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On the hyperplane conjecture for random convex sets

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Abstract

Let $N \geq n+1$, and denote by $\mathcal{K}$ the convex hull of N independent standard gaussian random vectors in $\mathbb{R}^n$. We prove that with high probability, the isotropic constant of $\mathcal{K}$ is bounded by a universal constant. Thus we verify the hyperplane conjecture for the class of gaussian random polytopes.

1 Introduction

The hyperplane conjecture suggests a positive answer to the following question: Is there a universal constant $c > 0$, such that for any dimension n and for any convex set $\mathcal{K} \subset \mathbb{R}^n$ of volume one, there exists at least one hyperplane $\mathcal{H} \subset \mathbb{R}^n$ with $\mathrm{Vol}_{n-1}(\mathcal{K} \cap \mathcal{H}) > c$? Here, of course, Vol_{n-1} denotes $(n-1)$-dimensional volume.

This seemingly innocuous question, considered two decades ago by Bourgain [3, 4], has not been answered yet. We refer the reader to, e.g., [2], [18] or [14] for partial results, history and for additional literature regarding this hyperplane conjecture. In particular, there are large classes of convex bodies for which an affirmative answer to the above question is known. These include unconditional convex bodies [3, 18], zonoids, duals to zonoids [2], bodies with a bounded outer volume ratio [18], unit balls of Schatten norms [15] and others (e.g., [11, 17]).

A potential counter-example to the hyperplane conjecture could have stemmed from random convex bodies, that typically belong to none of these classes. Recall that, starting with Gluskin's work [8], random polytopes are a major source of counter examples in high-dimensional convex geometry (in addition to the distance problem [8], one has, e.g., the basis problem [23] or the gaussian perimeter problem [19]). The goal of this short note is to show that gaussian random polytopes, and related models of random convex sets, do not constitute a counter-example to the hyperplane conjecture.

Suppose $\mathcal{K} \subset \mathbb{R}^n$ is a convex body. The isotropic constant of $\mathcal{K}$, denoted here by $L_{\mathcal{K}}$, is defined by

$$nL_{\mathcal{K}}^2 = \inf_{T:\mathbb{R}^n \to \mathbb{R}^n} \frac{1}{\mathrm{Vol}_n(\mathcal{K})^{1+\frac{2}{n}}} \int_{\mathcal{K}} |Tx|^2 dx, \tag{1}$$

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where the infimum runs over all volume-preserving affine maps $T : \mathbb{R}^n \to \mathbb{R}^n$, and $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^n$. Directly from the definition, the isotropic constant is invariant under affine transformations. It is well-known (see [10] or [18]) that when $\mathrm{Vol}_n(\mathcal{K}) = 1$,

$$\frac{c_1}{L_{\mathcal{K}}} \leq \inf_{T:\mathbb{R}^n\to\mathbb{R}^n} \sup_{\mathcal{H}\subset\mathbb{R}^n} \mathrm{Vol}_{n-1}(T\mathcal{K}\cap\mathcal{H}) \leq \frac{c_2}{L_{\mathcal{K}}}, \tag{2}$$

where the infimum runs over all volume-preserving affine maps $T : \mathbb{R}^n \to \mathbb{R}^n$, the supremum runs over all hyperplanes $\mathcal{H} \subset \mathbb{R}^n$, and $c_1, c_2 > 0$ are some universal constants. Throughout the text, the symbols c, C, c', C', c_1, c_2 etc. denote various positive universal constants, whose value may change from one line to the next.

Thus, according to (2), the hyperplane conjecture is equivalent to the existence of a universal upper bound for the isotropic constant of an arbitrary convex body in an arbitrary dimension. It is well-known that $L_{\mathcal{K}} > c$ for any finite-dimensional convex body $\mathcal{K}$ (see, e.g., [18]). The best known general upper bound is $L_{\mathcal{K}} < Cn^{1/4}$ for a convex body $\mathcal{K} \subset \mathbb{R}^n$ (see [14] and also [5], [6] and [7]).

There are two natural models for random convex bodies (a body is a compact with non-empty interior). In the centrally-symmetric context, the first model is the symmetric convex hull of N random independent points, while the second (its dual), is the intersection of N random strips. For the second model, however, it is quite easy to demonstrate the hyperplane conjecture. Indeed, if $N \geq 2n$, then a simple calculation shows that it has a bounded outer volume ratio, and hence has a bounded isotropic constant (see [18]). If $n \leq N < 2n$, then the hyperplane conjecture holds deterministically whenever the resulting set is a body, according to [12]. Thus we will focus on the first model.

We say that a random vector $X = (X_1, ..., X_n) \in \mathbb{R}^n$ is a standard gaussian vector if its coordinates $X_1, ..., X_n$ are independent, standard normal variables.

Theorem 1.1 *Let $n \geq 1, N \geq n$ and let $G_0, ..., G_N$ be independent standard gaussian vectors in $\mathbb{R}^n$. Denote*

$$\mathcal{K} = \mathrm{conv}\{G_0, ..., G_N\} \quad \textit{and} \quad \mathcal{T} = \mathrm{conv}\{\pm G_1, ..., \pm G_N\}$$

where conv *denotes convex hull. Then, with probability greater than $1 - Ce^{-cn}$,*

$$L_{\mathcal{K}} < C \quad \textit{and} \quad L_{\mathcal{T}} < C.$$

Here, $C, c > 0$ are universal constants.

Let us sketch the proof in the case that $N > n^2$ (for smaller N the argument is only slightly more opaque). It is easy to see that in this case the radius of the inscribed ball is, with high probability, $\geq c\sqrt{\log N}$ — just calculate the probability that in a given direction all points are inside a strip of width $\epsilon\sqrt{\log N}$ and do a union-bound over a dense net in S^{n-1}. On the other hand, with probability 1 all faces of $\mathcal{K}$ are $(n-1)$-dimensional simplices. Further,

with high probability all centers of gravity of all simplices are with distance $\leq C\sqrt{\log N}$ from 0 — the center of gravity of each simplex is a gaussian vector whose coordinates have variance $1/n$, and again all that is needed is to do a union-bound over all n-tuples of vertices. The concentration of the volume of a simplex around its center of gravity shows that almost all the mass is within distance of $\approx \sqrt{\log N}$ which implies the required estimate for $L_{\mathcal{K}}$.

On first glance it seems that some miracle is at work — why should we get the same $\sqrt{\log N}$ in the lower and upper bound? However, this is some manifestation of the phenomenon that the maximum of many independent variables is strongly concentrated. Of course, the different faces of our body are not independent, but it turns out that they are sufficiently independent to display a similar concentration phenomenon. Thus our proof is robust and would admit direct generalizations to other types of distributions, in place of the standard gaussian distribution. We will prove it for some other distributions that include the uniform distribution on the cube and on its corners $\{\pm 1\}^n$, see Theorem 3.2 below. The technique should also work for points uniform on S^{n-1}.

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2 Simplices

In this section, we assume that $N \geq n \geq 1$ are integers, and that $G_0, ..., G_N$ are independent random vectors in $\mathbb{R}^n$ which need not be identically distributed. We write $G_i = (G_{i,1}, ..., G_{i,n}) \in \mathbb{R}^n$, and we make the following assumptions regarding $G_0, ..., G_N$:

($*$a) The random variables $G_{i,j}$ $(i = 0, ..., N,\ j = 1, ..., n)$ are independent.

($*$b) For any $i = 0, ..., N$, $j = 1, ..., n$,

$$\mathbb{E}G_{i,j} = 0,\ \mathbb{E}G_{i,j}^2 = 1 \ \text{ and } \ \mathbb{E}\exp\left(G_{i,j}^2/10\right) \leq 10.$$

($*$c) The G_i $(i = 0, ..., N)$ are absolutely continuous.

The constant 10 plays no special rôle. Note that ($*$a), ($*$b) and ($*$c) hold when $G_0, ..., G_N$ are independent standard gaussian vectors. Our main technical tool is the following Bernstein's inequality for variables with exponential tail ("ψ_1"), see, e.g. [26, Section 2.2.2].

Theorem 2.1 *Let $L > 0$, let $m \geq 1$ be an integer, and let $X_1, ..., X_m$ be independent random variables with zero mean. Assume that*

$$\mathbb{E}\exp(|X_i|/L) \leq 20 \quad \textit{for} \ \ 1 \leq i \leq m.$$

Then, for any $t > 0$,

$$\mathbb{P}\left\{\left|\frac{1}{m}\sum_{i=1}^{m} X_i\right| > t\right\} \leq 2\exp\left(-cm\min\left\{\frac{t}{L}, \frac{t^2}{L^2}\right\}\right),$$

where $c > 0$ is a universal constant.

The following lemma is a consequence of Theorem 2.1.

Lemma 2.2 *Fix $0 \le k_1 < k_2 < \cdots < k_n \le N$. Then,*

(i) $\mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} G_{k_i}\right| > C\sqrt{\log\frac{2N}{n}}\right\} < \left(\frac{n}{10N}\right)^n,$

(ii) $\mathbb{P}\left\{\left|\frac{1}{N+1}\sum_{i=0}^{N} G_{i}\right| > C\sqrt{\log\frac{2N}{n}}\right\} < \left(\frac{n}{10N}\right)^n,$

(iii) $\mathbb{P}\left\{\frac{1}{n^2}\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n} G_{k_i,j}\right)^2 + \sum_{j=1}^{n} G_{k_i,j}^2\right] > C\log\frac{2N}{n}\right\} < \left(\frac{n}{10N}\right)^n.$

Here, $C > 0$ is a universal constant.

Proof. Fix $1 \le i, j \le n$. We have, for any t,

$$\mathbb{E}\exp(tG_{k_i,j}) \le \exp(\tfrac{5}{2}t^2)\mathbb{E}\exp(G_{k_i,j}^2/10) \overset{(*\mathrm{b})}{\le} 10\exp(\tfrac{5}{2}t^2).$$

By independence, for any $j = 1, .., n$ and t,

$$\mathbb{E}\exp\left(t\sum_{i=1}^{n} G_{k_i,j}\right) \le 10\exp\left(\tfrac{5}{2}nt^2\right),$$

so

$$\mathbb{E}\exp\left(\left|t\sum_{i=1}^{n} G_{k_i,j}\right|\right) \le \mathbb{E}\exp\left(t\sum_{i=1}^{n} G_{k_i,j}\right) + \exp\left(-t\sum_{i=1}^{n} G_{k_i,j}\right) \le 20\exp(\tfrac{5}{2}nt^2). \tag{3}$$

Denote $Y_j = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} G_{k_i,j}$ $(j = 1, ..., n)$. Then the Y_j are independent random variables of mean zero and variance one. Moreover, by (3), for $j = 1, ..., n$,

$$\mathbb{E}\exp\left(\frac{|Y_j^2 - 1|}{100}\right) \le 2\mathbb{E}\exp\left(\frac{Y_j^2}{100}\right) 2\int_0^\infty \mathbb{P}\left\{\frac{Y_j^2}{100} > \log t\right\} dt \le$$

$$\le 2 + 2\int_1^\infty t^{-10}\mathbb{E}e^{\sqrt{\log t}|Y_j|}\, dt \overset{(3)}{\le} 2 + 40\int_1^\infty t^{-15/2}\, dt < 9. \tag{4}$$

Hence, we may apply Theorem 2.1 for the independent random variables $Y_j^2 - 1$, with $L = 100$. We conclude that

$$\mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} G_{k_i}\right| > C\sqrt{\log\frac{2N}{n}}\right\}\mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{n} Y_j^2 > C^2\log\frac{2N}{n}\right\} \le$$

$$\le \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{n}[Y_j^2 - 1] > (C/2)^2\log\frac{2N}{n}\right\} \le 2\exp\left(-cn\frac{C^2}{400}\cdot\log\frac{2N}{n}\right) < \left(\frac{n}{10N}\right)^n,$$

for an appropriate choice of a large universal constant $C > 50$. This completes the proof of (i). The argument that leads to (ii) is similar. We define $\tilde{Y}_j = \frac{1}{\sqrt{N+1}} \sum_{i=0}^{N} G_{i,j} \quad (j = 1, ..., n)$. Arguing exactly as above, we find that for $j = 1, ..., n$,

$$\mathbb{E} \exp\left(\frac{|\tilde{Y}_j^2 - 1|}{100} \right) < 9.$$

We may invoke Theorem 2.1 for the independent random variables $\tilde{Y}_j^2 - 1$, with $L = 100$. This yields

$$\mathbb{P}\left\{ \left| \frac{1}{N+1} \sum_{i=0}^{N} G_i \right| > C\sqrt{\log \frac{2N}{n}} \right\} = \mathbb{P}\left\{ \frac{1}{N+1} \sum_{j=1}^{n} \tilde{Y}_j^2 > C^2 \log \frac{2N}{n} \right\} \leq$$
$$\leq \mathbb{P}\left\{ \frac{1}{n} \sum_{j=1}^{n} [\tilde{Y}_j^2 - 1] > (C/2)^2 \log \frac{2N}{n} \right\} \leq 2 \exp\left(-cn \frac{C^2}{400} \cdot \log \frac{2N}{n} \right) < \left(\frac{n}{10N} \right)^n,$$

provided that C is a sufficiently large universal constant. This proves (ii). It remains to prove (iii). The random variables $G_{i,j}^2 - 1$ $(i = 0, ..., N,\ j = 1, ..., n)$ are independent, have mean zero and they satisfy that

$$\mathbb{E} \exp\left[|G_{i,j}^2 - 1|/10 \right] \leq 2\mathbb{E} \exp(G_{i,j}^2/10) \leq 20,$$

according to (∗b). Hence, we may apply Theorem 2.1 for the independent random variables $G_{k_i,j}^2 - 1$, with $L = 10$. We conclude that for any $1 \leq j \leq n$,

$$\mathbb{P}\left\{ \frac{1}{n} \sum_{i=1}^{n} [G_{k_i,j}^2 - 1] > C \log \frac{2N}{n} \right\} \leq 2 \exp\left(-cn \frac{C}{10} \cdot \log \frac{2N}{n} \right) < \left(\frac{n}{20N} \right)^n, \tag{5}$$

for a large universal constant C. We sum (5) over $j = 1, ..., n$ and conclude that

$$\mathbb{P}\left\{ \frac{1}{n^2} \sum_{i,j=1}^{n} G_{k_i,j}^2 > (C+2) \log \frac{2N}{n} \right\} \tag{6}$$
$$\leq \sum_{j=1}^{n} \mathbb{P}\left\{ \frac{1}{n} \sum_{i=1}^{n} [G_{k_i,j}^2 - 1] > C \log \frac{2N}{n} \right\} \overset{(5)}{<} n \left(\frac{n}{20N} \right)^n < \frac{1}{2} \cdot \left(\frac{n}{10N} \right)^n.$$

Denote $Z_i = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} G_{k_i,j} \quad (i = 1, ..., n)$. Then the Z_i are independent and $\mathbb{E} Z_i^2 = 1$ for $i = 1, ..., n$. Repeating the argument from (3) and (4) (the only difference is that here we sum over j and before we summed over i), we obtain that $\mathbb{E} \exp\left(|Z_i^2 - 1|/100 \right) < 9$. Thus, we may use Theorem 2.1 for the random variables $Z_i^2 - 1$ with $L = 100$, and deduce that

$$\mathbb{P}\left\{ \frac{1}{n} \sum_{i=1}^{n} [Z_i^2 - 1] > C \log \frac{2N}{n} \right\} \leq 2 \exp\left(-cn \frac{C}{100} \log \frac{2N}{n} \right) < \frac{1}{2} \cdot \left(\frac{n}{10N} \right)^n. \tag{7}$$

The desired conclusion (iii) follows at once from (6) and (7). $\square$

The next lemma is a simple, concrete calculation for the regular $(n-1)$-simplex. We write $e_1, ..., e_n$ for the standard basis in $\mathbb{R}^n$, and denote

$$\triangle^{n-1} = \mathrm{conv}\{e_1, ..., e_n\} \subset \mathbb{R}^n,$$

the $(n-1)$-dimensional regular simplex.

Lemma 2.3 *Let $X = (X_1, ..., X_n)$ be a random vector that is distributed uniformly in $\triangle^{n-1}$. Then,*

$$\mathbb{E}X_iX_j = \frac{1+\delta_{i,j}}{n(n+1)}$$

where $\delta_{i,j}$ is Kronecker's delta.

Proof. Examine X without its last coordinate, $(X_1, ..., X_{n-1})$. This is distributed uniformly in the simplex

$$\left\{ x \in \mathbb{R}^{n-1} \; ; \; \sum_{i=1}^{n-1} x_i \leq 1, \; \forall i, x_i \geq 0 \right\}.$$

Consequently, the density of the random variable $Y = X_1 + \cdots + X_{n-1}$ is proportional to $t \mapsto t^{n-2}$ in the interval $(0,1)$, and is zero elsewhere. Hence,

$$\mathbb{E}(X_1 + \cdots + X_{n-1})^2 = \mathbb{E}Y^2 = \int_0^1 t^2 \cdot (n-1)t^{n-2}dt = \frac{n-1}{n+1}. \tag{8}$$

Note that $\sum_{i=1}^n X_i \equiv 1$. Therefore

$$1 = \mathbb{E}\left(\sum_{i=1}^n X_i\right)^2 = \sum_{i=1}^n \mathbb{E}X_i^2 + \sum_{i \neq j} \mathbb{E}X_iX_j. \tag{9}$$

From (8) and (9) we get that, when $n \geq 2$,

$$(n-1)\mathbb{E}X_1^2 + (n-1)(n-2)\mathbb{E}X_1X_2 = \frac{n-1}{n+1}, \quad n\mathbb{E}X_1^2 + n(n-1)\mathbb{E}X_1X_2 = 1,$$

and the lemma follows. $\square$

Corollary 2.4 *Fix $0 \leq k_1 < k_2 < ... < k_n \leq N$, and set $\mathcal{F} = \mathrm{conv}\{G_{k_1}, ..., G_{k_n}\}$. Denote $Z \frac{1}{N+1} \sum_{i=0}^N G_i$. Then with probability greater than $1 - 4\left(\frac{n}{10N}\right)^n$, the set $\mathcal{F}$ is an $(n-1)$-dimensional simplex that satisfies*

(i) $\displaystyle \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} |x|^2 dx < C \log \frac{2N}{n},$

(ii) $\displaystyle \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} |x - Z|^2 dx < C \log \frac{2N}{n}.$

Here, $C > 0$ is a universal constant.

Proof. The random vectors G_{k_i} are independent and absolutely continuous according to ($*$c). Hence, with probability one, the linear span of $G_{k_1}, ..., G_{k_n}$ equals $\mathbb{R}^n$, and the set $\mathcal{F}$ is an $(n-1)$-dimensional simplex in $\mathbb{R}^n$ whose vertices are the points $G_{k_1}, ..., G_{k_n}$. Denote $T(G_{k_i,j})_{i,j=1,...,n}$, an $n \times n$ matrix. Then, $\mathcal{F}T(\triangle^{n-1})$ and hence,

$$\frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} |x|^2 dx \frac{1}{\mathrm{Vol}_{n-1}(\triangle^{n-1})} \int_{\triangle^{n-1}} |Tx|^2 \, dx. \tag{10}$$

According to Lemma 2.3,

$$\frac{1}{\mathrm{Vol}_{n-1}(\triangle^{n-1})} \int_{\triangle^{n-1}} |Tx|^2 \, dx = \sum_{i=1}^{n} \sum_{j_1,j_2=1}^{n} G_{k_i,j_1} G_{k_i,j_2} \frac{1}{\mathrm{Vol}_{n-1}(\triangle^{n-1})} \int_{\triangle^{n-1}} x_{j_1} x_{j_2} \, dx \tag{11}$$

$$= \frac{1}{n(n+1)} \sum_{i=1}^{n} \sum_{j_1,j_2=1}^{n} G_{k_i,j_1} G_{k_i,j_2} (1+\delta_{j_1,j_2}) \frac{1}{n(n+1)} \sum_{i=1}^{n} \left[\left(\sum_{j=1}^{n} G_{k_i,j} \right)^2 + \sum_{j=1}^{n} G_{k_i,j}^2 \right].$$

We conclude from (10), (11) and Lemma 2.2(iii) that

$$\mathbb{P} \left\{ \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} |x|^2 dx > C \log \frac{2N}{n} \right\} < \left(\frac{n}{10N} \right)^n. \tag{12}$$

As for the second part of the corollary, according to Lemma 2.2(i) and Lemma 2.2(ii) we know that

$$\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} G_{k_i} \right| + \left| \frac{1}{N+1} \sum_{j=0}^{N} G_j \right| < 2C \sqrt{\log \frac{2N}{n}} \right\} \geq 1 - 2 \left(\frac{n}{10N} \right)^n. \tag{13}$$

Additionally,

$$\frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} |x - Z|^2 \, dx = |Z|^2 - 2 \left\langle \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} x \, dx, Z \right\rangle + \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} |x|^2 \, dx$$

$$= \left| \frac{1}{N+1} \sum_{j=0}^{N} G_j \right|^2 - 2 \left\langle \frac{1}{n} \sum_{i=1}^{n} G_{k_i}, \frac{1}{N+1} \sum_{j=0}^{N} G_j \right\rangle + \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} |x|^2 \, dx. \tag{14}$$

By combining (14) with (12) and (13), we obtain

$$\mathbb{P} \left\{ \frac{1}{\mathrm{Vol}_{n-1}(\mathcal{F})} \int_{\mathcal{F}} |x - Z|^2 \, dx > C \log \frac{2N}{n} \right\} < 3 \left(\frac{n}{10N} \right)^n. \tag{15}$$

From (12) and (15) the corollary follows. □

For a point $x \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$, we write $d(x, A) = \inf_{y \in A} |x - y|$.

Lemma 2.5 *Set*

$$\mathcal{K} = \mathrm{conv}\{G_0, ..., G_N\}, \quad \mathcal{T} = \mathrm{conv}\{\pm G_1, ..., \pm G_N\},$$

and denote $Z \frac{1}{N+1} \sum_{i=0}^{N} G_i$. *Then with probability greater than* $1 - Ce^{-cn}$,

(i) $\displaystyle \frac{1}{\mathrm{Vol}_n(\mathcal{T})} \int_{\mathcal{T}} |x|^2 dx < C \log \frac{2N}{n},$

(ii) $\displaystyle \frac{1}{\mathrm{Vol}_n(\mathcal{K})} \int_{\mathcal{K}} |x - Z|^2 dx < C \log \frac{2N}{n}.$

Here, $c, C > 0$ *are universal constants.*

Proof. The random vectors $G_{i,j}$ are absolutely continuous, by assumption ($*$c). Hence, with probability one, the points $G_0, ..., G_N$ are in general position in $\mathbb{R}^n$; that is, with probability one, no $n+1$ distinct points from $\{G_0, ..., G_N\}$ lie in the same affine hyperplane in $\mathbb{R}^n$. Consequently, all the $(n-1)$-dimensional facets of the polytopes $\mathcal{K}$ and $\mathcal{T}$ are simplices, with probability one. Note that, in the case of $\mathcal{T}$ we use the fact that G_i and $-G_i$ could never belong to the same face.

Let $\mathcal{F}_1, ..., \mathcal{F}_\ell$ be a complete list of the $(n-1)$-dimensional facets of $\mathcal{T}$. Since a facet is determined by n points from $\{\pm G_1, ..., \pm G_N\}$, then

$$\ell \leq \binom{2N}{n} \leq \left(\frac{2eN}{n}\right)^n.$$

According to Corollary 2.4(i), with probability greater than $1 - 4\left(\frac{2e}{10}\right)^n$,

$$\int_{\mathcal{F}_i} |x|^2 dx < C \left(\log \frac{2N}{n}\right) \cdot \mathrm{Vol}_{n-1}(\mathcal{F}_i), \quad \text{for } i = 1, ..., \ell. \tag{16}$$

Each point $x \in \mathcal{T}$ (except for the origin) may be uniquely represented as $x = ty$ with $0 < t \leq 1$ and $y \in \partial\mathcal{T}$. We integrate with respect to these standard polar coordinates, and obtain that

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})} \int_{\mathcal{T}} |x|^2 \, dx = \frac{1}{\mathrm{Vol}_n(\mathcal{T})} \int_{\partial\mathcal{T}} \int_0^1 |ty|^2 t^{n-1} \langle y, \nu_y \rangle \, dt \, dy \tag{17}$$

where ν_y is the unit outward normal to $\partial\mathcal{T}$ at y (ν_y is uniquely defined almost everywhere as $\mathcal{T}$ is convex). When $y \in \mathcal{F}_i$ for some $i = 1, ..., \ell$, we have that $\langle y, \nu_y \rangle = d(0, \mathrm{asp}\, \mathcal{F}_i)$ where $\mathrm{asp}\, \mathcal{F}_i$ is the affine subspace spanned by $\mathcal{F}_i$. Hence, from (17),

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})} \int_{\mathcal{T}} |x|^2 dx = \frac{1}{\mathrm{Vol}_n(\mathcal{T})} \sum_{i=1}^{\ell} \frac{d(0, \mathrm{asp}\, \mathcal{F}_i)}{n+2} \int_{\mathcal{F}_i} |y|^2 dy. \tag{18}$$

Recall that $\sum_{i=1}^{\ell} d(0, \operatorname{asp} \mathcal{F}_i) \cdot \operatorname{Vol}_{n-1}(\mathcal{F}_i) = n \operatorname{Vol}_n(\mathcal{T})$. We combine (18) with (16), and conclude that with probability greater than $1 - 4\left(\frac{2e}{10}\right)^n$,

$$\frac{1}{\operatorname{Vol}_n(\mathcal{T})} \int_{\mathcal{T}} |x|^2 dx < \frac{1}{\operatorname{Vol}_n(\mathcal{T})} \sum_{i=1}^{\ell} \frac{d(0, \operatorname{asp} \mathcal{F}_i) \cdot \operatorname{Vol}_{n-1}(\mathcal{F}_i)}{n+2} \cdot C \log \frac{2N}{n} < C \log \frac{2N}{n}.$$

This completes the proof of (i). The proof of (ii) is very similar; we supply some details. Let $\mathcal{G}_1, ..., \mathcal{G}_k$ denote the $(n-1)$-dimensional facets of $\mathcal{K}$. Observe that $Z \in \mathcal{K}$, and that any $x \in \mathcal{K}$ (except for the point Z) is uniquely represented as $x = Z + t(y - Z)$ with $0 < t \leq 1$ and $y \in \partial\mathcal{K}$. As before, integration in polar coordinates yields

$$\frac{1}{\operatorname{Vol}_n(\mathcal{K})} \int_{\mathcal{K}} |x - Z|^2 \, dx \frac{1}{\operatorname{Vol}_n(\mathcal{K})} \int_{\partial\mathcal{K}} \int_0^1 t^{n-1} |t(y - Z)|^2 \langle y - Z, \nu_y \rangle \, dt \, dy =$$

$$= \frac{1}{n+2} \sum_{i=1}^{k} \frac{d(Z, \operatorname{asp} \mathcal{G}_i)}{\operatorname{Vol}_n(\mathcal{K})} \int_{\mathcal{G}_i} |y - Z|^2 \, dy.$$

Again, $\sum_{i=1}^{k} d(Z, \operatorname{asp} \mathcal{G}_i) \operatorname{Vol}_{n-1}(\mathcal{G}_i) = n \operatorname{Vol}_n(\mathcal{K})$. Thus, in order to prove (ii), we may simply reproduce the argument from the proof of (i), with Corollary 2.4(ii) replacing the role of Corollary 2.4(i). This completes the proof. □

3 Random Polytopes

We summarize the results of Section 2 in the following corollary. Note that the convex bodies discussed in this corollary have diameter that is larger than $c\sqrt{n}$ with high probability. Nevertheless, it is still possible to prove a much better estimate regarding the second moment of the Euclidean norm.

Corollary 3.1 *Let $N \geq n \geq 1$ and suppose that $G_0, ..., G_N$ are independent random vectors in $\mathbb{R}^n$ that satisfy conditions $(*a)$ and $(*b)$ above. Set $\mathcal{K} = \operatorname{conv}\{G_0, ..., G_N\}, \mathcal{T} = \operatorname{conv}\{\pm G_1, ..., \pm G_N\}$, and denote $Z = \frac{1}{N+1} \sum_{i=0}^{N} G_i$. Then with probability greater than $1 - Ce^{-cn}$,*

$$\frac{1}{\operatorname{Vol}_n(\mathcal{T})} \int_{\mathcal{T}} |x|^2 dx < C \log \frac{2N}{n} \quad \text{and} \quad \frac{1}{\operatorname{Vol}_n(\mathcal{K})} \int_{\mathcal{K}} |x - Z|^2 dx < C \log \frac{2N}{n},$$

where $C, c > 0$ are universal constants.

Proof. Suppose first that $G_0, ..., G_N$ satisfy also $(*\mathrm{c})$; that is, assume that they are absolutely continuous random variables. Then the desired conclusion follows from Lemma 2.5. For the general case, note that the quantity

$$\mathbb{P}\left\{ \frac{1}{\operatorname{Vol}_n(\mathcal{T})} \int_{\mathcal{T}} |x|^2 dx < C \log \frac{2N}{n}, \quad \frac{1}{\operatorname{Vol}_n(\mathcal{K})} \int_{\mathcal{K}} |x - Z|^2 dx < C \log \frac{2N}{n} \right\} \tag{19}$$

depends continuously on the distribution of $G_0, ..., G_N$ in the weak topology at measures where $\mathbb{P}(\mathrm{Vol}_n(\mathcal{K}) = 0) = \mathbb{P}(\mathrm{Vol}_n(\mathcal{T}) = 0) = 0$. At other measures (19) may have a discontinuity of no more than $\mathbb{P}(\mathrm{Vol}_n(\mathcal{K}) = 0) + \mathbb{P}(\mathrm{Vol}_n(\mathcal{T}) = 0)$. The corollary follows by approximating $G_0, ..., G_N$ with absolutely continuous random vectors that satisfy ($*$a), ($*$b) and ($*$c) and noting that by ($*$a) and ($*$b) we have $\mathbb{P}(\mathrm{Vol}_n(\mathcal{K}) = 0)+\mathbb{P}(\mathrm{Vol}_n(\mathcal{T}) = 0) < Ce^{-cn}$, according to [22]. □

The next theorem is concerned with non-gaussian analogs of Theorem 1.1. The main new case covered by that theorem is that of random sign vectors, i.e., independent random vectors whose coordinates are independent, symmetric Bernoulli variables. We remark in passing that in the Bernoulli case the probability of $\mathcal{K}$ or $\mathcal{T}$ to be degenerate was known before [22]. See [13, 25].

Theorem 3.2 *Let $n \geq 1$ and $2n \leq N \leq 2^n$. Suppose that $G_1, ..., G_N$ are independent random vectors in $\mathbb{R}^n$ that satisfy conditions ($*a$) and ($*b$) above. Set $\mathcal{T} = \mathrm{conv}\{\pm G_1, ..., \pm G_N\}$. Then with probability greater than $1 - Ce^{-cn}$,*

$$L_{\mathcal{T}} < C$$

where $C, c > 0$ are universal constants.

Proof. We may clearly assume that n exceeds a certain universal constant. It was proved in [16, Theorem 4.8] that, under the assumptions of the present theorem,

$$(\mathrm{Vol}_n(\mathcal{T}))^{1/n} > C\sqrt{\frac{\log(2N/n)}{n}} \tag{20}$$

with probability greater than $1 - Ce^{-cn}$. From Corollary 3.1 we know that with probability larger than $1 - Ce^{-c'n}$,

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})} \int_{\mathcal{T}} |x|^2 dx < C \log \frac{2N}{n} \tag{21}$$

The theorem follows by substituting the estimates (20) and (21) into the definition (1). □

Generally speaking, the restrictions on N in Theorem 3.2 are typically quite easy to work around. For example, in the case of Bernoulli variables, if $2^n \leq N < 3^n$ then (20) and hence the conclusion of Theorem 3.2 hold with a different constant, while if $N \geq 3^n$ then with very high probability $\mathcal{T}$ is a hypercube. As for $n \leq N < 2n$, one can show (20) by noting that even a single simplex has enough volume — see e.g., [24].

Our next lemma shows the same volume estimates in the Gaussian case. It is standard and well-known.

Lemma 3.3 *Let $N \geq n \geq 1$ and suppose that $G_0, ..., G_N$ are independent standard gaussian vectors in $\mathbb{R}^n$. Denote*

$$\mathcal{K}\,\mathrm{conv}\{G_0, ..., G_N\} \quad \textit{and} \quad \mathcal{T} = \mathrm{conv}\{\pm G_1, ..., \pm G_N\}.$$

Then, with probability greater than $1 - Ce^{-cn}$,

$$\mathrm{Vol}_n(\mathcal{K})^{\frac{1}{n}} > c\sqrt{\frac{\log(2N/n)}{n}} \quad \textit{and} \quad \mathrm{Vol}_n(\mathcal{T})^{\frac{1}{n}} > c\sqrt{\frac{\log(2N/n)}{n}},$$

where $c, C > 0$ *are universal constants.*

Proof sketch. We start with the lower bound for $\mathrm{Vol}_n(\mathcal{T})$. For the range $N \geq 2n$, it is well-known (see, e.g., [9, 16] and references therein) that with probability greater than $1 - Ce^{-cn}$,

$$c\sqrt{\log \frac{2N}{n}} D^n \subseteq \mathcal{T} \tag{22}$$

where $D^n = \{x \in \mathbb{R}^n; |x| \leq 1\}$ is the unit Euclidean ball in $\mathbb{R}^n$. Since $\mathrm{Vol}^{1/n}(D^n) > c/\sqrt{n}$, the desired lower bound for $\mathrm{Vol}_n(\mathcal{T})$ follows from (22) in this case. It remains to deal with the range $n \leq N < 2n$. It turns out that in this range a single simplex supplies enough volume for our needs. We thus assume that $N = n$. Elementary Euclidean geometry shows that

$$\mathrm{Vol}_n(\mathcal{T}) = \frac{2^n}{n!} \prod_{i=1}^{n} d\left(G_i, \mathrm{sp}\{G_1, ..., G_{i-1}\}\right)$$

where sp stands for linear span (we define $\mathrm{sp}(\emptyset) = \{0\}$). Denote $\xi_i = d\left(G_i, \mathrm{sp}\{G_1, ..., G_{i-1}\}\right)$ for $i = 1, ..., n$. Then $\xi_1, ..., \xi_n$ are independent random variables, with ξ_i^2 being distributed chi-square with i degrees of freedom. Standard estimates (one may use, e.g., Bernstein's inequality above) show that

$$\mathbb{P}\left\{\mathrm{Vol}_n(\mathcal{T}) < \frac{(2c)^n}{\sqrt{n!}}\right\} \mathbb{P}\left\{\sum_{i=1}^{n} \left[-\log \frac{\xi_i}{\sqrt{i}}\right] > n \log \frac{1}{c}\right\} < Ce^{-c'n}$$

for a suitable choice of universal constants $c, C, c' > 0$. This completes the proof of the desired lower bound for $\mathrm{Vol}_n(\mathcal{T})$. Regarding $\mathrm{Vol}_n(\mathcal{K})$, denote $G_i' = G_i - G_0$ for $i = 1, ..., n$, and set $\mathcal{K}' = \mathrm{conv}\{\pm G_1', ..., \pm G_n'\}$. Then,

$$\mathrm{Vol}_n(\mathcal{K}) = \mathrm{Vol}_n(\mathrm{conv}\{0, G_1 - G_0, G_2 - G_0, ..., G_N - G_0\}) \geq 4^{-n} \mathrm{Vol}_n(\mathcal{K}') \tag{23}$$

by the Rogers-Shephard inequality [21]. With probability one, the vectors $G_1, ..., G_n$ are linearly independent. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be the unique linear map that satisfies $S(G_i) = G_i - G_0$ for $i = 1, ..., n$. Then $\mathcal{K}' = S(\mathcal{T})$. Hence,

$$\mathrm{Vol}_n(\mathcal{K}') = \det(S) \cdot \mathrm{Vol}_n(\mathcal{T}). \tag{24}$$

Let $v \in \mathbb{R}^n$ be such that $\langle v, G_i \rangle = 1$ for $i = 1, ..., n$. The vector v is independent of G_0. Moreover, $|v| \geq 1/|G_1|$, and with probability greater than $1 - Ce^{-cn}$ we have $|G_1| \leq C\sqrt{n}$. Consequently,

$$\mathbb{P}\{|v| < c/\sqrt{n}\} \leq Ce^{-cn}.$$

Clearly, $Sx = x - \langle x, v\rangle G_0$ for all $x \in \mathbb{R}^n$. Therefore $\det(S) = 1 - \langle v, G_0\rangle$. Conditioning on v, we see that $\det(S)$ is a gaussian random variable with mean 1 and variance $|v|^2$. Hence,

$$\mathbb{P}\{|\det(S)| < 2^{-n}\} = \mathbb{E}_v \int_{-2^{-n}}^{2^{-n}} \frac{1}{\sqrt{2\pi|v|^2}} \exp\left(-\frac{(t-1)^2}{2|v|^2}\right) dt \leq Ce^{-cn} + C\sqrt{n}2^{-n}. \qquad (25)$$

The desired lower bound for $\mathrm{Vol}_n(\mathcal{K})$ follows from (23), (24), (25) and from the lower bound for $\mathrm{Vol}_n(\mathcal{T})$, that was already proven. □

Proof of Theorem 1.1. From Corollary 3.1 and Lemma 3.3, we know that with probability greater than $1 - Ce^{-cn}$,

$$\frac{1}{\mathrm{Vol}_n(\mathcal{T})^{1+\frac{2}{n}}} \int_{\mathcal{T}} |x|^2 dx < Cn \quad \text{and} \quad \frac{1}{\mathrm{Vol}_n(\mathcal{K})^{1+\frac{2}{n}}} \int_{\mathcal{K}} |x - Z|^2 dx < Cn$$

for some point $Z \in \mathbb{R}^n$ depending on $\mathcal{K}$. The theorem follows from the definition (1). □

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