloor + 1}^{\lfloor \log_2 \frac{c_1 n}{\log^{(\mu)} n} \rfloor} \frac{2^{i/2}}{i^{2+\varepsilon}} K_{2^i}^\circ \tag{76}$$

(where an empty Minkowski sum, if one occurs, equals the empty set). Thus, to establish (72) it is sufficient to prove that

$$v.rad. \left(\sum_{\mu \geq 1} S_\mu \right) \leq C. \tag{77}$$

We will begin with estimating S_1, the most significant term. Let us fix a positive integer i such that $2^i \leq \frac{c_1 n}{\log^4 n}$. Because of (73), we may apply Lemma 3.6 for $k = 2^i$, for the body K_{2^i}, for $\varepsilon/6$ and for $t = i^{1+\varepsilon/2}$. By the conclusion of that lemma,

$$N\left(2^{i/2} K_{2^i}^\circ, C i^{1+\varepsilon/2} D\right) \leq \exp\left(\frac{Cn}{i^{(1+\varepsilon/2)(1-\varepsilon/6)}}\right) \leq \exp\left(\frac{C' n}{i^{1+\varepsilon/4}}\right).$$

Equivalently,

$$N\left(\frac{2^{i/2}}{i^{2+\varepsilon}} K_{2^i}^\circ, \frac{C}{i^{1+\varepsilon/2}} D\right) \leq \exp\left(\frac{C' n}{i^{1+\varepsilon/4}}\right). \tag{78}$$

The definition of S_1, and the estimates (78) and (56) imply a certain bound on $N\left(S_1, C \sum_i \frac{1}{i^{1+\varepsilon/2}} D\right)$. Note that $\sum_{i=1}^\infty \frac{1}{i^{1+\varepsilon/2}} < C'$. By (56), (75) and (78),

$$N(S_1, C' D) \leq \exp\left(\sum_i \tilde{C} \frac{n}{i^{1+\varepsilon/4}}\right) < e^{C'' n}. \tag{79}$$

Next, we analyze S_μ for any relevant $\mu \geq 2$. Pick a positive integer i such that $\frac{c_1 n}{\log^{(\mu-1)} n} < 2^i \leq \frac{c_1 n}{\log^{(\mu)} n}$. We may use Lemma 3.6 for $k = 2^i$, for the body K_{2^i}, for ε being $\frac{1}{3}$ and for $t = \sqrt{\log^{(\mu)} n}$ (note that (74) gives $t \leq c_{A,1/3} \sqrt{n/k}$, as required). By the conclusion of that lemma,

$$N\left(2^{i/2} K_{2^i}^\circ, C \sqrt{\log^{(\mu)} n} D\right) \leq \exp\left(\frac{C' n}{(\log^{(\mu)} n)^{1/3}}\right).$$

Equivalently (note that $c \log n \leq i \leq C \log n$ when $2^i \geq \frac{c_1 n}{\log^{(1)} n}$),

$$N\left(\frac{2^{i/2}}{i^{2+\varepsilon}} K_{2^i}^\circ, C \frac{\sqrt{\log^{(\mu)} n}}{\log^{2+\varepsilon} n} D\right) \leq \exp\left(\frac{C' n}{(\log^{(\mu)} n)^{1/3}}\right). \tag{80}$$

The number of summands in S_μ is bounded by

$$C\log\left(\log^{(\mu-1)} n\right) = C\left(\log^{(\mu)} n\right)^{1/4}$$

and each of these summands satisfies (80). Consequently, (80), (76) and (56) imply that for any $\mu \geq 2$,

$$N\left(S_\mu, C'\frac{\left(\log^{(\mu)} n\right)^{1/4}\sqrt{\log^{(\mu)} n}}{\log^{2+\varepsilon} n}D\right) \leq \exp\left(\tilde{C}n\frac{\left(\log^{(\mu)} n\right)^{1/4}}{\left(\log^{(\mu)} n\right)^{1/3}}\right). \tag{81}$$

Next, we will estimate the number of balls needed to cover $\sum_{\mu\geq 2} S_\mu$. By (81) and (56),

$$N\left(\sum_{\mu\geq 2} S_\mu, C'\left(\sum_{\mu\geq 2}\frac{\left(\log^{(\mu)} n\right)^{3/4}}{\log^{2+\varepsilon} n}\right)D\right) \leq \exp\left(\tilde{C}n\sum_{\mu\geq 2}\frac{1}{\left(\log^{(\mu)} n\right)^{1/12}}\right). \tag{82}$$

Both numerical sums in (82) are bounded by a universal constant. Thus (82) implies the simpler estimate,

$$N\left(\sum_{\mu\geq 2} S_\mu, \hat{C}D\right) \leq e^{C''n}. \tag{83}$$

The estimates (79) and (83), together with (56), imply that $N\left(\sum_{\mu\geq 1} S_\mu, C'D\right) \leq e^{Cn}$. Hence

$$v.rad.\left(\sum_{\mu\geq 1} S_\mu\right) \leq C$$

and (77) is proved. This completes the proof in the case where $f(0) = 1$ and f is isotropic.

The general case easily follows. Suppose $f : \mathbb{R}^n \to [0,\infty)$ is an even, log-concave function with $\int f = 1$. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that the function $f_u(x) = (\det u)\cdot f(u(x))$ satisfies $f_u(0) = \int f_u = 1$ and f_u is isotropic. Clearly $f(0) = \frac{1}{\det u}$. Moreover, by (29),

$$v.rad.\left(\hat{T}_\varepsilon(f)\right) = (\det u)^{-\frac{1}{n}}\, v.rad.\left(\hat{T}_\varepsilon(f_u)\right) = f(0)^{\frac{1}{n}} v.rad.\left(\hat{T}_\varepsilon(f_u)\right). \tag{84}$$

Note too that $L_{f_u} = L_f < A$. Thus we are in the case we have already treated, and according to the above discussion, $v.rad.(\hat{T}_\varepsilon(f_u)) > C$. The lemma now follows from (84). $\square$

4 Proof of the main result

In this section we complete the proofs of Theorem 1.1 and Theorem 1.3. Our first step is to remove the assumption that L_f is bounded.

Lemma 4.1 *Let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an isotropic, even, log-concave function with $\int f = 1$. Let $M > 0$ be such that $Cov(f) = M^2 Id$. Denote $g(x) = \frac{1}{(2\pi M^2)^{n/2}} \exp\left(-\frac{|x|^2}{2M^2}\right)$. Let $f' = f * g$, the convolution of f and g. Then,*

$$L_{f'} \leq 1 \quad \textit{and} \quad f'(0)^{\frac{1}{n}} \geq \frac{c}{M},$$

where $c > 0$ is a universal constant.

Proof: Since both f and g are even and log-concave, so is their convolution f'. Furthermore, according to our assumptions, $Cov(f) = Cov(g) = M^2 Id$. Consequently,

$$Cov(f') = Cov(f * g) = 2M^2 Id. \tag{85}$$

Next, note that

$$f'(0) = \int_{\mathbb{R}^n} f(x) g(-x) dx \leq \frac{1}{(2\pi M^2)^{n/2}} \int_{\mathbb{R}^n} f(x) dx = \left(\frac{1}{\sqrt{2\pi} M}\right)^n. \tag{86}$$

Clearly $\int f' = 1$. By (85), (86) and the definition (45),

$$L_{f'} = \left(\frac{\sup f'}{\int f'}\right)^{\frac{1}{n}} \det Cov(f')^{\frac{1}{2n}} = f'(0)^{\frac{1}{n}} \sqrt{2} M \leq \frac{1}{\sqrt{\pi}} \leq 1,$$

where we used once more the fact that an even, log-concave function attains its maximum at the origin. By Lemma 3.1, we know that $L_{f'} > c$. Since f' is isotropic, then (85) and the definition (45) imply that

$$f'(0)^{\frac{1}{n}} \geq \frac{c}{\det Cov(f')^{\frac{1}{2n}}} \geq \frac{c'}{M}.$$

The proof is complete. $\square$

Lemma 4.2 *Let $0 < \varepsilon \leq 1$, let $n \geq 4$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, log-concave function with $\int f = 1$. Denote by μ the measure whose density is f. Then,*

$$v.rad.\left(\hat{T}_\varepsilon(f)\right) > c_\varepsilon \frac{f(0)^{\frac{1}{n}}}{L_f}.$$

Furthermore, there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $1 \leq t \leq \sqrt{n} \log^{2+\varepsilon/2} n$,

$$\mu\left(\{x \in \mathbb{R}^n ; |\varphi(x)| > t \|\varphi\|_{L_1(\mu)}\}\right) < e^{-c_\varepsilon \frac{t^2}{\log^{4+\varepsilon}(t+1)}}. \tag{87}$$

Here $c_\varepsilon > 0$ is a constant depending only on ε.

Proof: In this proof, c, C, c' etc. denote constants that depend solely on ε. As we argued before, we may replace f with $f \circ u$, for any $u \in SL_n(\mathbb{R})$, without altering the validity of the conclusions of the lemma. We thus replace f with an appropriate $f \circ u$, and assume from now on that f is isotropic.

Let $M = f(0)^{-\frac{1}{n}} L_f$. Then $Cov(f) = M^2 Id$. Denote $g(x) = \frac{1}{(2\pi M^2)^{n/2}} \exp\left(-\frac{|x|^2}{2M^2}\right)$, and set

$$f' = f * g.$$

Then f' is an even, log-concave function with $\int f' = 1$. Furthermore, $L_{f'} \leq 1$ and $f'(0)^{\frac{1}{n}} \geq c/M$, according to Lemma 4.1. We invoke lemma 3.7, for f', ε and $A = 1$. By the conclusion of that lemma,

$$v.rad.\left(\hat{T}_\varepsilon(f')\right) > \frac{c}{M} = c\frac{f(0)^{\frac{1}{n}}}{L_f}. \tag{88}$$

Note that $\Upsilon(f') = \Upsilon(f) + \Upsilon(g) \geq \Upsilon(f)$. Consequently, $\hat{T}_\varepsilon(f') \subset \hat{T}_\varepsilon(f)$ and hence

$$v.rad.(\hat{T}_\varepsilon(f)) > c\frac{f(0)^{\frac{1}{n}}}{L_f}. \tag{89}$$

This completes the proof of the first part of the lemma. It remains to prove the second part of the lemma. To that end, denote

$$T = \{x \in \mathbb{R}^n; \Upsilon f(x) \leq \log 2\}. \tag{90}$$

Suppose $x \in T$. Since f is even, then

$$\int_{\mathbb{R}^n} \langle x, y\rangle^2 f(y)dy = 2\int_{\{y \in \mathbb{R}^n; \langle x,y\rangle \geq 0\}} \langle x, y\rangle^2 f(y)dy \leq 4\int_{\mathbb{R}^n} e^{\langle x,y\rangle} f(y)dy \leq 4\log 2,$$

where we used the fact that $t^2 \leq 2e^t$ for $t > 0$. Consequently, $x \in T$ implies that $\langle Cov(f)x, x\rangle \leq 4\log 2$. We conclude that

$$v.rad.(T) \leq \sqrt{4\log 2}\frac{1}{\det Cov(f)^{\frac{1}{2n}}} = C\frac{f(0)^{1/n}}{L_f}, \tag{91}$$

where the last equality follows from the definition (45). By comparing (89) with (91), we conclude that

$$\hat{T}_\varepsilon(f) \not\subset c'T.$$

The sets $\hat{T}_\varepsilon(f)$ and T are convex and centrally-symmetric. In particular, there exists $0 \neq x \in \mathbb{R}^n$ such that

$$\forall s \in \mathbb{R}, \quad sx \in T \ \Rightarrow \ c'sx \in \hat{T}_\varepsilon(f). \tag{92}$$

We fix $0 \neq x \in \mathbb{R}^n$ that satisfies (92). Let

$$M = \|\langle\cdot, x\rangle\|_{L_1(\mu)} = \int |\langle x, y\rangle| f(y)dy. \tag{93}$$

It is well-known that

$$\mu\left(\{y \in \mathbb{R}^n; |\langle x, y\rangle| \leq M\}\right) \geq c \tag{94}$$

for some universal constant $c > 0$ (see, e.g., [3, section 3]). Recall Borell's lemma, the ψ_1-estimate (17). From (17) and (93),

$$\int_{\mathbb{R}^n} \exp\left(\frac{\langle x, y\rangle}{CM}\right) f(y)dy \leq 2.$$

Thus, $\frac{x}{CM} \in T$, by the definition (90). According to (92),

$$\frac{x}{C'M} \in \hat{T}_\varepsilon(f). \tag{95}$$

We may apply Lemma 3.2, based on (95). We conclude that for any $1 \leq t \leq \sqrt{n}\log^{2+\varepsilon/2} n$,

$$\mu\left(\{y \in \mathbb{R}^n; |\langle x, y\rangle| > CMs\}\right) \leq e^{-c'\frac{t^2}{\log^{4+\varepsilon}(t+1)}}. \tag{96}$$

By using (93) and (94), we see that (96) is equivalent to the desired estimate (87). This completes the proof. $\square$

We still need to take care of the range $t \geq C\sqrt{n}\log^{2+\varepsilon/2} n$. The following lemma serves this purpose.

Lemma 4.3 *Let $\alpha, A > 0$, let $n \geq \frac{2}{A}$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an αn-concave function with $\int f = 1$ whose barycenter lies at the origin. Let $x \in \mathbb{R}^n, b > 0$, and denote $K = \{y \in \mathbb{R}^n; |\langle x, y\rangle| \geq b\}$. Suppose that*

$$\int_K f(x)dx \leq e^{-An}.$$

Then

$$\sup_{y \in Supp(f)} |\langle x, y\rangle| \leq C_{A,\alpha} b$$

where $C_{A,\alpha} > 0$ is a constant depending solely on A and α, and as usual, $Supp(f)$ is the closure of the set $\{x \in \mathbb{R}^n; f(x) \neq 0\}$.

Proof: For $t \in \mathbb{R}$ denote

$$\psi(t) = \left(\int_{\{y \in \mathbb{R}^n; \langle x, y\rangle \geq t\}} f(y)dy\right)^{\frac{1}{(\alpha+1)n}}.$$

According to (18), the function ψ is concave on its support. The function f is log-concave, $\int f = 1$ and the barycenter of f lies at the origin. It is well-known that a hyperplane through the origin divides $\mathbb{R}^n$ into two half-spaces, on each of which the integral of f is not smaller than $\frac{1}{e}$. This fact, essentially going back to Grünbaum and to

Hammer [15], is proven e.g. in [3, Lemma 3.3]. Thus, we know that $\psi(0) \geq \left(\frac{1}{e}\right)^{1/(\alpha+1)n}$. Our assumptions imply that

$$\psi(0) \geq \left(\frac{1}{e}\right)^{\frac{1}{(\alpha+1)n}} \geq e^{-\frac{A}{2(\alpha+1)}}, \quad \psi(b) \leq e^{-\frac{A}{\alpha+1}}. \tag{97}$$

Since ψ is concave on its support, then (97) implies that

$$\psi\left(\left(1 - e^{-\frac{A}{2(\alpha+1)}}\right)^{-1} b\right) = 0.$$

Thus $Supp(f) \subset \{y \in \mathbb{R}^n; \langle x, y\rangle \leq C_{A,\alpha} b\}$, for $C_{A,\alpha} = \left(1 - e^{-\frac{A}{2(\alpha+1)}}\right)^{-1}$. Repeating the argument for $-x$ in place of x, the lemma follows. □

Lemma 4.4 *Let $0 < \varepsilon \leq 1, \alpha > 0$, let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an even, αn-concave function with $\int f = 1$. Denote by μ the measure whose density is f. Then there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $t \geq 1$,*

$$\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| > t\|\varphi\|_{L_1(\mu)}\}\right) < e^{-c_{\varepsilon,\alpha}\frac{t^2}{\log^{4+\varepsilon}(t+1)}} \tag{98}$$

where $c_{\varepsilon,\alpha} > 0$ is a constant depending only on ε and α.

Proof: In this proof, $c_1, c_2, C, C', \tilde{C}$ etc. denote positive constants that depend solely on ε and α. An αn-concave function is clearly log-concave. First, suppose that $n \geq 4$. Thus we may apply Lemma 4.2. By the conclusion of that lemma, there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $1 \leq t \leq \sqrt{n}\log^{2+\varepsilon/2} n$,

$$\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| > t\|\varphi\|_{L_1(\mu)}\}\right) \leq e^{-c\frac{t^2}{\log^{4+\varepsilon}(t+1)}}. \tag{99}$$

Substituting $t = \sqrt{n}\log^{2+\varepsilon/2} n$ in (99), we obtain

$$\mu\left(\left\{x \in \mathbb{R}^n; |\varphi(x)| > \sqrt{n}(\log n)^{2+\varepsilon/2}\|\varphi\|_{L_1(\mu)}\right\}\right) \leq e^{-c_1 n}. \tag{100}$$

Consider first the case $n \geq \frac{2}{c_1}$, where c_1 is the constant from (100). We may apply Lemma 4.3, based on (100), and deduce that

$$\mu\left(\left\{x \in \mathbb{R}^n; |\varphi(x)| \geq C_2\sqrt{n}(\log n)^{2+\varepsilon/2}\|\varphi\|_{L_1(\mu)}\right\}\right) = 0. \tag{101}$$

From (99) and (101) we conclude that for any $t \geq 1$,

$$\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| > t\|\varphi\|_{L_1(\mu)}\}\right) \leq e^{-C'\frac{t^2}{\log^{4+\varepsilon}(t+1)}}, \tag{102}$$

for an appropriate constant C' depending only on α and ε. The inequality (102) is precisely the desired estimate (98). The lemma is thus proven, for the case where $n \geq \max\{\frac{2}{c_1}, 4\}$.

To deal with the case $1 \leq n \leq \max\{\frac{2}{c_1}, 4\}$, simply note that $\|\varphi\|_{L_\infty(\mu)} \leq \tilde{C}n\|\varphi\|_{L_1(\mu)}$, and hence (98) trivially holds in this degenerate case. □

Next, we remove the assumption that the functions are even.

Theorem 4.5 *Let $0 < \varepsilon \leq 1$, let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $\int f = 1$. Denote by μ the measure whose density is f. Then there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $1 \leq t \leq \sqrt{n} \log^{2+\varepsilon/2} n$,*

$$\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| > t\|\varphi\|_{L_1(\mu)}\}\right) < e^{-c_\varepsilon \frac{t^2}{\log^{4+\varepsilon}(t+1)}} \tag{103}$$

where $c_\varepsilon > 0$ is a constant depending only on ε.

Proof: In this proof, $c, C, c', \tilde{c}$ etc. represent constants depending only on ε. Let $f'(x) = f(-x)$, and consider the convolution $g = f * f'$. The function $g : \mathbb{R}^n \to [0, \infty)$ is an even, log-concave function with $\int g = 1$. Denote by ν the measure whose density is g. We may suppose that $n \geq 4$ (see the final sentence in the proof of Lemma 4.4). According to Lemma 4.2, there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $1 \leq t \leq \sqrt{n} \log^{2+\varepsilon/2} n$,

$$\nu\left(\{x \in \mathbb{R}^n; |\varphi(x)| > t\|\varphi\|_{L_1(\nu)}\}\right) < e^{-c \frac{t^2}{\log^{4+\varepsilon}(t+1)}}. \tag{104}$$

Let $M > 0$ be such that

$$\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| \leq M\}\right) = \frac{2}{3}.$$

By Borell's lemma (17), we know that $c\|\varphi\|_{L_2(\mu)} \leq M$. Consequently,

$$M \geq c\|\varphi\|_{L_2(\mu)} \geq c\sqrt{Var_\mu(\varphi)} = c\sqrt{\frac{Var_\nu(\varphi)}{2}} = \frac{c}{\sqrt{2}}\|\varphi\|_{L_2(\nu)} \geq \frac{c}{\sqrt{2}}\|\varphi\|_{L_1(\nu)}, \tag{105}$$

where for any probability measure ρ, we write $Var_\rho(\varphi) = \int_{\mathbb{R}^n} \varphi^2(y) d\rho(y) - \left(\int_{\mathbb{R}^n} \varphi(y) d\rho(y)\right)^2$, the variance of φ with respect to the measure ρ. Let X and Y be independent random vectors, that are distributed according to the densities f and f', respectively. Then the random vector $X + Y$ is distributed according to ν. Note that $Prob\{|\varphi(Y)| \leq M\} = \frac{2}{3}$. Consequently, by (105), for any $1 \leq t \leq \sqrt{n} \log^{2+\varepsilon/2} n$,

$$\begin{aligned} &\frac{2}{3} Prob\{|\varphi(X)| \geq (t+1)M\} \qquad (106) \\ = \;& Prob\{|\varphi(X)| \geq (t+1)M, |\varphi(Y)| \leq M\} \leq Prob\{|\varphi(X+Y)| \geq tM\} \\ \leq \;& Prob\left\{|\varphi(X+Y)| \geq ct\|\varphi\|_{L_1(\nu)}\right\} \leq e^{-\tilde{c}\frac{t^2}{\log^{4+\varepsilon}(t+1)}}, \end{aligned}$$

where we used (104) for the last inequality. From (106) we deduce (103), for $C < t < c\sqrt{n} \log^{2+\varepsilon/2} n$. Recall that $\mu(\{x \in \mathbb{R}^n; |\varphi(x)| \geq \|\varphi\|_{L^1(\mu)}\}) \leq 1 - c$ (see, e.g., the methods in [3, section 3]). Thus, we may adjust the constants, so that (103) will hold for all $1 \leq t \leq \sqrt{n} \log^{2+\varepsilon/2} n$. The proof is complete. $\square$

Theorem 1.3 is the case $\varepsilon = 1$ of the following theorem.

Theorem 4.6 *Let $\alpha > 0, 0 < \varepsilon \leq 1$, let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an αn-concave function with $\int f = 1$. Denote by μ the probability measure whose density is f. Then there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $t \geq 1$,*

$$\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| > t\|\varphi\|_{L_1(\mu)}\}\right) < e^{-c_{\alpha,\varepsilon}\frac{t^2}{\log^{4+\varepsilon}(t+1)}} \tag{107}$$

where $c_{\alpha,\varepsilon} > 0$ is a constant that depends only on α and ε.

Proof: In this proof, c, c_1, C, c' etc. denote positive constants that depend only on α and ε. Let $f'(x) = f(-x)$, and consider the convolution $g = f * f'$. The function $g : \mathbb{R}^n \to [0, \infty)$ is an even, $(2\alpha + 1)n$-concave function.

Denote by ν the measure whose density is g. According to Lemma 4.4, there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $t \geq 1$,

$$\nu\left(\{x \in \mathbb{R}^n; |\varphi(x)| > t\|\varphi\|_{L_1(\nu)}\}\right) < e^{-c\frac{t^2}{\log^{4+\varepsilon}(t+1)}}.$$

Let $M > 0$ be such that $\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| \leq M\}\right) = \frac{2}{3}$. Arguing as in the proof of Theorem 4.5, we find that $M \geq c\|\varphi\|_{L_1(\nu)}$. Let X and Y be independent random vectors, that are distributed according to the densities f and f', respectively. Then the random vector $X + Y$ is distributed according to the density g. As in the proof of Theorem 4.5, we conclude that for any $t \geq 1$,

$$\frac{2}{3} Prob\{|\varphi(X)| \geq (t+1)M\} \leq Prob\left\{|\varphi(X+Y)| \geq ct\|\varphi\|_{L_1(\nu)}\right\} \leq e^{-\tilde{c}\frac{t^2}{\log^{4+\varepsilon}(t+1)}}.$$

By adjusting the constants, we conclude that for any $t \geq 1$,

$$\mu\left(\{x \in \mathbb{R}^n; |\varphi(x)| > t\|\varphi\|_{L_1(\mu)}\}\right) < e^{-\hat{c}\frac{t^2}{\log^{4+\varepsilon}(t+1)}}.$$

The theorem is thus proven. $\square$

Theorem 1.1 is the case $\varepsilon = 1$ of the following theorem.

Theorem 4.7 *Let $0 < \varepsilon \leq 1$, let $n \geq 1$ be an integer, and let $K \subset \mathbb{R}^n$ be a convex body of volume one. Then there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that for any $t \geq 1$,*

$$Vol_n\left(\{x \in K; |\varphi(x)| > t\|\varphi\|_{L_1(K)}\}\right) < e^{-c_\varepsilon\frac{t^2}{\log^{4+\varepsilon}(t+1)}}. \tag{108}$$

Equivalently, denote $\psi_{2^-}(t) = \exp\left(\frac{t^2}{\log^{4+\varepsilon}(t+5)}\right) - 1$. Then the linear functional φ satisfies

$$\|\varphi\|_{L_{\psi_{2^-}}(K)} \leq C_\varepsilon\|\varphi\|_{L_1(K)}.$$

Here, $C_\varepsilon, c_\varepsilon > 0$ are constants that depend only on ε.

Proof: The function 1_K is an n-concave function. Thus, according to Theorem 4.6, there exists a non-zero linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ that satisfies (108). To establish the second part, note that for $M = \|\varphi\|_{L_1(K)}$,

$$\begin{aligned}
&\int_K \exp\left[\left(\frac{\varphi(x)}{C_1 M}\right)^2 \Big/ \log^{4+\varepsilon}\left(\frac{|\varphi(x)|}{C_1 M}+5\right)\right] dx \\
= &\int_0^\infty Vol_n\left(\left\{x \in K; \left(\frac{\varphi(x)}{C_1 M}\right)^2 \Big/ \log^{4+\varepsilon}\left(\frac{|\varphi(x)|}{C_1 M}+5\right) > \log t\right\}\right) dt \\
\leq &\int_0^\infty \min\left\{1, \frac{1}{t^{cC_1}}\right\} dt \leq 2
\end{aligned}$$

for a suitable constant $C_1 > 0$. The theorem is thus proven. □

5 Using ℓ-position

In this section we employ the fundamental properties of the ℓ-position, see e.g. [31, sections 2 and 3]. We will prove variants of Theorem 1.1 and Theorem 1.3, where rather than asserting the existence of a linear functional with a certain property, we will show that a "random" linear functional possesses this property. See Theorem 5.1 and Theorem 5.2 below for the exact formulation.

Denote by σ_n the unique rotation invariant probability measure on the unit sphere S^{n-1}. Let $K \subset \mathbb{R}^n$ be a centrally-symmetric convex body. For $x \in \mathbb{R}^n$ we denote $\|x\|_K = \inf\{\lambda > 0; x \in \lambda K\}$. Then $\|\cdot\|_K$ is the norm whose unit ball is K. We define

$$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_n(x).$$

A well-known consequence of Hölder's inequality is that $M(K)v.rad.(K) \geq 1$. In general, $M(K)$ may be much larger than $1/v.rad.(K)$. However, a useful theorem (see, e.g., [27, sections 14 and 15] or [31, sections 2 and 3]) states that for any centrally-symmetric convex body $K \subset \mathbb{R}^n$, there exists a linear transformation $u \in SL_n(\mathbb{R})$ such that

$$M(u(K)) \leq C \frac{\log n}{v.rad.(u(K))}. \tag{109}$$

It is customary to say that this linear map u transforms K into ℓ-position, or that the body $u(K)$ is in ℓ-position.

Theorem 5.1 *Let $0 < \varepsilon \leq 1$, let $n \geq 1$, and let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $\int f = 1$, such that $\int x f(x) dx = 0$. Then there exists $u \in SL_n(\mathbb{R})$ for which the following holds.*

Denote $\tilde{f} = f \circ u$, and let μ be the measure whose density is $\tilde{f}$. Then, there exists $\Theta \subset S^{n-1}$ with $\sigma_n(\Theta) \geq \frac{4}{5}$, such that for any $\theta \in \Theta$, and for any $1 \leq t \leq \sqrt{n}\log^{3+\varepsilon/2} n$,

$$\mu\left(\{x \in \mathbb{R}^n; |\langle x, \theta\rangle| > t\|\langle \cdot, \theta\rangle\|_{L_1(\mu)}\}\right) < e^{-c_\varepsilon \frac{t^2}{\log^2(n+1)\log^{4+\varepsilon}(t+1)}}$$

where $c_\varepsilon > 0$ is a constant depending only on ε.

Proof: In this proof, the letters $c, C, \tilde{C}$ etc. stand for constants depending solely on ε. We define $f'(x) = f(-x)$ and $S(f)(x) = (f * f')(x) = \int_{\mathbb{R}^n} f(y)f(y-x)dy$. Then $S(f)$ is an even, log-concave function with $\int S(f) = 1$. Clearly, for any map $u \in SL_n(\mathbb{R})$ we have $S(f) \circ u = S(f \circ u)$. Recall the definition (46) of $\hat{T}_\varepsilon$. Then

$$\hat{T}_\varepsilon(S(f \circ u)) = u^* \left(\hat{T}_\varepsilon(S(f)) \right).$$

The set $\hat{T}_\varepsilon(S(f))$ is convex and centrally-symmetric. Let $u \in SL_n(\mathbb{R})$ be such that $u^* \left(\hat{T}_\varepsilon(S(f)) \right)$ is in ℓ-position. We write $\tilde{f} = f \circ u$.

To simplify the notation, we denote $g = S(\tilde{f})$. According to Lemma 4.2,

$$v.rad.(\hat{T}_\varepsilon(g)) \geq c \frac{g(0)^{1/n}}{L_g}.$$

By our assumption, $\hat{T}_\varepsilon(g)$ is in ℓ-position. Therefore,

$$M(\hat{T}_\varepsilon(g)) \leq C \frac{L_g \log n}{g(0)^{1/n}}. \tag{110}$$

The inequality (110) implies the existence of a set $\Theta_1 \subset S^{n-1}$ with $\sigma_n(\Theta_1) \geq \frac{9}{10}$, such that for each $\theta \in \Theta_1$,

$$c \frac{g(0)^{1/n}}{L_g \log n} \theta \in \hat{T}_\varepsilon(g). \tag{111}$$

Denote $T = \{x \in \mathbb{R}^n ; \Upsilon g(x) \leq \log 2\}$. Arguing as in (90) and (91), we know that

$$v.rad.(T) \leq c \frac{g(0)^{1/n}}{L_g}.$$

The upper bound on $v.rad.(T)$ implies the existence of $\Theta_2 \subset S^{n-1}$ with $\sigma_n(\Theta_2) \geq \frac{9}{10}$, such that for each $\theta \in \Theta_2$,

$$C \frac{g(0)^{1/n}}{L_g} \theta \notin T \tag{112}$$

(otherwise, the convex hull of $T \cap C\frac{g(0)^{1/n}}{L_g} S^{n-1}$ and the origin, that is contained in T, would have too large a volume). Let us set $\Theta = \Theta_1 \cap \Theta_2$. Then $\sigma_n(\Theta) \geq \frac{4}{5}$. We compare (111) and (112), and conclude that for any $\theta \in \Theta$, we have

$$\forall s \in \mathbb{R}, \quad s\theta \in T \;\Rightarrow\; \frac{c's}{\log n}\theta \in \hat{T}_\varepsilon(g). \tag{113}$$

All that remains for us to show, is that for any $\theta \in \Theta$, and for any $1 \leq t \leq \sqrt{n} \log^{3+\varepsilon} n$,

$$\mu\left(\{x \in \mathbb{R}^n ; |\langle x, \theta \rangle| > t \|\langle \cdot, \theta \rangle\|_{L_1(\mu)}\}\right) < e^{-c \frac{t^2}{\log^2(n+1) \log^{4+2\varepsilon}(t+1)}}. \tag{114}$$

We thus focus our attention on establishing (114). Fix $\theta \in \Theta$ and set

$$M = \|\langle \cdot, \theta \rangle\|_{L_1(\mu)} = \int_{\mathbb{R}^n} |\langle x, \theta \rangle| \tilde{f}(x) dx.$$

Denote by ν the measure whose density is g. Arguing as in (105) above, we find that

$$M \geq C \|\langle \cdot, \theta \rangle\|_{L_1(\nu)}.$$

Borell's lemma (17) implies now that $\frac{\theta}{CM} \in T$, and by (113) we deduce that

$$\frac{\theta}{C' M \log n} \in \hat{T}_\varepsilon(g). \tag{115}$$

We may apply Lemma 3.2, based on (115). We conclude that for any $1 \leq s \leq \sqrt{n} \log^{2+\varepsilon/2} n$,

$$\nu \left(\{ y \in \mathbb{R}^n ; |\langle x, y \rangle| > CM(\log n) s \} \right) \leq e^{-C' \frac{s^2}{\log^{4+\varepsilon}(s+1)}}. \tag{116}$$

The estimate (116) is very close to our desired inequality (114), the only difference is that the measure μ is replaced by ν. Next, we repeat the argument (106) from the proof of Theorem 4.5, and deduce (114) from (116).

Let X and Y be independent random vectors that are distributed according to the densities $\tilde{f}$ and $\tilde{f}'(x) = \tilde{f}(-x)$, respectively. Then the random vector $X + Y$ is distributed according to the density g. Note that $Prob\{|\langle Y, \theta \rangle| \leq 3M\} \geq \frac{2}{3}$. Consequently, for any $C \log^2 n \leq s \leq \sqrt{n} \log^{3+\varepsilon} n$,

$$Prob \left\{ |\langle X, \theta \rangle| \geq (s+1) M \right\} \leq \frac{3}{2} Prob \left\{ |\langle X + Y, \theta \rangle| \geq sM \right\} \leq e^{-C' \frac{s^2}{\log^2(n+1) \log^{4+2\varepsilon}(s+1)}}.$$

Now (114) follows easily, in the range $C \log^2 n \leq t \leq c\sqrt{n} \log^{3+\varepsilon} n$. Note that (114) in the range $1 \leq t \leq C \log^2 n$ is redundant, and (17) provides a better estimate in that range. The proof is complete. □

Theorem 5.2 *Let $0 < \varepsilon \leq 1, \alpha > 0$, let $n \geq 1$ be an integer, and let $f : \mathbb{R}^n \to [0, \infty)$ be an αn-concave function with $\int f = 1$, such that $\int x f(x) dx = 0$. Then there exists $u \in SL_n(\mathbb{R})$ for which the following holds.*

Denote $\tilde{f} = f \circ u$, and let μ be the measure whose density is $\tilde{f}$. Then, there exists $\Theta \subset S^{n-1}$ with $\sigma_n(\Theta) \geq \frac{4}{5}$, such that for any $\theta \in \Theta$, and for any $t \geq 1$,

$$\mu \left(\{ x \in \mathbb{R}^n ; |\langle x, \theta \rangle| > t \|\langle \cdot, \theta \rangle\|_{L_1(\mu)} \} \right) < e^{-c_{\varepsilon,\alpha} \frac{t^2}{\log^2(n+1) \log^{4+\varepsilon}(t+1)}}$$

where $c_{\varepsilon,\alpha} > 0$ is a constant depending only on ε and α.

Proof: The argument is similar to the proof of Lemma 4.4. We outline the details. In this proof we denote by $c, C, c', \hat{c}$ etc. constants depending only on ε and α. An αn-concave function is also log-concave. We thus may employ Theorem 5.1. Let $u \in SL_n(\mathbb{R})$ be the linear map from the conclusion of that theorem. We denote $\tilde{f} = f \circ u$, and note that the barycenter of $\tilde{f}$ lies at the origin.

By the conclusion of Theorem 5.1, there exists $\Theta \subset S^{n-1}$ with $\sigma_n(\Theta) \geq \frac{4}{5}$, such that for any $\theta \in \Theta$, and for any $1 \leq t \leq \sqrt{n} \log^{3+\varepsilon/2} n$,

$$\mu\left(\{x \in \mathbb{R}^n; |\langle x, \theta \rangle| > t \|\langle \cdot, \theta \rangle\|_{L_1(\mu)}\}\right) < e^{-c \frac{t^2}{\log^2(n+1) \log^{4+\varepsilon}(t+1)}}. \tag{117}$$

We will substitute $t = \sqrt{n} \log^{3+\varepsilon/2} n$ in (117) and then use Lemma 4.3 (similarly to (100) and (101)). By the conclusion of that lemma, for any $\theta \in \Theta$,

$$\mu\left(\left\{x \in \mathbb{R}^n; |\langle x, \theta \rangle| \geq C_2 \sqrt{n} (\log n)^{3+\varepsilon/2} \|\langle \cdot, \theta \rangle\|_{L_1(\mu)}\right\}\right) = 0, \tag{118}$$

under the assumption that $n \geq C$. The theorem follows From (117) and (118). $\square$

Recall that the characteristic function of a convex set is trivially n-concave. The following corollary is equivalent to the case $\varepsilon = \alpha = 1$ in Theorem 5.2.

Corollary 5.3 *Let $n \geq 1$ be an integer, and let $K \subset \mathbb{R}^n$ be a convex body of volume one, whose barycenter lies at the origin. Then there exists $u \in SL_n(\mathbb{R})$ for which the following holds.*

Denote $\tilde{K} = u(K)$. Then, there exists $\Theta \subset S^{n-1}$ with $\sigma_n(\Theta) \geq \frac{4}{5}$, such that for any $\theta \in \Theta$, and for any $t \geq 1$,

$$Vol_n\left(\left\{x \in \tilde{K}; |\langle x, \theta \rangle| > t \|\langle \cdot, \theta \rangle\|_{L_1(\tilde{K})}\right\}\right) < e^{-c \frac{t^2}{\log^2(n+1) \log^5(t+1)}}$$

where $c > 0$ is a universal constant.

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