Volumes in High Dimensions - Remarks on the Home Assignments

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In this file I want to address some points about the assignments you have submitted.

Week 1

- In questions 1, many students had problems with the tail estimation. Since the condition is $|\theta_i| \leq 4/\sqrt{n}$ some of the θ_i can be very small. Hence, using the estimation we saw in class without modifications will not work.

  A simple fix for this, using the fact that $\theta \in S^{n-1}$, and so, many of the θ_i's are not too small. For example, let
  $$k = \# \left\{ i;\ |\theta_i| \leq \frac{1}{10\sqrt{n}} \right\}.$$
  Without loss of generality, assume $0 \leq \theta_1 \leq \ldots \leq \theta_n$ we have
  $$1 = \sum_{i=1}^{n} \theta_i^2 = \sum_{i=1}^{k} \theta_i^2 + \sum_{i=k+1}^{n} \theta_i^2 \leq k\frac{1}{100n} + (n-k)\frac{25}{n}.$$
  From this we have $k \leq n/30$.

  To finish, we note that the estimation does not change if a small fraction are big, since $|\mathrm{sinc}(t)| \leq 1$.

- Question 3 can be found as Theorem 1 (which is a special case of Theorem 3) in [1].

Week 2

The solutions were very nice. A remark about question 4. In class you proved concentration around the *median* and the *expectation*. In order to prove concentration around other values (such as the L_2 norm) it is enough to show that it is close to one of those values. Since $f \geq 0$ we have

$$\begin{aligned}\left| \int_{S^{n-1}} f d\sigma_{n-1} - \sqrt{\int_{S^{n-1}} f^2 d\sigma_{n-1}} \right| &\leq \left| \int_{S^{n-1}} f d\sigma_{n-1} - \sqrt{\int_{S^{n-1}} f^2 d\sigma_{n-1}} \right| \left| \int_{S^{n-1}} f d\sigma_{n-1} + \sqrt{\int_{S^{n-1}} f^2 d\sigma_{n-1}} \right| \\ &= \left| \left(\int_{S^{n-1}} f d\sigma_{n-1} \right)^2 - \int_{S^{n-1}} f^2 d\sigma_{n-1} \right| = \mathrm{Var}(f).\end{aligned}$$

Hence, we may use Poincaré's inequality.

Week 3

No remarks about the submitted assignments. I do recommend that you look at the questions that are not for submission, and give yourself an outline of the proof.

Week 4

In question 3 you had to calculate the expectation of $\max\{X_i\}$ where X is a uniform vector in the sphere. We know that

$$\frac{(G_1,\dots,G_n)}{\sqrt{G_1^2+\cdots+G_n^2}},$$

where $G_1,\dots,G_n$ are independent Gaussians, sn equal in distrubiton to X and by concentration we get that X is very close to $(G_1,\dots,G_n)/\sqrt{n}$. This can be very useful for estimating probabilities of some events and to give us intuition, but when calculating expectation we have another tool that can give us a cleaner and more acurate result.

Proposition 1. *Let f be a $p-$homogeneous function. Let θ be a uniform random vector on the sphere, and let G be a random vector with independent standard Gaussian entries. Then,*

$$\mathbb{E}f(\theta) = C_{n,p}\mathbb{E}f(G),$$

where $C_{n,p} \approx n^{-p/2}$

Proof. We use integration in polar coordinates:

$$\mathbb{E}f(G) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-|x|^2/2}dx = \frac{n\kappa_n}{(2\pi)^{n/2}}\int_0^\infty r^{n-1}\int_{S^{n-1}} f(r\theta)e^{-r^2/2}d\sigma_{n-1}(\theta)dr,$$

where κ_n is the volume of the unit ball. Since f is homogeneous, we have

$$\mathbb{E}f(G) = \frac{n\kappa_n}{(2\pi)^{n/2}}\int_0^\infty r^{n-1+p}e^{-r^2/2}dr\int_{S^{n-1}} f(\theta)d\sigma_{n-1}(\theta) = C_{p,n}^{-1}\mathbb{E}f(\theta).$$

Setting $t = r^2/2$ we get,

$$\int_0^\infty r^{n+p-1}e^{-r^2/2}dr = 2^{n/2+p/2-1}\Gamma(n/2+p/2).$$

Remembering that $\kappa_n = \pi^{n/2}/\Gamma(n/2+1)$, we have

$$C_{p,n}^{-1} = \frac{2^{p/2-1}n\Gamma(n/2+p/2)}{\Gamma(n/2+1)}.$$

The Stirling formula finishes the proof. ☐

Note that changing G to $G/\sqrt{n}$ (Gaussian with variance $1/n$) gives us a constant that is approximately one.

Week 5

In question 1, there was a typo. It should have been,

$$\mathbb{P}\left(\left||\mathrm{Proj}_F(y_j)|^2 - \frac{k}{n}\right| \ge \varepsilon\frac{k}{n},\ \text{for some } j=1,\dots,N\right) \le e^{-c\varepsilon^2 k}.$$

Also, I noticed some confusion about the formulation:

> Let $y_1,\dots,y_N \in S^{n-1}$ where $N \le e^{c\varepsilon^2 k}$.

This means, that you need to show that for some universal constant $c>0$ (that does not depend on n, N, ε, or any other parameter), the statement holds true for any $N \le e^{c\varepsilon^2 k}$.

Week 6

The solutions where good. Please note that in the instructions (and your solutions) of question 1, there was an hidden assumption the the process is finite, that is a Gaussian vector in $\mathbb{R}^N$ for some $N > 0$.

As in many other proofs in this subject, we start by proving for finite set T, and show that the cardinality of T does not effect the result. Then we need to take supremum in order to move to infinite sets (question 2 in the homework).

Week 7

As we wrote to you, questions 3 had a mistake. The correct set to consider is

$$T = \left\{ \frac{e_i}{\sqrt{1 + \log i}};\ 1 \leq i \leq n \right\}.$$

To see a solution, please look at page 44 in [2]. There are some notations you should know before you read this.

1. The letter L denotes some universal constant.
2. The cardinality N_s is defined by $N_s = 2^{2^s}$.
3. Entropy numbers $e_n(T)$. You can define them in two ways:

   $$e_n(T) = \inf_{T_n} \sum_{t \in T} d(t, T_n),$$

   where the infimum is taken over all sets T_n with cardinality at most N_n. Another definition, is by covering numbers.

   $$e_n(T) = \inf \left\{ \varepsilon;\ N(T, d, \varepsilon) \leq N_n \right\}.$$

   This connection is exactly the one we used to move from the usual *chaining* argument to *generic chaining*.

Week 8

You should note that in question 2 part 3, we use step functions to find a net for Lipschitz functions. Since step functions are not Lipschitz this does not directly gives us a bound on the covering numbers. Denote this external covering number by $N_{\text{ext}}(T, d, \varepsilon)$ (this is covering that allows centers outside of the set T). By the triangle inequality we have,

$$N(T, d, \varepsilon) \leq N_{\text{ext}}(T, d, \varepsilon/2).$$

In addition, trivially we have

$$N_{\text{ext}}(T, d, \varepsilon) \leq N(T, d, \varepsilon).$$

References

[1] Franck Barthe, Olivier Guédon, Shahar Mendelson, and Assaf Naor. A probabilistic approach to the geometry of the l_p^n-ball. *Ann. Probab.*, 33(2):480–513, 2005.

[2] Michel Talagrand. *The generic chaining.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005. Upper and lower bounds of stochastic processes.