Rapid Steiner symmetrization of most of a convex body and the slicing problem

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Abstract

For an arbitrary n-dimensional convex body, at least almost n Steiner symmetrizations are required in order to symmetrize the body into an isomorphic ellipsoid. We say that a body $T \subset \mathbb{R}^n$ is "quickly symmetrizable" if for any $\varepsilon > 0$ there exist only $\lfloor \varepsilon n \rfloor$ symmetrizations that transform T into a body which is $c(\varepsilon)$-isomorphic to an ellipsoid, where $c(\varepsilon)$ depends solely on ε. In this note we ask, given a body $K \subset \mathbb{R}^n$, whether it is possible to remove a small portion of its volume and obtain a body $T \subset K$ which is quickly symmetrizable? We show that this question, for a large variety of $c(\varepsilon)$, is equivalent to the slicing problem.

1 Introduction

We work in $\mathbb{R}^n$, endowed with the usual scalar product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $| \cdot |$. Let $K \subset \mathbb{R}^n$ be a convex body, and let $H = \{x \in \mathbb{R}^n; \langle x, h \rangle = 0\}$ be a hyperplane through the origin in $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ there exists a unique decomposition $x = y + th$ where $y \in H, t \in \mathbb{R}$, so we can refer to (y, t) as coordinates in $\mathbb{R}^n$. The result of a "Steiner symmetrization of K with respect to h" is the body:

$$S_H(K) = \left\{ \ (x, t) \ ; \ K \cap (x + \mathbb{R}h) \neq \emptyset \ , \ |t| \leq \frac{1}{2} Meas\{K \cap (x + \mathbb{R}h)\} \ \right\}$$

where $Meas$ is the one-dimensional Lebesgue measure in the line $x + \mathbb{R}h$. Steiner symmetrization is a well-known operation in convexity. It

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preserves the volume of a body and transforms convex sets to convex sets (e.g. [BF]). A suitably chosen finite sequence of Steiner symmetrizations may transform an arbitrary convex body into a body that is close to a Euclidean ball. Less expected is the fact that relatively few symmetrizations suffice for obtaining a body that is close to a Euclidean ball. The following theorem, which improves a previous result of [BLM], appears in [KM] ($D = \{x \in \mathbb{R}^n; |x| \leq 1\}$ is the standard Euclidean ball in $\mathbb{R}^n$):

Theorem 1.1 *For any $n \geq 2$ and any convex body $K \subset \mathbb{R}^n$ with $Vol(K) = Vol(D)$, there exist $3n$ Steiner symmetrizations that transform the body K into $\tilde{K}$ such that:*

$$\frac{1}{c}D \subset \tilde{K} \subset cD$$

where $c > 0$ is a numerical constant.

Given a convex body $K \subset \mathbb{R}^n$, we define its "geometric distance" from a convex body $T \subset \mathbb{R}^n$ as

$$d_G(K,T) = \inf\left\{ab; \frac{1}{a}T \subset K \subset bT,\ a,b > 0\right\}$$

and we set $d_G(K) = d_G(K,D)$, the geometric distance of K from a Euclidean ball. The Banach-Mazur distance of K from a Euclidean ball is $d_{BM}(K) = \inf_T d_G(TK)$, where the infimum runs over all invertible linear transformations. d_{BM} measures the geometric distance of K from an ellipsoid. Notice that we do not allow translations of the convex body when defining the distances.

The constant "3" in Theorem 1.1 is not optimal (see more accurate results in [KM]). However, for bodies such as the cross-polytope $B_1^n = \{x \in \mathbb{R}^n; \sum |x_i| \leq 1\}$, at least $n - C\log n$ symmetrizations are required in order to symmetrize B_1^n into a body which is $\sqrt{C/2}$-close to an ellipsoid (see [KM]). Therefore it is impossible to symmetrize a general convex body in $\mathbb{R}^n$ into an isomorphic ellipsoid, using significantly less than n symmetrizations. Let us consider another example: the cube $B_\infty^n = \{x \in \mathbb{R}^n; \forall i\ |x_i| \leq 1\}$ has a very short symmetrization process. For any $\varepsilon > 0$, there exist $\lfloor \varepsilon n \rfloor$ symmetrizations that transform B_∞^n into a body whose distance from a Euclidean ball is smaller than $c\sqrt{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}}$ for some numerical constant $c > 0$. Given a convex body $K \subset \mathbb{R}^n$, we say that "K is $c(\varepsilon)$-symmetrizable" if for any $\varepsilon > 0$, there exist $\lfloor \varepsilon n \rfloor$ symmetrizations that transform K into $\tilde{K}$ with $d_{BM}(K) < c(\varepsilon)$. Using this terminology, the cube is $c(\varepsilon)$-symmetrizable for $c(\varepsilon) = c\sqrt{\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}}$. Note that here $c(\varepsilon)$ does not depend on the dimension n, and grows polynomially in $\frac{1}{\varepsilon}$ as ε tends to zero. Here we ask whether an arbitrary convex body $K \subset \mathbb{R}^n$ contains a large part which is $c(\varepsilon)$-symmetrizable, with $c(\varepsilon)$ being a polynomial in $\frac{1}{\varepsilon}$, whose coefficients do not depend on the dimension n.

Question 1.2 *Does there exist a function $c(\varepsilon)$, which is a polynomial in $\frac{1}{\varepsilon}$, such that for any dimension n, for any convex body $K \subset \mathbb{R}^n$, there exists a convex body $T \subset K$ with $Vol(T) > \frac{9}{10}Vol(K)$ such that T is $c(\varepsilon)$-symmetrizable?*

The number "$\frac{9}{10}$" has no special meaning, and may be replaced with any $\alpha < 1$. An apriori unrelated question is concerned with the isotropic constant. Let $K \subset \mathbb{R}^n$ be a convex body. K has an affine image $\tilde{K}$, which is unique up to orthogonal transformations, such that the barycenter of $\tilde{K}$ is at the origin, $Vol(\tilde{K}) = 1$, and

$$\int_{\tilde{K}} \langle x, \theta \rangle^2 dx = L_K^2 |\theta|^2$$

for any $\theta \in \mathbb{R}^n$, where L_K does not depend on θ (see [MP1]). We say that L_K is the isotropic constant of K. A fundamental question in asymptotic convex geometry is the following:

Question 1.3 *Does there exist a constant $c > 0$ such that for any integer n, for any convex body $K \subset \mathbb{R}^n$ we have $L_K < c$?*

The main goal of this note is to show that Question 1.3 and Question 1.2 are equivalent.

Theorem 1.4 *Question 1.2 and Question 1.3 have the same answer.*

Theorem 1.4 connects two properties of the class of all convex bodies in all dimensions, yet formally it does not say anything about an individual body $K \subset \mathbb{R}^n$. We also obtain here results that are applicable to individual bodies. Proposition 4.4 states that given a body $K \subset \mathbb{R}^n$ that contains a large portion which is $c(\varepsilon)$-symmetrizable for some polynomial $c(\varepsilon)$, the isotropic constant of K may be bounded by a quantity that depends solely on the polynomial $c(\varepsilon)$. See also Proposition 3.2 for the opposite direction.

Before turning to the details of the proofs, let us shed some light on the concept of a $c(\varepsilon)$-symmetrizable body $T \subset \mathbb{R}^n$, for a polynomial $c(\varepsilon)$. Assume that T is such a body, for $c(\varepsilon) < c_1 \left(\frac{1}{\varepsilon}\right)^{c_2}$, where $c_1, c_2 > 0$ do not depend on n. Then for any $\varepsilon > 0$ there exist $\lfloor \varepsilon n \rfloor$ symmetrizations of T with respect to special $v_1, .., v_{\lfloor \varepsilon n \rfloor}$, that transform

T into $\tilde{T}$ that is $c(\varepsilon)$-close to an ellipsoid. Denote by E the subspace $\{v_1, .., v_{\lfloor \varepsilon n \rfloor}\}^{\perp}$. By Lemma 2.4 in [KM],

$$Proj_E(T) = Proj_E(\tilde{T}) \quad \Longrightarrow \quad d_{BM}(Proj_E(T)) < c_1 \left(\frac{1}{\varepsilon}\right)^{c_2}$$

where $Proj_E$ is the orthogonal projection onto E in $\mathbb{R}^n$. Therefore $T \subset \mathbb{R}^n$ has, for any $\varepsilon > 0$, projections to subspaces of dimension $\lceil (1-\varepsilon)n \rceil$ whose distance from an ellipsoid is smaller than some polynomial in $\frac{1}{\varepsilon}$. In fact, as will be explained later, by Theorem 1.1 a body is $c(\varepsilon)$-symmetrizable for a polynomial $c(\varepsilon)$ if and only if it has large projections which are polynomially close to an ellipsoid. Since the latter notion is clearly linearly invariant, then also $c(\varepsilon)$-symmetrizability with a polynomial $c(\varepsilon)$ is a linearly invariant property.

Throughout the paper we denote by $c, c', \tilde{c}, C$ etc. some positive universal constants whose value is not necessarily the same on different appearances. Whenever we write $A \approx B$, we mean that there exist universal constants $c, c' > 0$ such that $cA < B < c'A$. Also, $Vol(T)$ denotes the volume of a set $T \subset \mathbb{R}^n$ relative to its affine hull. A random k-dimensional subspace in $\mathbb{R}^n$ is chosen according to the unique rotation invariant probability measure in the Grassman manifold $G_{n,k}$.

2 An M-position of order α and the isotropic position

For $K, T \subset \mathbb{R}^n$ denote the covering number of K by T as

$$N(K,T) = \min \left\{ N; \exists x_1, .., x_N \in \mathbb{R}^n, K \subset \bigcup_{i=1}^{N} x_i + T \right\}.$$

Let $K \subset \mathbb{R}^n$ be a convex body. An ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ is an M-ellipsoid of K with constant $c > 0$ if

$$\max\{N(K, \mathcal{E}), N(\mathcal{E}, K)\} < e^{cn}.$$

If $\mathcal{E} = D$, we say that K is in M-position with constant $c > 0$. A result by Milman states that any centrally symmetric (i.e. $K = -K$) convex body has a linear image in M-position with some absolute constant (see [M1], or chapter 7 in the book [P]). Furthermore, we say that K is in M-position of order α with constants c_α, c'_α if for all $t > 1$

$$max\{N(K, tc_\alpha D), N(D, tc_\alpha K)\} < e^{c'_\alpha \frac{n}{t^\alpha}}.$$

Another common terminology to describe this property is α-regular M-position with the appropriate constants. By a duality theorem [AMS], if K is centrally-symmetric and is in M-position of order α, then also

$$max\{N(K^\circ, c_\alpha tD), N(D, c_\alpha tK^\circ)\} < e^{\tilde{c}_\alpha \frac{n}{t^\alpha}}$$

where $K^\circ = \{y \in \mathbb{R}^n; \forall x \in K, \ \langle x, y\rangle \leq 1\}$. A theorem of Pisier [P] states that given a centrally-symmetric $K \subset \mathbb{R}^n$, for any $\alpha < 2$, there exists a linear image of K which is in M-position of order α with some constants that depend solely on α.

The assumption of central symmetry in the above discussion is not crucial. In [M2, MP2] it is proven that any convex body whose barycenter lies at the origin has a linear image in M-position with some absolute constant. However, the literature seems to contain no discussion on the existence of regular M-positions for non-symmetric convex bodies. Next, we deduce that a regular M-position exists for any convex body. Begin with a lemma in the spirit of [K].

Lemma 2.1 *Let $K \subset \mathbb{R}^n$ be a convex body whose barycenter is at the origin. Let $E \subset \mathbb{R}^n$ be a subspace, $dim(E) = \lambda n$. Then there exists a convex body $T \subset E$ whose barycenter is at the origin such that*

$$c_1 \lambda Proj_E(K) \subset T \subset c_2 Proj_E(K)$$

where $c_1, c_2 > 0$ are universal constants.

Proof: The proof is just a minor adaptation of the proof of Proposition 5.2 in [K] and we omit its details. For $x \in E$ we define $f(x) = Vol(K \cap [E^\perp + x])$. Then f is a log-concave function, and hence $T = \left\{x \in E; \int_0^\infty f(rx) r^n dr \geq 1\right\}$ is a convex set whose barycenter is at the origin. Since f is a $(1-\lambda)n$-concave function on a λn dimensional space, and its support is $Proj_E(K)$, Lemma 2.2 in [K] completes the proof. $\square$

Proposition 2.2 *Let $K \subset \mathbb{R}^n$ be a convex body whose barycenter is at the origin. Then there exists a linear transformation L such that $\tilde{K} = L(K)$ satisfies, for any $t > 1$,*

$$\max\{N(K, tD), N(D, tK), N(K^\circ, tD), N(D, tK^\circ)\} < exp\left(c\frac{n}{t^{1/6}}\right)$$

where $c > 0$ is a numerical constant.

Proof: Consider the centrally-symmetric convex bodies $\underline{K} = K \cap (-K)$ and $\overline{K} = conv(K, -K)$ where *conv* denotes convex hull. Then $\underline{K} \subset K \subset \overline{K}$ and also $\overline{K^\circ} = (\underline{K})^\circ$. Let $E \subset \mathbb{R}^n$ be a subspace, and denote $k = \lambda n = dim(E)$. Since $Proj_E(\overline{K}) = \overline{Proj_E(K)}$, by [RS]

$$Vol(Proj_E(\overline{K}))^{\frac{1}{k}} \leq 4 Vol(Proj_E(K))^{\frac{1}{k}}. \quad (1)$$

The barycenter of $Proj_E(K)$ may be different from zero. However, by Lemma 2.1 there exists a convex body $T \subset E$ whose barycenter lies at the origin, such that

$$c_1 \lambda Proj_E(K) \subset T \subset c_2 Proj_E(K).$$

For a k-dimensional body T we denote $v.rad.(T) = \left(\frac{Vol(T)}{Vol(D_k)}\right)^{\frac{1}{k}}$ where D_k is a k-dimensional Euclidean unit ball. By Santalo inequality (e.g. [MeP]) $v.rad.(T)v.rad.(T^\circ) < C$ and hence

$$v.rad.(Proj_E(K^\circ))v.rad.(K \cap E) < \frac{c}{\lambda} \tag{2}$$

for any convex body K whose barycenter is at the origin. Next, by [R] and Theorem 1 in [F],

$$v.rad.(\overline{K} \cap E) < C\frac{1}{\lambda} \sup_{x \in E^\perp} v.rad.(K \cap (E+x)) < C' \left(\frac{1}{\lambda}\right)^2 v.rad.(K \cap E).$$

By the reverse Santalo inequality [BM],

$$v.rad.(Proj_E(\underline{K})) > \frac{c}{v.rad.((\underline{K})^\circ \cap E)} = \frac{c}{v.rad.(\overline{K^\circ} \cap E)} > c'\lambda^2 \frac{1}{v.rad.(K^\circ \cap (E+x))}$$

where $x \in \mathbb{R}^n$ is the barycenter of K°. By (2) and (1),

$$v.rad.(Proj_E(\underline{K})) > \tilde{c}\lambda^3 v.rad.(Proj_E(K)) > C\lambda^3 v.rad.(Proj_E(\overline{K})).$$

Let us assume that $\underline{K}$ is in 1-regular position. Denote by $\mathcal{E}$ a 1-regular ellipsoid of $\overline{K}$ (i.e. if $L(\overline{K})$ is in M-position of order 1, then $\mathcal{E}$ is defined so that $L(\mathcal{E}) = D$). Since $N(Proj_E(\mathcal{E}), \frac{1}{\lambda} Proj_E(\overline{K})) < exp(c\lambda n)$, we get that

$$v.rad.(Proj_E(\mathcal{E})) < \frac{c}{\lambda} v.rad.(Proj_E(\overline{K})) < \frac{C}{\lambda^4} v.rad.(Proj_E(\underline{K})).$$

Because $N(Proj_E(\underline{K}), \frac{1}{\lambda} Proj_E(D)) < exp(c\lambda n)$, we conclude that

$$v.rad.(Proj_E(\mathcal{E})) < \frac{C}{\lambda^4} v.rad.(Proj_E(\underline{K})) < \frac{C'}{\lambda^5}.$$

This is true for any λn-dimensional subspace E, for any $0 < \lambda < 1$ such that λn is an integer. By standard estimates for the covering number of an ellipsoid by Euclidean balls (e.g. Remark 5.15 in [P]), we get that for any $t > 1$,

$$N(\mathcal{E}, tD) < exp\left(cnt^{-\frac{1}{5}}\right), \quad N(D, t\mathcal{E}^\circ) < exp\left(cnt^{-\frac{1}{5}}\right)$$

and hence

$$N(K,tD) \le N(\overline{K},tD) \le N\left(\overline{K},t^{\frac{1}{6}}\mathcal{E}\right) N\left(\mathcal{E},t^{\frac{5}{6}}D\right) < exp\left(c\frac{n}{t^{\frac{1}{6}}}\right),$$

$$N(D,tK^\circ) \le N\left(D,t(\overline{K})^\circ\right) \le N\left(D,t^{\frac{1}{6}}\mathcal{E}^\circ\right) N\left(\mathcal{E}^\circ,t^{5/6}(\overline{K})^\circ\right) < exp\left(c\frac{n}{t^{\frac{1}{6}}}\right).$$

Trivially $N(D,tK) < N(D,t\underline{K}) < exp(c\frac{n}{t})$ and also $N(K^\circ,tD) < N((\underline{K})^\circ,tD) < exp(c\frac{n}{t})$. We conclude that D is an M-ellipsoid of K of order $\frac{1}{6}$. □

Remark: The power "$\frac{1}{6}$" in Proposition 2.2 is clearly non-optimal and may be improved. We do not know what the optimal power is.

If K is in M-position, then proportional sections of K typically have a small diameter, and proportional projections of K typically contain a large Euclidean ball. If K is also in M-position of order α, then typical sections of dimension $\lfloor(1-\varepsilon)n\rfloor$ have a diameter which is smaller than some polynomial in $\frac{1}{\varepsilon}$, as follows from the next theorem (see e.g. [GM]).

Theorem 2.3 *Let $K \subset \mathbb{R}^n$ be a convex body in M-position of order α with constants c_α, c'_α. Let E be a random subspace of dimension $(1-\varepsilon)n$. Then with probability larger than $1-e^{-c'\varepsilon n}$,*

$$K \cap E \subset \left(\frac{c(c_\alpha,c'_\alpha)}{\varepsilon^{\frac{1}{2}+\frac{1}{\alpha}}}\right) D$$

$$\left(\frac{\varepsilon^{\frac{1}{2}+\frac{1}{\alpha}}}{c(c_\alpha,c'_\alpha)}\right) D \cap E \subset Proj_E(K)$$

where $c' > 0$ is a numerical constant, and $c(c_\alpha,c'_\alpha)$ depends neither on K nor on n, but solely on its arguments.

Assume that Question 1.3 has an affirmative answer. Our next proposition proves the existence of large projections that contain large Euclidean balls as in Theorem 2.3, for bodies in isotropic position (compare with Proposition 5.4 in [K]).

Proposition 2.4 *Assume a positive answer to Question 1.3. Let $K \subset \mathbb{R}^n$ be a convex isotropic body with volume one whose barycenter is at the origin. Then for any integer $k = (1-\varepsilon)n$ where $0 < \varepsilon < 1$, there exists a subspace E of dimension k with*

$$c\varepsilon^\beta \sqrt{n} D \cap E \subset Proj_E(K)$$

where $c > 0$ depends only on the constant in Question 1.3, and $\beta \le 13$ is a numerical constant. If in addition K is centrally-symmetric, then $\beta \le 3$.

Proof: We shall use the following observation which appears in [MP1] (Proposition 3.11 there) and in [Ba]. Although it is stated there for centrally-symmetric bodies, the generalization to the non-symmetric case is straightforward (a formulation appears in [BKM]). A positive answer to Question 1.3 yields that for any subspace F of dimension k,

$$c_1 < Vol(K \cap F)^{\frac{1}{n-k}} < c_2 \tag{3}$$

where c_1, c_2 depend only on the constant in Question 1.3. Since the barycenter of K is at the origin, then $Vol(K \cap F)Vol(Proj_{F^\perp}(K)) \geq Vol(K) = 1$ for any subspace F (see [Sp]). By (3),

$$Vol(Proj_{F^\perp}(K))^{\frac{1}{n-k}} > \frac{1}{c_2}. \tag{4}$$

Assume for simplicity that K is centrally-symmetric. Let $\mathcal{E}$ be an M-ellipsoid of order 1 of K, i.e.

$$\max\{N(K, t\mathcal{E}), N(\mathcal{E}, tK)\} < e^{c\frac{n}{t}}$$

where $c > 0$ is a numerical constant. Let $0 < \lambda_1 \leq ... \leq \lambda_n$ be the axes of $\mathcal{E}$. Let $0 < \delta < 1$, and denote by F_1 the subspace spanned by the shortest $\lfloor \delta n \rfloor$ axes of $\mathcal{E}$. Since $N(Proj_{F_1}(K), tProj_{F_1}(\mathcal{E})) < e^{c\frac{n}{t}}$, we obtain that

$$Vol(Proj_{F_1}(K)) < e^{\frac{cn}{t}} \left(t\lambda_{\lfloor \delta n \rfloor}\right)^{\lfloor \delta n \rfloor} Vol(D \cap E)$$

for any $t > 0$. Using (4) and the fact that $Vol(D \cap E)^{\frac{1}{\lfloor \delta n \rfloor}} \approx \frac{1}{\sqrt{\delta n}}$, when we set $t = \frac{1}{\delta}$ we get that $\lambda_{\lfloor \delta n \rfloor} > c'\sqrt{n}\delta^{3/2}$. Assume that $\lfloor \delta n \rfloor = \lfloor \frac{\varepsilon n}{2} \rfloor$ (hence $\delta \leq \frac{1}{2}$), and let F_2 denote the subspace of the longest $\lceil (1-\delta)n \rceil$ axes of $\mathcal{E}$. Since

$$N(Proj_{F_2}(K), t(\mathcal{E} \cap F_2)) \leq N(K, t\mathcal{E}) < e^{c\frac{n}{t}} < e^{2c\frac{(1-\delta)n}{t}}$$

and since a similar inequality holds for $N(\mathcal{E} \cap F_2, tProj_{F_2}(K))$, then $\mathcal{E} \cap F_2$ is an M-ellipsoid of order 1 of $Proj_{F_2}(K)$. Also $c'\delta^{3/2}\sqrt{n}D \cap F_2 \subset \mathcal{E} \cap F_2$. By Theorem 2.3, there exists a subspace $E \subset F_2$ of dimension $(1-2\delta)n \geq (1-\varepsilon)n$, with

$$c\varepsilon^3\sqrt{n}D \cap E \subset c'\varepsilon^{3/2}\mathcal{E} \cap E \subset Proj_E(Proj_{F_2}(K)) = Proj_E(K)$$

and the proposition is proved for centrally-symmetric bodies. Regarding non-symmetric convex bodies, we may repeat the argument using a $\frac{1}{6}$-regular M-ellipsoid, whose existence is guaranteed by Proposition 2.2. We obtain the same conclusion as in the symmetric case, but with a different power of ε. □

Remark: Even in the centrally-symmetric case, our bound $\beta \leq 3$ in Proposition 2.4 is not optimal, and may be improved by considering M-ellipsoids of higher order. We do not know what the best β is.

3 Slicing implies rapid symmetrization

The following lemma is standard. For completeness, we include its proof, which is trivial for centrally-symmetric bodies. We would like to remind the reader that our definitions of distances forbid translations of the bodies.

Lemma 3.1 *Let $K \subset \mathbb{R}^n$ be a convex body and let $E \subset \mathbb{R}^n$ be a subspace such that:*

1. *$Proj_E(K) = K \cap E$ and $d_{BM}(K \cap E) < A$ for some $A \geq 1$.*
2. *$d_{BM}(K \cap E^\perp) < B$ for some $B \geq 1$.*

Then $d_{BM}(K) < cAB$ where $c > 0$ is a numerical constant.

Proof: Applying a linear transformation inside E if necessary, we may assume that $D \subset K \cap E \subset AD$. Let $x \in Proj_E(K)$ be any point. We claim that

$$-x + \left[K \cap \left(x + E^\perp\right)\right] \subset (A+1)K \cap E^\perp.$$

Indeed, since $|x| \leq A$ and since $-\frac{x}{A} \in K$, by convexity of K

$$\frac{-x + \left[K \cap \left(x + E^\perp\right)\right]}{A+1} \subset conv\left[-\frac{x}{A}, K \cap \left(x + E^\perp\right)\right] \cap E^\perp \subset K \cap E^\perp.$$

Therefore, $Proj_{E^\perp}(K) \subset (A+1)K \cap E^\perp$. Now, let $\mathcal{E}$ be an ellipsoid, symmetric with respect to E, such that $\mathcal{E} \cap E \subset K \cap E \subset A\mathcal{E} \cap E$ and $\mathcal{E} \cap E^\perp \subset K \cap E^\perp \subset B\mathcal{E} \cap E^\perp$. Then,

$$\frac{1}{\sqrt{2}}\mathcal{E} \subset conv(K \cap E, K \cap E^\perp) \subset K \subset Proj_E(K) \times Proj_{E^\perp}(K)$$

$$\subset K \cap E \times (A+1)K \cap E^\perp \subset (A\mathcal{E} \cap E) \times \left[(A+1)B\mathcal{E} \cap E^\perp\right] \subset \sqrt{2}(A+1)B\mathcal{E}$$

and the lemma is proven. □

Let $K \subset \mathbb{R}^n$ be a convex body, and assume that for any $\varepsilon > 0$ there exists a subspace E of dimension $\lfloor \varepsilon n \rfloor$ such that $d_{BM}(Proj_{E^\perp}(K)) < c(\varepsilon)$, for some function $c(\varepsilon)$. Consider the body $K \cap E$. According to Theorem 1.1, after $\lfloor 3\varepsilon n \rfloor$ symmetrizations $K \cap E$ may be transformed into an isomorphic Euclidean ball. Apply the same symmetrizations to K, to obtain $\tilde{K}$. Since these symmetrizations include symmetrizations with respect to an orthogonal basis of E, elementary properties of the symmetrization (e.g. [KM]) together with Lemma 3.1 imply that

$$d_{BM}(\tilde{K}) < c'c(\varepsilon).$$

We conclude, as was mentioned in the introduction, that a convex body is $c(\varepsilon)$-symmetrizable with a $c(\varepsilon)$ which is polynomial in $\frac{1}{\varepsilon}$ if and only if it has projections to dimension $\lfloor (1-\varepsilon)n \rfloor$ whose distance from an ellipsoid is smaller than some polynomial in $\frac{1}{\varepsilon}$.

Before proving one direction of Theorem 1.4, which assumes a positive answer to Question 1.3, let us prove a weaker statement (with an exponential dependence, rather than polynomial), one that is applicable to an individual body $K \subset \mathbb{R}^n$, and does not require uniform boundness of the isotropic constant.

Proposition 3.2 *Let $\varepsilon > 0$, and let $K \subset \mathbb{R}^n$ be a convex body with $L_K < A$ for some $A > 0$. Then there exists a body $T \subset K$ with $Vol(T) > \frac{9}{10} Vol(K)$ and $\lfloor \varepsilon n \rfloor$ Steiner symmetrizations that transform T into $\tilde{T}$ such that*

$$d_{BM}(\tilde{T}, D) < (cA)^{\frac{1}{\varepsilon}}$$

where $c > 0$ is a universal constant.

Proof: Assume that the barycenter of K is at the origin. Let $\mathcal{E}$ be the isotropy ellipsoid of K normalized so that $Vol(\mathcal{E}) = Vol(K)$ (i.e. if $\tilde{K} = L(K)$ is isotropic for a linear operator L, then $\mathcal{E}$ is defined so that $L(\mathcal{E})$ is a Euclidean ball of volume one). Let $T = K \cap cA\mathcal{E}$. By Borell lemma (e.g. Theorem III.3 in [MS]) $Vol(T) > \frac{9}{10} Vol(K)$, if $c > 0$ is suitably chosen. Note that $T \subset cA\mathcal{E}$, and

$$\left(\frac{Vol(cA\mathcal{E})}{Vol(T)} \right)^{1/n} < c' A.$$

By a theorem of Szarek and Tomczak-Jaegermann ([Sz], [SzT]) there exists a subspace E of dimension $\lceil (1-\varepsilon)n \rceil$ such that

$$d_G(Proj_E(T), Proj_E(\mathcal{E})) < (cA)^{\frac{1}{\varepsilon}}.$$

Let us apply the $3\varepsilon n$ symmetrizations that suit $T \cap E^{\perp}$ according to Theorem 1.1, to the body T, and obtain the body $\tilde{T}$. By Lemma 2.1 and Lemma 2.4 from [KM] (these $3\varepsilon n$ symmetrizations include symmetrizations with respect to an orthogonal basis),

$$\tilde{T} \cap E = Proj_E(\tilde{T}) = Proj_E(T)$$

and also $\tilde{T} \cap E^{\perp}$ has a universally bounded distance from a Euclidean ball. By Lemma 3.1,

$$d_{BM}(\tilde{T}, D) < (c' A)^{\frac{1}{\varepsilon}}$$

which completes the proof. □

The following proposition proves one part of Theorem 1.4.

Proposition 3.3 *Assume that Question 1.3 has a positive answer. Let* $\varepsilon > 0$*, and let* $K \subset \mathbb{R}^n$ *be a convex body. Then there exists a body* $T \subset K$ *with* $Vol(T) > \frac{9}{10} Vol(K)$ *and* $\lfloor \varepsilon n \rfloor$ *Steiner symmetrizations that transform* T *into* $\tilde{T}$ *such that*

$$d_{BM}(\tilde{T}) < c \frac{1}{\varepsilon^\beta}$$

where $c > 0$ *is a constant that depends only on the constant in Question 1.3 and* $0 < \beta < 13$ *is a numerical constant.*

Proof: Assume that $Vol(K) = 1$ and that the barycenter of K is at the origin. Let $\mathcal{E}$ be the isotropy ellipsoid of K normalized so that $Vol(\mathcal{E}) = Vol(K)$, and denote $T = K \cap c\mathcal{E}$, where $c > 0$ depends linearly on the constant in Question 1.3. As before, by Borell lemma, $Vol(T) > \frac{9}{10}$. Also, if $c > 0$ is chosen properly, then the isotropy ellipsoid $\mathcal{F}$ of T satisfies $d_G(\mathcal{F}, \mathcal{E}) < c_1$ (e.g. [Bou]). By Proposition 2.4 there exists a subspace E of dimension $> (1 - \varepsilon)n$ with

$$c\varepsilon^\beta Proj_E(\mathcal{E}) \subset c'\varepsilon^\beta Proj_E(\mathcal{F}) \subset Proj_E(T) \subset c'' Proj_E(\mathcal{E})$$

where $c, c'' > 0$ depend only on the constant in Question 1.3. By Theorem 1.1 there exist some special $\lfloor 3\varepsilon n \rfloor$ symmetrizations designed specific to the body $T \cap E^\perp$. Apply these $\lfloor 3\varepsilon n \rfloor$ symmetrizations to T itself. Reasoning as in Proposition 3.2, we obtain a body $\tilde{T}$ with

$$d_{BM}(\tilde{T}) < c \frac{1}{\varepsilon^\beta}.$$

□

3.1 Dual symemtrization

Let $K \subset \mathbb{R}^n$ be a convex body and let H be a hyperplane in $\mathbb{R}^n$. For simplicity, assume that K is centrally-symmetric. The result of a dual Steiner symmetrization of K is the body

$$S_H^\circ(K) = [S_H(K^\circ)]^\circ \, ,$$

i.e. we symmetrize the dual body with respect to H. Next, we propose an alternative short symmetrization process for an arbitrary convex body $K \subset \mathbb{R}^n$. Rather than cutting a small portion of the volume, we combine symmetrizations of two kinds: Steiner symmetrization and dual Steiner symmetrization.

Theorem 3.4 *Let* $K \subset \mathbb{R}^n$ *be a centrally-symmetric convex body. Then there exists* $\tilde{K}$*, a linear image of* K*, such that for any* $0 < \varepsilon < 1$

there exist εn Steiner symmetrizations that transform $\tilde{K}$ into K_1, and εn dual Steiner symmetrizations that transform K_1 into K_2 such that

$$d_G(K_2) < \frac{c}{\varepsilon^3}$$

where $c > 0$ is a numerical constant.

Proof: Assume that $\varepsilon < \frac{1}{2}$. Let $\tilde{K}$ be a linear image of K which is in M-position of order 1. By Theorem 2.3 there exists a subspace E of dimension $\lfloor \varepsilon n \rfloor$ such that

$$c\varepsilon^{3/2} D \cap E^{\perp} \subset Proj_{E^{\perp}}(\tilde{K}). \tag{5}$$

Also, since $N\left(D \cap E, \frac{1}{\varepsilon} Proj_E(\tilde{K})\right) < exp(c\varepsilon n)$, then

$$\left(\frac{Vol(Proj_E(\tilde{K}))}{Vol(\varepsilon D \cap E)}\right)^{\frac{1}{dim(E)}} > C.$$

We apply $\lfloor 3\varepsilon n \rfloor$ symmetrization to $\tilde{K}$, all in the subspace E according to Theorem 1.1, to obtain the body K_1. The body K_1 satisfies

$$c\varepsilon D \cap E \subset K_1 \cap E.$$

In addition, $K_1 \cap E^{\perp} = Proj_{E^{\perp}}(K_1) = Proj_{E^{\perp}}(\tilde{K})$ (see e.g. [KM]). By (5) we conclude that

$$c\varepsilon^{3/2} D \subset K_1. \tag{6}$$

Note that (6) also remains true if we replace K_1 with a dual Steiner symmetrization of K_1. Next, as in the proof of Proposition 2.4, we have that $D \cap E^{\perp}$ is an M-ellipsoid of order 1 for $Proj_{E^{\perp}} \tilde{K} = Proj_{E^{\perp}} K_1 = K_1 \cap E^{\perp}$. By Theorem 2.3 there exists a subspace F of dimension $\lfloor 2\varepsilon n \rfloor$ that contains E such that

$$K_1 \cap F^{\perp} \subset \frac{c}{\varepsilon^{3/2}} D \cap F^{\perp}.$$

Note that all Steiner symmetrizations were carried out with respect to vectors inside F and hence the volume of $\tilde{K} \cap F$ is preserved. Reasoning as before, since $\tilde{K}$ is in M-position of order 1,

$$\left(\frac{Vol(K_1 \cap F)}{Vol\left(\frac{1}{\varepsilon} D \cap F\right)}\right)^{\frac{1}{dim(F)}} = \left(\frac{Vol(\tilde{K} \cap F)}{Vol\left(\frac{1}{\varepsilon} D \cap F\right)}\right)^{\frac{1}{dim(F)}} < C.$$

We apply $\lfloor 2\varepsilon n \rfloor$ dual Steiner symmetrizations to K_1, all in the subspace F according to Theorem 1.1, to obtain the body K_2. As before, we obtain that the body K_2 satisfies

$$Proj_F K_2 \subset \frac{c}{\varepsilon} D \cap F, \quad Proj_{F^{\perp}} K_2 \subset \frac{c}{\varepsilon^{3/2}} D \cap F^{\perp}.$$

Combining this with (6) we get that

$$c\varepsilon^{3/2}D \subset K_2 \subset \frac{C}{\varepsilon^{3/2}}D$$

and the proof is complete. □

Remark: It is possible to avoid the use of a linear image in Theorem 3.4, at the cost of replacing the geometric distance with a Banach-Mazur distance. i.e. For any centrally-symmetric convex body $K \subset \mathbb{R}^n$ there exist $\lfloor \varepsilon n \rfloor$ Steiner symmetrizations followed by $\lfloor \varepsilon n \rfloor$ dual Steiner symmetrizations that transform K into a body which is $\frac{c}{\varepsilon^3}$ close to an ellipsoid.

4 Rapid symmetrization implies slicing

It remains to prove the second implication in Theorem 1.4, that a positive answer to Question 1.2 implies a positive answer to Question 1.3. We begin with a few lemmas, the first of which is standard and well-known, and is proved here only for completeness.

Lemma 4.1 *Let $\mathcal{E}$ be an ellipsoid in $\mathbb{R}^n$. Then among all k-dimensional sections of $\mathcal{E}$, the intersection of $\mathcal{E}$ with the subspace spanned by the shortest k axes of the ellipsoid has a minimal volume.*

Proof: Choose orthogonal coordinates such that $\mathcal{E} = TD$ for a diagonal matrix T. Let $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the numbers on the diagonal. Let V be a matrix of k rows and n columns such that its rows are orthonormal vectors in $\mathbb{R}^n$. Writing volumes as determinants, we need to show that

$$\sqrt{det(VT^2V^t)} \geq \prod_{i=1}^{k} \lambda_i.$$

We will use the Cauchy-Binet formula. The sums in the next formula are over all subsets $A \subset \{1, .., n\}$ with exactly k elements. For such A, we write V_A for the matrix obtained from V by taking the columns whose indices are in A. Then,

$$det(VT^2V^t) = \sum_A det(V_A T^2 (V_A)^t) = \sum_A \left(\prod_{i \in A} \lambda_i^2 \right) det(V_A (V_A)^t)$$

$$\geq \left(\prod_{i=1}^{k} \lambda_i^2 \right) \sum_A det(V_A (V_A)^t) = \left(\prod_{i=1}^{k} \lambda_i^2 \right) det(VV^t) = \prod_{i=1}^{k} \lambda_i^2.$$

□

Lemma 4.2 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin. Assume that K is in isotropic position, and denote $d = d_{BM}(K)$. Then for any subspace E of dimension εn,*

$$Vol(K \cap E)^{1/n} > \left(\frac{c}{d}\right)^{\varepsilon}$$

where $c > 0$ is a numerical constant.

Proof: Let $\mathcal{E}$ be such that $\mathcal{E} \subset K \subset d\mathcal{E}$, and select an orthonormal basis $\{e_1, .., e_n\}$ and $0 < \lambda_1 \leq ... \leq \lambda_n$ such that $\mathcal{E} = \left\{x \in \mathbb{R}^n; \sum \frac{\langle x, e_i \rangle^2}{\lambda_i^2} \leq 1\right\}$. Since $K \subset d\mathcal{E}$,

$$c \sum_{i=1}^{n} \frac{1}{\lambda_i^2} < L_K^2 \sum_{i=1}^{n} \frac{1}{\lambda_i^2} = \int_K \sum_{i=1}^{n} \frac{\langle x, e_i \rangle^2}{\lambda_i^2} dx \leq d^2.$$

Therefore, by the Geometric-Harmonic means inequality,

$$\left(\prod_{i=1}^{\varepsilon n} \lambda_i\right)^{\frac{1}{\varepsilon n}} \geq \sqrt{\frac{\varepsilon n}{\sum_{i=1}^{\varepsilon n} \frac{1}{\lambda_i^2}}} \geq \sqrt{\frac{c \varepsilon n}{d^2}} > c' \frac{\sqrt{\varepsilon n}}{d}.$$

Let E_ε denote the subspace spanned by the shortest εn axes, $e_1, .., e_{\varepsilon n}$. By Lemma 4.1, $Vol(\mathcal{E} \cap E) \geq Vol(\mathcal{E} \cap E_\varepsilon)$ and

$$Vol(K \cap E)^{1/n} \geq Vol(\mathcal{E} \cap E)^{1/n} \geq Vol(\mathcal{E} \cap E_\varepsilon)^{1/n}.$$

Since

$$Vol(\mathcal{E} \cap E_\varepsilon)^{1/n} > \left(\prod_{i=1}^{\varepsilon n} \lambda_i\right)^{1/n} \frac{c}{(\sqrt{\varepsilon n})^{\varepsilon}} > \left(c' \frac{\sqrt{\varepsilon n}}{d\sqrt{\varepsilon n}}\right)^{\varepsilon}$$

the lemma is proven. □

Let $K \subset \mathbb{R}^n$ be a convex body, and let $E \subset \mathbb{R}^n$ be a subspace of dimension k. We define the Schwartz symmetrization of K with respect to E, as the unique body $S_E(K)$ such that:

(i) For any $x \in E^{\perp}$, $Vol(K \cap (x + E)) = Vol(S_E(K) \cap (x + E))$.

(ii) For any $x \in E^{\perp}$, the body $S_E(K) \cap (x + E)$ is a Euclidean ball centered at $E^{\perp}$.

We replace any section of K parallel to E with a Euclidean ball of the same volume. Schwartz symmetrization is a limit of a sequence of Steiner symmetrizations, and preserves volume and convexity. The following lemma is a reformulation of Theorem 2.5 in [BKM]. For a convex body $K \subset \mathbb{R}^n$ of volume one whose barycenter is at the origin, denote by M_K the operator defined by

$$\forall u, v \in \mathbb{R}^n, \quad \langle u, M_K v \rangle = \int_K \langle x, u \rangle \langle x, v \rangle dx.$$

Define also $Iso(K) = L_K M_K^{-1/2} K$. Then $Iso(K)$ is the unique isotropic image of K under a positive definite linear transformation.

Lemma 4.3 *Let $K \subset \mathbb{R}^n$ be a convex body of volume one whose barycenter is at the origin, and let $E \subset \mathbb{R}^n$ be a k-dimensional invariant subspace of M_K. Then,*

$$\left(\frac{1}{c}\frac{k}{n}\right)^{\frac{k}{n}} < \frac{L_{S_E(K)}}{L_K^{1-\frac{k}{n}}\ Vol(Iso(K) \cap E)^{\frac{1}{n}}} < \left(c\frac{n}{k}\right)^{\frac{k}{n}}$$

where $c > 0$ is a numerical constant.

Proof: If K is isotropic, then the lemma is just a particular case of Theorem 2.5 in [BKM]. Otherwise, since E is an invariant subspace of M_K,

$$Iso(S_E(Iso(K))) = Iso(S_E(K))$$

and hence the isotropic constant of $S_E(K)$ equals the isotropic constant of $S_E(Iso(K))$, and the lemma follows. □

Assume that there exist k Steiner symmetrizations that transform T into a body $\tilde{T}$ with $d_{BM}(\tilde{T}) < A$. Then also a Schwartz symmetrization of T with respect to a k-dimensional subspace that contains these k symmetrization vectors, transforms T into $\tilde{\tilde{T}}$ with $d_{BM}(\tilde{\tilde{T}}) < A$.

Proposition 4.4 *Let $K \subset \mathbb{R}^n$ be a convex body. Assume that there exists $T \subset K$ with $Vol(T) > \frac{9}{10}Vol(K)$, such that for any $\varepsilon > 0$ there exist $\lfloor \varepsilon n \rfloor$ symmetrizations, that transform T into $\tilde{T}$ with $d_{BM}(\tilde{T}) < c_1 \frac{1}{\varepsilon^{c_2}}$, where c_1, c_2 are independent of ε.*
Then $L_K < c(c_1, c_2)$ where $c(c_1, c_2)$ depends solely on its arguments.

Proof: By the discussion at the end of Section 1, we may assume that the barycenter of T is at the origin, that $Vol(T) = 1$ and that T is isotropic (symmetrizability is an affine invariant property). Also, for any $\varepsilon > 0$, there exists a subspace $E_{\varepsilon n} \subset \mathbb{R}^n$ of dimension $\lfloor \varepsilon n \rfloor$ such that the Schwartz symmetrization of T with respect to any subspace that contains $E_{\varepsilon n}$ is $\frac{c_1}{\varepsilon^{c_2}}$-close to an ellipsoid. Let us denote $\log^{(0)} n = n$ and $\log^{(i+1)} n = \log \max\{\log^{(i)} n, e\}$. Substitute $\delta_i = \frac{1}{(\log^{(i)} n)^2}$, and for i such that $\delta_i < \frac{1}{2}$ let

$$F_i = sp\{E_{\delta_1 n}, ..., E_{\delta_i n}\}$$

where sp denotes linear span. Denote $\varepsilon_i = \frac{1}{n} dim(F_i)$. Then $\frac{1}{(\log^{(i)} n)^2} \leq \varepsilon_i \leq \sum_{j=1}^{i} \frac{1}{(\log^{(j)} n)^2} < \frac{2}{(\log^{(i)} n)^2}$. Let T_i denote the Schwartz symmetrization of T with respect to F_i. Since $F_{i-1} \subset F_i$ we can think of

T_i as the Schwartz symmetrization of T_{i-1} with respect to F_i. According to our assumptions,

$$cL_{T_i} \leq d_{BM}(T_i) < c_1 \left(\frac{1}{\varepsilon_i}\right)^{c_2} < c_1 \left(\log^{(i)} n\right)^{2c_2}$$

where the left-most inequality appears in [MP1]. By Lemma 4.2, since $\varepsilon_{i+1} < \frac{2}{\left(\log^{(i+1)} n\right)^2}$,

$$Vol(Iso(T_i) \cap E_{i+1})^{1/n} > \left(\frac{c}{c_1(\log^{(i)} n)^{2c_2}}\right)^{\frac{2}{(\log^{(i+1)} n)^2}} > C^{\frac{1}{\log^{(i+1)} n}}$$

and hence by Lemma 4.3, since F_{i+1} is an invariant subspace of M_{T_i} (recall that T is isotropic, and symmetrizations were applied only with respect to subspaces contained in F_{i+1}),

$$L_{T_{i+1}} > \left(\frac{c}{\left(\log^{(i+1)} n\right)^2}\right)^{\frac{2}{\left(\log^{(i+1)} n\right)^2}} L_{T_i}^{1-\frac{2}{\left(\log^{(i+1)} n\right)^2}} C^{\frac{1}{\log^{(i+1)} n}}$$

and since $L_{T_i} < c\left(\log^{(i)} n\right)^{2c_2}$,

$$L_{T_{i+1}} > c^{\frac{1}{\log^{(i+1)} n}} L_{T_i} > \ldots > c^{\sum_{j=1}^{i+1} \frac{1}{\log^{(j)} n}} L_T.$$

Let i^* be the largest integer such that $\varepsilon_i < \frac{1}{2}$. Then T_{i^*} has a bounded distance from an ellipsoid, and $L_{T_{i^*}} < c(c_1, c_2)$ (see [MP1]). Therefore,

$$L_T < c^{\sum_{j=1}^{t^*} \frac{1}{\log^{(j)} n}} c(c_1, c_2) < c'(c_1, c_2)$$

and since $L_K \approx L_T$ (e.g. [Bou] or Borell lemma), the proposition is proved. □

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