Size of line intersections in high dimensional L_p-balls and product measures

Bo'az Klartag
Weizmann Institute of Science

Joint work with Dor Elboim

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Segments in convex bodies in high dimensions

What's the first thing you note in high dim. Euclidean geometry? Maybe:

The unit cube $[0,1]^n \subseteq \mathbb{R}^n$ has volume one, yet it contains long segments "in diagonal", of length $\sqrt{n}$.

- The lengthscale of $\sqrt{n}$ is typical in high dim., it is roughly the radius of the Euclidean ball of volume one.

Theorem (Classical isodiametric inequality)

Any convex $K \subseteq \mathbb{R}^n$ of volume one, contains a segment of length at least

$$\left(\sqrt{\frac{2}{\pi e}} + o(1) \right) \cdot \sqrt{n},$$

the minimum is attained for the Euclidean ball.

Removing a little bit of mass

Question

Can one remove 1% of the mass of the convex body, and avoid these long segments?

Trivial answer: remove points with a rational coordinate or so.

Better formulation

Let $K \subseteq \mathbb{R}^n$ be convex, volume one. Does there exist a subset $A \subseteq K$ with $|A| = 1/2$ such that

$$|A \cap \ell| < C$$

for all lines in $\mathbb{R}^n$? Here $C > 0$ means a universal constant.

- The constant $1/2$ can be replaced by any $c \in (0, 1)$.
- If K is the Euclidean ball, the answer is YES.
  We select $A = K \setminus (1 - \frac{c}{n})K$, a thin spherical sphell.

The curvature of the Euclidean sphere

The case of the centered Euclidean ball $K \subseteq \mathbb{R}^n$ of volume one

Set $A = K \setminus (1 - \frac{1}{n})K$, so that $|A| = 1 - (1 - \frac{1}{n})^n \approx 1 - 1/e$. Fix $x \in \mathbb{R}^n$ and a unit vector θ. We need all $t \in \mathbb{R}$ such that

$$\left(1 - \frac{1}{n}\right)^2 r_n^2 \leq |x + t\theta|^2 \leq r_n^2$$

with $r_n = Vol_n(B^n)^{-1/n} \sim \sqrt{n}$. This has a bounded measure.

- For this K and for $A \subseteq K$ with $|A| = 1/2$, there is always a line ℓ with $|A \cap \ell| > c$, by Fubini's theorem.
- Slightly less trivial: The same is true for any convex body $K \subseteq \mathbb{R}^n$ of volume one.
- Consider $B_p^n = \left\{x \in \mathbb{R}^n \,;\, \|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \leq \kappa_{p,n}\right\}$ for $1 \leq p \leq \infty$, where $\kappa_{p,n} = \frac{\Gamma(1+n/p)^{1/n}}{2\Gamma(1+1/p)} = \Theta(n^{1/p})$.

The quantity $L(\mu, a)$

Definition

For a probability measure μ on $\mathbb{R}^n$ and $0 < a < 1$ define

$$L(\mu, a) := \inf_{\mu(A) \geq a} \sup_{\ell \text{ line}} \left| \ell \cap A \right|,$$

where the inf runs over all Borel sets $A \subseteq \mathbb{R}^n$ with $\mu(A) = a$.

- For $K \subseteq \mathbb{R}^n$ of volume one, abbreviate $L(K, a) = L(\lambda|_K, a)$, where λ is the Lebesgue measure.

Theorem 1

Let μ be the uniform measure on the unit cube, or the regular simplex, or the standard Gaussian measure γ_n in $\mathbb{R}^n$. Then,

$$L(\mu, 1/2) = \Theta(n^{1/4}).$$

- Here, the extremal set A is a thin spherical shell.

What's going on with these L_p-balls?

- Let X be a random vector, $X \sim Unif(B_p^n)$. For $p \neq 2$,

$$Var(\|X\|_2) = \Theta_p(1)$$

while for $p = 2$ we have $Var(\|X\|_2) = \Theta(1/n)$.
The thin spherical shell does not explain everything:

Theorem 2

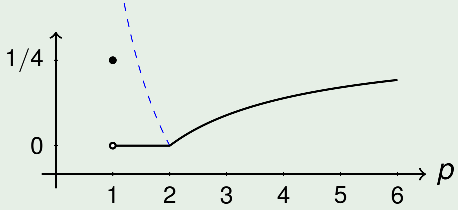
Fix $1 \leq p \leq \infty$. Then as $n \to \infty$,

$$L(B_p^n, 1/2) = \begin{cases} \Theta\left(n^{1/4}\right) & p = 1, \infty \\ \Theta_p\left((\log n)^{\frac{2-p}{2p}}\right) & 1 < p \leq 2 \\ \Theta_p\left(n^{\frac{p-2}{4p+2}}\right) & 2 \leq p < \infty \end{cases}$$

- The constant $1/2$ can be replaced by any fixed $a \in (0, 1)$.

Phase transitions of different types

The exponent of n as a function of p (and the exponent of $\log n$ in blue)



- Phase transitions of different types... But do we have universality in some class? Isn't $n^{1/4}$ somewhat universal?

Definition

A product measure $\frac{d\mu}{dx} = \prod_{i=1}^n \rho_i(x_i)$ in $\mathbb{R}^n$ is "admissible" if

(i) Each ρ_i is smooth and positive, at least in the interval $(-1/2, 1/2)$, with uniform estimates.

(ii) Each ρ_i has a sub-Gaussian tail, with uniform estimates.

Universality for product measures

Theorem 3

Let $n \geq 1, 0 < a < 1$ and let μ be an admissible product probability measure in $\mathbb{R}^n$. Then,

$$L(\mu, a) = \begin{cases} \Theta\left(a \cdot n^{1/4}\right) & e^{-n} \leq a \leq 1/2 \\ \Theta\left(n^{1/4} \cdot |\log(1-a)|^{1/4}\right) & 1/2 \leq a \leq 1 - e^{-n} \end{cases}$$

- Using Chen's recent bound for the thin shell constant, relying on Eldan's stochastic localization:

Proposition 4

Let $K \subseteq \mathbb{R}^n$ be a convex body of volume one in isotropic position (i.e., scalar covariance matrix). Then, uniformly over K,

$$L(K, 1/2) = O\left(n^{1/4+o(1)}\right).$$

Can $L(\mu, 1/2)$ be much larger than $n^{1/4}$?

- Outside the realm of convexity and product measures:

Radially-symmetric example for μ in $\mathbb{R}^n$ with $L(\mu, 1/2) = \Theta(\sqrt{n})$

Take μ to be the law of the random vector

$$X + UY$$

where X, Y, U are independent, with X, Y standard Gaussians in $\mathbb{R}^n$ and U uniform in $[0, 1]$.

Proof of the lower bound: suppose that $\mu(A) \geq 1/2$. Then,

$$\mathbb{P}(X + UY \in A, |Y| \geq \sqrt{n}/2) \geq \mathbb{P}(X + UY \in A) - Ce^{-cn} \geq 1/3.$$

Hence there exist $x, y \in \mathbb{R}^n$ s.t. for the line $\ell = x + \mathbb{R}y$,

$$|\ell \cap A| \geq |y| \cdot \mathbb{P}(x + Uy \in A) \geq \sqrt{n}/6. \quad \square$$

What mathematics does this story remind us of?

A sample of mathematical directions, maybe of a similar spirit:

1. Density **Hales-Jewett Theorem.** Any subset $A \subseteq \{1, \ldots, D\}^n$ of positive density contains a combinatorial line (a row, a column or a "diagonal"). Here D is fixed, $n \to \infty$. This gives "length one", not $n^{1/4}$.
2. **The lower-dimensional Busemann-Petty problem.** Let $K, T \subseteq \mathbb{R}^n$ be convex, volume one, centrally-symmetric. For $2 \leq \ell \leq n-1$, is there an ℓ-dimensional subspace E with
$$|K \cap E| \geq |T \cap E|?$$
3. The **Radon transform** $R(1_A)$ with $A \subseteq [0,1]^n$ of positive density, satisfies $\|R(1_A)\|_\infty \geq cn^{1/4}$ ("reverse Kakeya"?)
4. **Szemerédi-Trotter Theorem.** Given n points and m lines in the plane, there are at least $(nm)^{2/3}$ incidences.

Proofs – Needle decomposition

Our idea for the lower bound: need to approximate μ by a mixture of uniform measures on **long segments.**

Consider the case where $\mu = \gamma_n$ is Gaussian.

A computation (hint: use relative entropy and Pinsker's inequality)

Let $X, Y \sim \gamma_n$ be independent. Then for $r < cn^{-1/4}$,

$$d_{TV}(X, X + rY) \leq 1/10.$$

Hence $d_{TV}(X, X + rUY) \leq 1/10$ for $U \sim Unif([0, 1]), r = cn^{-1/4}$.

- A family of uniform measures on intervals

$$[X, X + rY]$$

  of length $\sim n^{1/4}$ approximates the Gaussian measure well.
- There is a related effect for the uniform measure on the cube.

The exponent in the case $p > 2$

- Given $X \sim Unif(B_p^n)$. Need $d_{TV}(Y, X) \leq 1/4$ with

$$Y_i = X_i + rU\delta_i\psi(X_i)$$

where $\delta \in \{\pm 1\}$ i.i.d symmetric Bernoulli, $U \sim Unif([0, 1])$.

Only small coordinates move, since $(t^p)''$ is small near zero. We take φ to be a bump function in $[1/2, 2]$, $\beta \geq \alpha$ and

$$Y_i = X_i \pm n^{-\beta}\varphi(n^\alpha X_i)$$

Requirements

1. Not too many coordinates enter the interval $[-n^{-\alpha}, n^{-\alpha}]$. This gives the constraint $1 + \alpha - 2\beta \leq (1 - \alpha)/2$.
2. The change in $\|X\|_p^p$ is at most a constant. This gives the constraint $\alpha(p - 1) + 2\beta \geq 1$.

Extremal set is $B_p^n \cap \{|\sum_i h(x_i) - E| \leq C\}$ for certain convex h.

The behaviour in the case $1 < p < 2$

Differences with the previous case:

1. Here $(t^p)''$ is small for large t. So better to move large coordinates.
2. The extremal set, without large line intersections, is
$$A = B_p^n \cap \{\kappa_{p,n}^p - C \leq \|x\|_p^p \leq \kappa_{p,n}^p\} \cap \{\forall i, |x_i| \leq C(\log n)^{1/p}\}.$$
3. The contribution from $(t^p)'$ should be "cancelled". Here the terms are too big, so we move the coordinates in pairs, and in opposite direction, to improve cancellation.
4. We randomly move coordinate X_i with $|X_i| \sim R = (\log n)^{1/p}$ to distance δ so that
$$\#(points) \cdot R^{p-2} \cdot \delta^2 \leq C.$$
The segment length is about $\sqrt{\#(points)} \cdot \delta$.

The end

Thank you!