The Santaló point of a function, and a functional form of Santaló inequality

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Abstract

Let $\mathcal{L}(f)$ denotes the Legendre transform of a function $f : \mathbb{R}^n \to \mathbb{R}$. We generalize a theorem of K. Ball about even functions, and prove that for any measurable function $f \geq 0$, there exists a translation $\tilde{f}(x) = f(x-a)$ such that

$$\int_{\mathbb{R}^n} e^{-\tilde{f}} \int_{\mathbb{R}^n} e^{-\mathcal{L}(\tilde{f})} \leq (2\pi)^n. \tag{1}$$

If we select a as to minimize the left hand side of (1), then equality in (1) is satisfied if and only if e^{-f} is the distribution of a Gaussian random variable. This inequality immediately implies Santaló inequality for convex bodies, as well as a new concentration inequality for the Gaussian measure.

1 Introduction

In this paper we present some developments in the study of the geometry of log-concave functions. This approach is further continued in [KM]. Log-concave functions have been investigated intensively by many authors, and we view this paper as a step in the "geometrization" of log-concave functions. We find that the intuition coming from the study of convex bodies enables one to formulate functional inequalities which turn out to be of independent interest. Also, the functional inequalities can sometimes be applied to functions related to a convex body, and give back strong inequalities for convex bodies. This scheme was pursued for example in [K].

*The first and third authors were partially supported by a BSF grant. The second named author was supported by NSF grant DMS-0111298 and the Bell Companies Fellowship.

A measure μ on $\mathbb{R}^n$ is called log-concave if for any measurable $A, B \subset \mathbb{R}^n$ and any parameter $0 < \lambda < 1$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}, \tag{2}$$

where $A + B = \{a + b : a \in A, b \in B\}$ is the Minkowski sum of A and B and $\lambda A = \{\lambda a : a \in A\}$ is the λ-homothety of A.

The first example of a log-concave measure is the standard Lebesgue measure Vol_n on $\mathbb{R}^n$. The log-concavity of the Lebesgue measure follows from the Brunn-Minkowski inequality (see (5) below). Similarly, a uniform measure on a convex body is log-concave. More examples for log-concave measures stem from Brunn's concavity principle [Br]. This principle states that any lower dimensional marginal of a uniform measure on a convex body is a log-concave measure. Moreover, marginals of uniform measures on convex bodies are essentially the only source for log-concave measures, as these marginals form a dense subset in the class of all log-concave measures.

A function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if $\log f$ is concave on the support of f. The two notions are closely related, as was shown in [Bo]: a measure μ on $\mathbb{R}^n$ whose support is not contained in any affine hyperplane is log-concave if and only if it is absolutely continuous with respect to the Lebesgue measure, and its density is a log-concave function. Therefore the standard Gaussian measure on $\mathbb{R}^n$ with density $(\sqrt{2\pi})^{-n/2}\exp\{-|x|^2/2\}$, is a log-concave measure, where $|\cdot|$ is the standard euclidean norm in $\mathbb{R}^n$.

One of the fundamental tools in convex geometry is that of duality. For a convex body $K \subset \mathbb{R}^n$ (a compact convex set with the origin in its interior), its polar is defined by

$$K^\circ = \{x \in \mathbb{R}^n : \sup_{y \in K}\langle x, y\rangle \leq 1\},$$

where $\langle x, y\rangle$ is the standard scalar product in $\mathbb{R}^n$. Generalizing the theory of convex bodies to log-concave functions, one of the first tasks is to understand what is the correct definition of the dual of a function. A hint may arise from the fact that log-concave functions are essentially marginals of convex bodies, and we may find a way to induce the notion of duality from convex bodies to their marginals, the log-concave measures. This idea leads, as explained in Section 3, to the following definition. For a log-concave function $f : \mathbb{R}^n \to [0, \infty)$ we define its dual (or polar) as

$$f^\circ(x) = \inf_{y \in \mathbb{R}^n}\left[e^{-\langle x,y\rangle}/f(y)\right]. \tag{3}$$

To better understand this definition recall the classical Legendre transform (see e.g. [Ar]): For a function $\varphi : \mathbb{R}^n \to \mathbb{R}$, its Legendre transform is defined by

$$\mathcal{L}\varphi(x) = \sup_{y\in\mathbb{R}^n} \left[\langle x, y\rangle - \varphi(y)\right].$$

The above definition of polarity is simply Legendre transform in the logarithm, as $-\log f^\circ = \mathcal{L}(-\log f)$. Definition (3) makes sense for any non-negative function f, not necessarily log-concave. As the Legendre transform of any function is a convex function, f° is always log-concave. Also, if φ is convex and lower semi-continuous (see [R]) then $\mathcal{L}\mathcal{L}\varphi = \varphi$. Translating to our language, if f is a log-concave upper semi continuous function, then $f^{\circ\circ} = f$. The semi continuity assumptions are rather technical. Any log-concave function may be modified on a set of Lebesgue measure zero, to become upper semi continuous and log-concave.

As an example, consider a norm $\|\cdot\|_K$ on $\mathbb{R}^n$ with unit ball K. It is easy to check that for the function $\varphi(x) = \frac{1}{2}\|x\|_K^2$ (which is convex) one has $(\mathcal{L}\varphi)(x) = \frac{1}{2}\|x\|_{K^\circ}^2$; that is, Legendre transform has a clear relation to usual duality of convex bodies. Denote by $\|\cdot\|^*$ the dual norm. Then the functions $\exp\{-\frac{1}{2}\|x\|^2\}$ and $\exp\{-\frac{1}{2}\left(\|x\|^*\right)^2\}$ are dual functions, in the sense of 3. In fact, for every pair $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ the relation

$$\left(\exp\left\{-\frac{1}{p}\|x\|^p\right\}\right)^\circ = \exp\left\{-\frac{1}{q}(\|x\|^*)^q\right\} \tag{4}$$

holds true. Also,

$$(1_K)^\circ = \exp\{-\|x\|_{K^\circ}\},$$

where 1_K denotes the indicator function of the convex body K, that is, a function which is 1 on K and 0 on its complement. Finally, it is not difficult to see (e.g. Lemma 2.2) that the *only* self dual function is the Gaussian $g(x) = \exp\{-\frac{1}{2}|x|^2\}$, and therefore it will play the role which, in the theory of convex bodies, is played by the unique self dual body - the euclidean ball, which we denote by D_n.

Another fundamental tool in convex geometry is Minkowski addition. To define the right analogue for addition and multiplication by scalar of functions, we consider another known operation, which is related to the Legendre transform, namely the Asplund product. Given two functions $f, g : \mathbb{R}^n \to [0, \infty)$, their Asplund product is defined by

$$(f \star g)(x) = \sup_{x_1+x_2=x} f(x_1)g(x_2).$$

It is easy to check that $1_K \star 1_T = 1_{K+T}$ and $(f \star g)^\circ = f^\circ g^\circ$, so the dual of an Asplund product is the usual product of the dual functions. We argue that Asplund product of log-concave functions is the right analogue for Minkowski addition of convex bodies.

To define the λ-homothety of a function $f(x)$, which we denote $(\lambda \cdot f)(x)$ we use

$$(\lambda \cdot f)(x) = f^{\lambda}\left(\frac{x}{\lambda}\right).$$

Note that for a log-concave f one has indeed $f \star f = 2 \cdot f$, and that $(\lambda \cdot f)^{\circ} = (f^{\circ})^{\lambda}$.

To check whether the definitions of Minkowski addition, of homothety and of duality for log-concave functions are meaningful and make sense, it is natural to ask whether the basic inequalities for convex bodies such as the Brunn-Minkowski inequality and the Santaló inequality remain true. The role of "volume" will be played of course by the integral. We discuss first the Brunn-Minkowski inequality.

In its dimension free form, Brunn-Minkowski inequality states that for any two bodies $A, B \subset \mathbb{R}^n$ and for any $0 < \lambda < 1$ one has

$$Vol(\lambda A + (1-\lambda)B) \geq Vol(A)^{\lambda} Vol(B)^{1-\lambda}. \tag{5}$$

We have ready made a functional analogue of the Brunn-Minkowski inequality, namely the Prékopa-Leindler inequality (see e.g. [P]). In the above notation it states precisely that

Theorem 1.1 *(Prékopa-Leindler) Given $f, g : \mathbb{R}^n \to [0, \infty)$ and $0 < \lambda < 1$,*

$$\int (\lambda \cdot f) \star ((1-\lambda) \cdot g) \geq \left(\int f\right)^{\lambda} \left(\int g\right)^{1-\lambda}.$$

(notice that the multiplication is in homothety sense and not standard multiplication!)

The standard Brunn-Minkowski inequality follows directly from Prékopa-Leindler by considering indicator functions of sets.

Let us turn to Santaló inequality. Santaló inequality for convex bodies says that for a centrally-symmetric set $K \subset \mathbb{R}^n$ (i.e. $K = -K$) with a finite positive volume, one has

$$Vol(K)Vol(K^{\circ}) \leq (Vol(D_n))^2. \tag{6}$$

Equality holds if and only if K is an ellipsoid. A non-centrally symmetric version also holds true, although there one needs to choose the right center with respect to which the polarity is defined, as there is no natural "zero" and in principal the polar body may be unbounded. More precisely, Santaló proved that (see [MeP2] for a simpler proof)

Theorem 1.2 *For any convex body K there exists a point x_0 such that, denoting $\tilde{K} = K - x_0$, one has*

$$\mathrm{Vol}(\tilde{K})\mathrm{Vol}(\tilde{K}^{\circ}) \leq (\mathrm{Vol}(D_n))^2. \tag{7}$$

It is possible to set x_0 as the center of mass of K. The minimum over x_0 of the left hand side is equal $(\mathrm{Vol}(D_n))^2$ if and only if K is an ellipsoid.

The point x_0 for which the minimal product in (7) is attained, is called the Santaló point of the body K. This point is zero if and only if the barycenter of the dual body lies at the origin.

The corresponding inequality for log-concave functions can be stated as follows

Theorem 1.3 *Let $f : \mathbb{R}^n \to [0,\infty)$ be any function such that $0 < \int f < \infty$. Then, for some vector x_0, defining $\tilde{f}(x) = f(x - x_0)$, one has*

$$\int_{\mathbb{R}^n} \tilde{f} \int_{\mathbb{R}^n} \tilde{f}^\circ \le (2\pi)^n. \tag{8}$$

If f is log-concave, we may choose $x_0 = \frac{\int x f(x)}{\int f(x)}$, the center of mass of f. The minimum over x_0 of the left hand side product equals $(2\pi)^n$ if and only if f is a gaussian.

Note that (8) is precisely inequality (1) stated in the abstract.

The Santaló point of a function f is the point x_0 that minimizes $\int_{\mathbb{R}^n} (f(x - x_0))^\circ \, dx$. As in the case of convex bodies, this point is zero if and only if the barycenter of f° lies at the origin.

The main part of this paper is devoted to the proof of Theorem 1.3. It has recently come to our attention that a statement equivalent to inequality (8), for the case of an even function f, appeared in the Ph.D. thesis of K. Ball [Ba1]. We reproduce his proof in Section 2 below for the convenience of the reader.

As one would expect, Santaló inequality for convex bodies follows easily from Theorem 1.3. Indeed, a standard computation gives that (also in the case of non-symmetric K, which includes the origin, in which case $\|x\|_K$ is defined by $\inf\{r : x \in rK\}$)

$$\int_{\mathbb{R}^n} e^{-\frac{\|x\|_K^2}{2}} dx = c_n \mathrm{Vol}(K), \tag{9}$$

where $c_n = (2\pi)^{n/2}/\mathrm{Vol}(D_n)$ (see e.g. [P] p. 11). Using together (9), (8), and (4) we get Santaló inequality (7).

The rest of the paper is organized as follows. In Section 2 we present the argument which appeared in [Ba1]. In Section 3 we elaborate on the definition of the polar function, and in Section 4 we prove inequality (8). The equality case in Theorem 1.3 is settled in Section 5, and in Section 6 we present applications to Gaussian concentration. We thank G. Schechtmann for pointing to us the connection of our inequality with the paper of B. Maurey [Mau].

2 The even case

Theorem 2.1 *(K. Ball, [Ba1]) Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be an even convex function. Assume that $0 < \int e^{-\varphi} < \infty$. Then*

$$\int e^{-\varphi} \int e^{-\mathcal{L}\varphi} \le (2\pi)^n. \tag{10}$$

Proof: Note that for any $x, y \in \mathbb{R}^n$,

$$\varphi(x) + \mathcal{L}\varphi(y) \ge \langle x, y \rangle. \tag{11}$$

Abbreviate $[\varphi < t] = \{x \in \mathbb{R}^n : \varphi(x) < t\}$. If $\mathcal{L}\varphi(x) < s$, then for any $y \in [\varphi < t]$ by (11) we have $\langle x, y \rangle < s + t$. Hence for any $s, t \in \mathbb{R}$,

$$[\mathcal{L}\varphi < s] \subset (s + t)[\varphi < t]^\circ.$$

Since φ is even, the set $[\varphi < t]$ is centrally-symmetric, and by Santaló inequality (6),

$$Vol([\mathcal{L}\varphi < s]) \le (s + t)^n Vol([\varphi < t]^\circ) \le (s + t)^n \frac{Vol(D_n)^2}{Vol([\varphi < t])}.$$

Denote $f(t) = Vol([\varphi < t])$ and $g(s) = Vol([\mathcal{L}\varphi < s])$. Then for any $s, t \in \mathbb{R}$,

$$e^{-s} g(s) e^{-t} f(t) \le e^{-(s+t)} (s + t)^n Vol(D_n)^2.$$

In our notation, if $F(t) = e^{-t} f(t)$, $G(s) = e^{-s} g(s)$ and $H(u) = Vol(D_n) e^{-u} (2u)^{\frac{n}{2}}$, then $(\frac{1}{2} \cdot F) \star (\frac{1}{2} \cdot G) \le H$. The one dimensional Prékopa-Leindler inequality (Theorem 1.1 here) implies that

$$\int_{\mathbb{R}} e^{-t} Vol([\varphi < t]) \int_{\mathbb{R}} e^{-s} Vol([\mathcal{L}\varphi < s]) = \int F \int G \le \left(\int H \right)^2.$$

A straightforward calculation (e.g. [P], page 11) yields that $\left(\int H\right)^2 = (2\pi)^n$. Since $\int_{\mathbb{R}^n} e^{-\psi} = \int_{\mathbb{R}} e^{-t} Vol([\psi < t]) dt$ for any function ψ, the proof is complete. □

We are not aware of a straightforward generalization of Ball's argument to the non-even case. The difficulty lies in the fact that if f is not even, the one has to apply different translations to $[\varphi < t]$ for different values of t. It will be useful for us later on to note that in the one-dimensional case, Ball's proof demonstrates that gaussians are the only even functions satisfying equality in (10). For a convex even $\varphi : \mathbb{R} \to \mathbb{R}$, we say that it is a maximizer if $\int e^{-\varphi} \int e^{-\mathcal{L}\varphi} = 2\pi$.

Lemma 2.2 *Assume that φ is a maximizer. Then there exist $a > 0$ and $b \in \mathbb{R}$ such that $\varphi(t) = b + at^2$.*

Proof: Note first that if φ is a maximizer then for any $b \in \mathbb{R}$ and $a > 0$ also the function $t \mapsto \varphi(ta) + b$ is a maximizer. Indeed, $\mathcal{L}(\varphi(ta) + b) = \mathcal{L}\varphi\left(\frac{t}{a}\right) - b$ and hence

$$\int e^{-(\varphi(ta)+b)} \int e^{-\mathcal{L}(\varphi(ta)+b)} = \int e^{-\varphi} \int e^{-\mathcal{L}\varphi} = 2\pi.$$

Choosing the correct a and b, we may assume that $\int e^{-\varphi} = \int e^{-\mathcal{L}\varphi}$ and that $\varphi(0) = \mathcal{L}\varphi(0) = 0$. In order for equality in (10) to hold, the functions F, G and H from the above proof must satisfy the equality conditions in Prékopaá-Leindler inequality. These appear, e.g., in [Bar] and state that there exists $x \in \mathbb{R}$ such that $\frac{F}{\int F} = \frac{G(\cdot - x)}{\int G}$. According to our assumptions $\int F = \int G$, hence F is a translation of G. Also since φ and $\mathcal{L}\varphi$ are convex even functions, their minimum is $\varphi(0) = \mathcal{L}\varphi(0) = 0$. Therefore $F(t), G(t) = 0$ for $t < 0$ and $F(t), G(t) > 0$ for $t > 0$. Since F is a translation of G, necessarily $F(t) = G(t)$ and $Vol([\varphi < t]) = Vol([\mathcal{L}\varphi < t])$ for any $t \in \mathbb{R}$. Since the functions are one-dimensional and even, we conclude that $\varphi = \mathcal{L}\varphi$. By the definition of the Legendre transform, for any $s, t \in \mathbb{R}$,

$$\varphi(s) + \varphi(t) \geq st$$

and in particular, $\varphi(t) \geq \frac{t^2}{2}$ and hence $\mathcal{L}\varphi(t) \leq \frac{t^2}{2}$. Since $\varphi = \mathcal{L}\varphi$, we conclude that $\varphi(t) = \frac{t^2}{2}$. □

Remark: We want to remark that another proof of K. Ball, for Santaló inequality for convex bodies (see [Ba1],[MeP1]) can be directly generalized to our setting. The proof uses Steiner symmetrizations, and we can define the Steiner symmetrization of a function: For a function $f(x)$ defined on $\mathbb{R}^n$ and a hyperplane $H = e^{\perp}$ in $\mathbb{R}^n$ we denote, for $y \in H$ and $t \in \mathbb{R}$,

$$f_{(y,+)}(t) = f(y + te), \qquad f_{(y,-)}(t) = f(y - te)$$

and define its Steiner symmetrization by

$$(S_H f)(y + te) = \left(\frac{1}{2} \cdot f_{(y,+)}\right) \star \left(\frac{1}{2} \cdot f_{(y,-)}\right)(t).$$

It can be shown that for an even function f the product $s(f) = \int f \int f^{\circ}$ grows when a Steiner symmetrization is performed, and this eventually reduces the question to a one dimensional one.

3 Duality in the space of log-concave functions

A function $f : \mathbb{R}^n \to [0, \infty)$ is called s-concave if $f^{\frac{1}{s}}$ is concave on the support of f. (Note the perhaps non-standard definition, with

$\frac{1}{s}$ replaced by s). Any s-concave function, for $s > 0$, is also log-concave. It is also possible to approximate any log-concave function $f : \mathbb{R}^n \to [0, \infty)$ with an s-concave function f_s as follows:

$$f_s(x) = \left(1 + \frac{\log f(x)}{s}\right)_+^s \tag{12}$$

where $x_+ = \max\{x, 0\}$. The log-concavity of f implies the s-concavity of f_s. Note also that $f_s \le f$ for any $s > 0$, and since a log-concave function is continuous on its support one has $f_s \overset{s\to\infty}{\longrightarrow} f$ locally uniformly on $\mathbb{R}^n$.

Let $s > 0$ be an integer. By the Brunn concavity principle a function on $\mathbb{R}^n$ is s-concave if and only if it is a marginal of a uniform measure on a convex body in $\mathbb{R}^{n+s}$. One is thus tempted to define the polar of an s-concave function f, as the marginal of K° where $K \subset \mathbb{R}^{n+s}$ is some convex body whose marginal is f. This approach is problematic, as the body K whose marginal is f is not unique. However, having this idea in mind, we define

$$\mathcal{L}_s f(x) = \inf_{\{y: f(y)>0\}} \frac{\left(1 - \frac{\langle x,y\rangle}{s}\right)_+^s}{f(y)}.$$

Clearly $\mathcal{L}_s f \le f^\circ$. Define

$$K_s(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^s : \sqrt{s}x \in \overline{\text{supp} f}, \quad |y| \le f^{1/s}(\sqrt{s}x))\}.$$

Up to some rescaling, the function f is the marginal on $\mathbb{R}^n$ of the uniform measure on the body $K_s(f)$. In particular we have, denoting the volume of the unit euclidean ball in $\mathbb{R}^s$ by $\kappa_s = Vol(D_s) = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$, that

$$Vol(K_s(f)) = \int_{\mathbb{R}^n} \kappa_s f\left(\sqrt{s}x\right) dx = \frac{\kappa_s}{s^{n/2}} \int_{\mathbb{R}^n} f.$$

The following Lemma clarifies the relations between our various definitions.

Lemma 3.1 *For any* $f : \mathbb{R}^n \to [0, \infty)$,

$$(K_s(f))^\circ = K_s(\mathcal{L}_s(f)).$$

Proof: Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^s$. Then $(x, y) \in (K_s(f))^\circ$ if and only if for any $(u, v) \in \mathbb{R}^n \times \mathbb{R}^s$ such that $\sqrt{s}u \in \overline{\text{supp} f}$ and $|v| \le f^{1/s}(\sqrt{s}u)$ one has

$$\langle x, u\rangle + \langle y, v\rangle \le 1.$$

This is equivalent to the fact that for any $u' \in \overline{\text{supp} f}$ one has $\langle x, u'\rangle/\sqrt{s} + |y| f^{1/s}(u') \leq 1$. We conclude that $(x, y) \in (K_s(f))^\circ$ if and only if

$$|y|^s \leq \inf_{u' \in \overline{\text{supp} f}} \frac{\left(1 - \frac{\langle x, u'\rangle}{\sqrt{s}}\right)^s}{f(u')}.$$

Whenever the infimum is non-negative, it equals $\mathcal{L}_s f(\sqrt{s}x)$, and it is non-negative if and only if $\sqrt{s}x \in \overline{\text{supp}\mathcal{L}_s f}$. This completes the proof. □

If f is upper semi continuous, s-concave, and $f(0) > 0$, then $K_s(f)$ is closed, convex and contains the origin. Hence $K_s(f)^{\circ\circ} = K_s(f)$. By Lemma 3.1 we conclude that $\mathcal{L}_s \mathcal{L}_s f = f$ as $K_s(f)$ determines f and

$$K_s(f) = (K_s(f)^\circ)^\circ = K_s(\mathcal{L}_s(f))^\circ = K_s(\mathcal{L}_s(\mathcal{L}_s(f))).$$

However, for log-concave functions which are not s-concave, the transform $\mathcal{L}_s$ is less natural. For a fixed $s > 0$, the class of s-concave functions is rather restricted among the log-concave functions. This class becomes larger and larger as s approaches infinity, and the increasing union of all these classes is a dense subset in the space of log-concave functions. In view of this, it is natural that the notion of duality for log-concave functions we defined earlier, is the limit of $\mathcal{L}_s$ when $s \to \infty$. The proofs of the following lemmas are given in the Appendix.

Lemma 3.2 *Let $f, f_1, f_2, \ldots : \mathbb{R}^n \to [0, \infty)$ be log-concave functions such that $f_n \to f$ on a dense subset $A \subset \mathbb{R}^n$. Then,*

1. *$\int f_n \to \int f$.*
2. *If $\int f < \infty$ then $\int x f_n \to \int x f$.*
3. *$f_n^\circ \to f^\circ$ locally uniformly on the interior of the support of f°.*

Lemma 3.3 *Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function. Assume that $x_s \in \mathbb{R}^n$ and $x_s \overset{s\to\infty}{\longrightarrow} 0$. Then for $\widetilde{f}_s(x) = f_s(x - x_s)$,*

$$\liminf_{s\to\infty} \int_{\mathbb{R}^n} \mathcal{L}_s \widetilde{f}_s \geq \int_{\mathbb{R}^n} f^\circ.$$

where f_s is the s-concave function associated to f via (12).

Remark: A variant of our duality notions appeared many years ago in [Mi1] and [Mi2]. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a positive convex function. Let (y, t) be coordinates in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. Then for any $\mu > 0$, there exists a unique norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $\left\|\left(y, \frac{1}{\sqrt{\mu}}\right)\right\| = \frac{1+\mu f(y)}{\sqrt{\mu}}$ for $y \in \mathbb{R}^{n-1}$. Then, $\left\|\left(x, \frac{1}{\sqrt{\mu}}\right)\right\|^* = \frac{1+\mu \mathcal{T}_\mu f(x)}{\sqrt{\mu}}$ where

$$(\mathcal{T}_\mu f)(x) = \sup_{y \in \mathbb{R}^{n-1}} \frac{\langle x, y\rangle - f(y)}{1 + \mu f(y)}.$$

Note that $\lim_{\mu \to 0} \mathcal{T}_\mu = \mathcal{L}$.

4 Proof of functional Santaló

We are now ready for the proof of Theorem 1.3. The first Corollary is a direct consequence of the Santaló inequality for convex bodies and the considerations of the previous section.

Corollary 4.1 *Let f be an s-concave function on $\mathbb{R}^n$, with $0 < \int f < \infty$, and whose center of mass is at the origin (i.e. $\int x f(x) = 0$). Then*

$$\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} \mathcal{L}_s(f) \le \frac{s^n \kappa_{n+s}^2}{\kappa_s^2}$$

with equality if and only if f is a marginal of the uniform distribution of an $(n+s)$ dimensional ellipsoid.

Proof: Noticing that the center of mass of the convex body $K_s(f)$ remains at the origin, by Theorem 1.2

$$\int f \int \mathcal{L}_s(f) = \frac{s^n}{\kappa_s^2} \mathrm{Vol}(K_s(f)) \mathrm{Vol}((K_s(f))^\circ) \le \frac{s^n \kappa_{n+s}^2}{\kappa_s^2}.$$

Since 0 is the Santaló point of $K_s(f)^\circ$, equality holds if and only if $K_s(f)^\circ$ is an ellipsoid, i.e., if $K_s(f)$ is an ellipsoid and f is a marginal of the uniform distribution of an ellipsoid. □

Using the convergence of $\mathcal{L}_s f$ to f° we can infer the functional Santaló inequality.

Proof of Theorem 1.3: Since for any function $f^{\circ\circ} \ge f$, we may assume to begin with that $f : \mathbb{R}^n \to [0, \infty)$ is log-concave. Translating f if necessary, we also assume that the barycenter of f is at the origin. Denote by x_s the barycenter of f_s, and denote $\widetilde{f_s} = f_s(x - x_s)$. By Lemma 3.2, $x_s \to 0$ as $s \to \infty$. By Lemma 3.2, Lemma 3.3 and Corollary 4.1,

$$\int f \int f^\circ \le \liminf_{s \to \infty} \int \widetilde{f_s} \int \mathcal{L}_s \widetilde{f_s} \le \lim_{s \to \infty} \frac{s^n \kappa_{n+s}^2}{\kappa_s^2} = (2\pi)^n.$$

□

Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function with $0 < \int f < \infty$. Denote $F(z) = \int_{\mathbb{R}^n} (f(x-z))^\circ dx$, and assume that $\int f^\circ = \min_{z \in \mathbb{R}^n} F(z)$. Since $(f(x-z))^\circ = e^{-\langle x,z \rangle} f^\circ(x)$, then $F(z) = \int e^{-\langle x,z \rangle} f^\circ(x) dx$. We conclude that

$$0 = \nabla F(z)|_{z=0} = -\int_{\mathbb{R}^n} x f^\circ(x) dx.$$

The differentiation under the integral sign is allowed as the derivative is locally bounded and integrals converge (see the proof of Lemma 3.3 below). Hence, if the Santaló point of a function lies at the origin, so is the center of gravity of the dual function.

5 The equality case

Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function, such that $\int f^\circ < \infty$ and $\int x f^\circ(x) dx = 0$. By Theorem 1.3, $\int f \int f^\circ \leq (2\pi)^n$. We say that f is a maximizer if

$$\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} f^\circ = (2\pi)^n.$$

In this section we prove the following:

Proposition 5.1 *The function f is a maximizer if and only if it is a gaussian function, i.e. $f(x) = ce^{-\langle Ax,x \rangle}$ for a positive-definite A and some number $c > 0$.*

Let $f_n : \mathbb{R}^n \to [0, \infty)$ and $g_m : \mathbb{R}^m \to [0, \infty)$ be maximizers. Then the function $h : \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty)$ defined by

$$h(x,y) = f_n(x) g_m(y)$$

is also a maximizer. Indeed, h is log-concave, $h^\circ(x,y) = f_n^\circ(x) g_m^\circ(y)$ and hence the barycenter of h° is at the origin. Also, we have

$$\int_{\mathbb{R}^{n+m}} h \int_{\mathbb{R}^{n+m}} h^\circ = \int_{\mathbb{R}^n} f_n \int_{\mathbb{R}^n} f_n^\circ \int_{\mathbb{R}^m} g_m \int_{\mathbb{R}^m} g_m^\circ = (2\pi)^{n+m}.$$

Next we show that the set of maximizers is also closed under the operation of taking a marginal. We start with a variant of a result due to Meyer and Pajor [MeP2]. Recall that the Steiner symmetrization of a body K with respect to a hyperplane $H = v^\perp$ is the body T such that for any $x \in H$ the segment $T \cap [x + \mathbb{R}v]$ is centered at x and has the same length as $K \cap [x + \mathbb{R}v]$

Lemma 5.2 *Let $v^\perp = H \subset \mathbb{R}^n$ be a hyperplane ($v \in S^{n-1}$). Let $K \subset \mathbb{R}^n$ be a convex body. Then the Steiner symmetrization of K with respect to $H = v^\perp$, which we denote by $T \subset \mathbb{R}^n$, satisfies*

$$\min_z Vol((T-z)^\circ) \geq \min_z Vol((K-z)^\circ).$$

Proof: We use the terminology and results of [MeP2]. For $a \in \mathbb{R}^n$ denote, as in [MeP2], the set $(K-a)^\circ + a$ by K^a. Denote also $H_t = \{x : \langle x, v\rangle = t\}$, and choose any $z \in H$ such that $z + \mathbb{R}v$ intersects the interior of K. We claim that there exists a scalar $v(z) \in \mathbb{R}$ such that $H_{v(z)}$ is a medial hyperplane for $K^{z+v(z)v}$, in the sense that it partitions the body into two parts of equal volume. Indeed, put $f(t) = \frac{Vol(K^{z+tv} \cap H_t^+)}{Vol(K^{z+tv})}$, where $H_t^+ = \{x : \langle x, v\rangle \geq t\}$. Since K is convex, $K \cap (z + \mathbb{R}v)$ is a segment of the form $[z + t_1 v, z + t_2 v]$ for $t_1 < t_2$. Then $f(t) \to 0$ as $t \to t_1$ and $f(t) \to 1$ as $t \to t_2$. By continuity, there exist $v(z) \in \mathbb{R}$ such that $f(v(z)) = 1/2$. Let T be the Steiner symmetrization of K with respect to H. By Lemma 7 from [MeP2], for any $z \in H$,

$$Vol(T^z) \geq Vol(K^{z+v(z)v}) \tag{13}$$

(notice that symmetrizing with respect to parallel hyperplanes results in translates of the same body). Let $z_0 \in \mathbb{R}^n$ be such that $Vol(T^{z_0}) = \min_x Vol(T^x)$. Since T is invariant under reflections with respect to v necessarily $z_0 \in H$ (see Lemma 2 in [MeP2]). We see that

$$\min_x Vol\,(T^x) = Vol(T^{z_0}) \geq Vol(K^{z_0+v(z_0)v}) \geq \min_x Vol(K^x).$$

□

Remark: The lemma implies that the expression $Vol(K) min_x Vol(K^x)$ increases when a Steiner symmetrization is applied. Also, the Santaló point of a Steiner symmetrization of K with respect to H, is the orthogonal projection onto H of the Santaló point of K. By taking the limit of an appropriate sequence of Steiner symmetrizations, all with respect to vectors in $E^\perp$, we conclude that the expression $Vol(K) min_x Vol(K^x)$ increases also when a Schwartz symmetrization is applied. Recall that the Schwartz symmetral of K with respect to $E^\perp$ is the body T such that for any $x \in E$, the body $T \cap (x + E^\perp)$ is a euclidean ball centered at x and $Vol(T \cap (x + E^\perp)) = Vol(K \cap (x + E^\perp))$.

Lemma 5.3 *Let $f : \mathbb{R}^n \to [0, \infty)$ be a maximizer. Let $m < n$ and let $E \subset \mathbb{R}^n$ be an m-dimensional subspace. Define $g : E \to [0, \infty)$ by*

$$g(x) = \int_{x+E^\perp} f.$$

Then g is a maximizer.

Proof: Let x_s be the Santaló point of f_s (with respect to $\mathcal{L}_s$-duality) and denote $\tilde{f}_s(x) = f(x - x_s)$. By Corollary 4.1, for any $s > 0$,

$$\int \tilde{f}_s \int \mathcal{L}_s \tilde{f}_s \leq (2\pi)^n.$$

We claim that it is impossible that $|x_s| \to \infty$. Indeed, otherwise since $f_s \le f$ we have $\tilde{f}_s \to 0$ locally uniformly and by Lemma 3.3 we have $\liminf \int \mathcal{L}_s \tilde{f}_s = \infty$. Hence we may choose a converging subsequence, $x_{s_k} \to x_0$. Without loss of generality, we assume that $x_s \to x_0$. Denote $\tilde{f}(x) = f(x - x_0)$. Since f is a maximizer,

$$\liminf_{s\to\infty} \int \tilde{f}_s \int \mathcal{L}_s \tilde{f}_s \ge \int \tilde{f} \int \tilde{f}^\circ \ge \int f \int f^\circ = (2\pi)^n.$$

Hence

$$\liminf_{s\to\infty} \frac{s^n}{\kappa_s^2} Vol(K_s(f_s)) Vol((K_s(f_s) - x_s)^\circ) \ge (2\pi)^n.$$

This expression only becomes larger when x_s is replaced by any other point. Let s_0 be such that for $s > s_0$,

$$Vol(K_s(f_s)) \min_z Vol((K_s(f_s) - z)^\circ) > [(2\pi)^n - \varepsilon] \frac{\kappa_s^2}{s^n}.$$

Let $g^s : E \to [0, \infty)$ be the marginal of the uniform measure on $K_s(f_s)$. Then g^s is $s+n-m$-concave. The body $K_{s+n-m}(g^s)$ is the Schwartz symmetral with respect to $E^\perp$ of the body $K_s(f_s)$. By Lemma 5.2 and the remark following it

$$Vol(K_{s+n-m}(g^s)) Vol(K_{s+n-m}(g^s)^\circ) \ge Vol(K_s(f_s)) \min_z Vol((K_s(f_s) - z)^\circ)$$

which in turn exceeds $[(2\pi)^n - \varepsilon] \frac{\kappa_s^2}{s^n}$, and hence

$$\int_E g^s \int_E \mathcal{L}_{s+n-m}(g^s) > [(2\pi)^n - \varepsilon] \frac{\kappa_s^2}{s^n} \frac{(s+n-m)^m}{\kappa_{s+n-m}^2} \stackrel{s\to\infty}{\longrightarrow} (2\pi)^m - \frac{\varepsilon}{(2\pi)^{n-m}}.$$

Since $\varepsilon > 0$ was an arbitrary number, we conclude that

$$\liminf_{s\to\infty} \int_E g^s \int_E (g^s)^\circ \ge \liminf_{s\to\infty} \int_E g^s \int_E \mathcal{L}_{s+n-m}(g^s) \ge (2\pi)^m,$$

as $\mathcal{L}_{s+n-m} g^s \le (g^s)^\circ$. Since $g_s \to g$ pointwise, and all functions are log-concave, by Lemma 3.2 we conclude that $\int_E g \int_E g^\circ \ge (2\pi)^m$. It remains to prove that the Santaló point of g lies at the origin. Since $f_s \to f$, by Lemma 3.2 we have $\int x f_s^\circ \to 0$ and the Santaló point of f_s tends to zero. Since the Santaló point of g_s is an orthogonal projection of the Santaló point of f_s, the lemma follows. □

Lemma 5.4 *Let $f : \mathbb{R} \to [0, \infty)$ be a log-concave function whose barycenter is at zero, and assume that $\int f \int f^\circ = 2\pi$. Then f is a gaussian.*

Proof: Define $g(x) = f(-x)$. By the above $f(x)g(y)$ is a 2-dimensional maximizer, and so is its marginal on the line $l = \{(x,x) \in \mathbb{R}^2; x \in \mathbb{R}\}$, which is in turn equivalent to the convolution

$$\tilde{f}(x) = [f * g](x) = \int_{\mathbb{R}} f(y)f(y-x)dy.$$

The function $\tilde{f}$ is an even one dimensional maximizer, and by Lemma 2.2, $\tilde{f}$ must be a gaussian. According to Cramér Theorem (e.g. Page 1 of [LO]), if the convolution of two densities is gaussian, then both of them are gaussian. We conclude that f is gaussian. □

Proof of Proposition 5.1: By Lemma 5.3, all one-dimensional marginals of f are maximizers, and hence by Lemma 5.4 all of the one dimensional marginals of f are gaussians. A classical fact is that if all marginals of a function are gaussians, then the function itself is a gaussian. This completes the proof. □

6 Applications

In the remarkable paper [Mau], following a not less remarkable result by M. Talagrand [T], B. Maurey defined the property (τ) (for Talagrand) of a couple (μ, w), where μ is a probability measure on $\mathbb{R}^n$ and w a positive function on $\mathbb{R}^n$. A couple (μ, w) is said to satisfy (τ) if for every bounded function φ on $\mathbb{R}^n$

$$\int e^{-\varphi} d\mu \int e^{\inf\{\varphi(x-y)+w(y):y\in\mathbb{R}^n\}} \leq 1$$

(where we agree that $0\cdot\infty \leq 1$). The expression in the exponent of the second integral is called the *inf convolution* of φ and w, and denoted by $\varphi \square w$. Note that $-\log(f \star g) = (-\log f) \square (-\log g)$.

Maurey shows, in particular, that the couple $(\gamma_n, |x|^2/4)$ satisfies property (τ), where γ_n denotes the n-dimensional gaussian density. As a corollary he gets the gaussian concentration inequality: for every 1-Lip. function φ on $\mathbb{R}^n$, for two independent standard gaussian vectors X and Y, and any $\lambda > 0$

$$\mathbb{E}e^{\frac{\lambda}{\sqrt{2}}(\varphi(X)-\varphi(Y))} \leq e^{\lambda^2/2}.$$

This inequality is optimal in the sense that if φ is a linear functional, it is in fact an equality.

We will show below that for an *even* function φ a stronger concentration inequality holds true, namely the factor $\sqrt{2}$ on the left hand side can be omitted. Also, as a consequence of our Theorem 1.3 we will get a similar concentration inequality for general functions, without $\sqrt{2}$, however with an additional linear functional inserted, which corresponds to the right choice of a Santaló point.

We begin by defining the corresponding property which we call (even τ). A couple (μ, w) is said to satisfy (even τ) if for every bounded even function φ on $\mathbb{R}^n$

$$\int e^{-\varphi} d\mu \int e^{\varphi \Box w} d\mu \leq 1.$$

Theorem 6.1 *The pair* $(\gamma_n, |x|^2/2)$ *satisfies* (even τ).

Proof We need to show

$$\int e^{-(|x|^2/2+\varphi(x))} dx \int e^{-(|x|^2/2-\varphi\Box w)} \leq (2\pi)^n.$$

Let $f = e^{-(|x|^2/2+\varphi(x))}$, and check that

$$\mathcal{L}(-\log f) = |x|^2/2 - \inf_y \left\{ \frac{|x-y|}{2}^2 + \varphi(y) \right\} = |x|^2/2 - (\varphi \Box w)(x).$$

Thus the inequality we need is precisely $\int f \int f^\circ \leq (2\pi)^n$ for even functions, which follows from K. Ball's Theorem 2.1. $\Box$

From Theorem 6.1 we can deduce a strong gaussian concentration inequality for even functions. Namely we prove

Corollary 6.2 *For every* φ *on* $\mathbb{R}^n$ *which is* 1*-Lip. and even, for* X, Y *independent normalized gaussian vectors in* $\mathbb{R}^n$*, and any* $\lambda > 0$*,*

$$\mathbb{E} e^{\lambda(\varphi(X)-\varphi(Y))} \leq e^{\lambda^2/2}.$$

Proof The proof is identical to the one in [Mau]. We repeat it for the convenience of the reader. Denote $\psi = (\lambda\varphi)\Box w$, where φ is 1-Lip. and even, and $w(x) = |x|^2/2$. Then for some y

$$\begin{aligned} \psi(x) &= \lambda\varphi(y) + |x-y|^2/2 \\ &\geq \lambda\varphi(x) - \lambda|x-y| + |x-y|^2/2 \\ &\geq \lambda\varphi(x) - \lambda^2/2. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E} e^{\lambda(\varphi(X)-\varphi(Y))} &\leq e^{\lambda^2/2} \mathbb{E} e^{\psi(X)} \mathbb{E} e^{-\lambda\varphi(Y)} \\ &= e^{\lambda^2/2} \mathbb{E} e^{(\lambda\varphi\Box w)(X)} \mathbb{E} e^{-\lambda\varphi(Y)} \leq e^{\lambda^2/2}. \end{aligned}$$

This completes the proof. $\Box$

We can similarly deduce theorems regarding general functions φ, but with an extra correcting linear factor, corresponding to the Santaló point. Namely we can prove

Corollary 6.3 *For every φ on $\mathbb{R}^n$ which is 1-Lip, for X, Y independent normalized gaussian vectors in $\mathbb{R}^n$, and any $\lambda > 0$, there exists a linear functional ϕ_0 such that*

$$\mathbb{E}e^{\lambda(\varphi(X)-\varphi(Y))-\phi_0(X)} \le e^{\lambda^2/2}.$$

This follows from the following

Theorem 6.4 *For every φ on $\mathbb{R}^n$ there exists an $x_0 \in \mathbb{R}^n$ such that*

$$\int e^{-\varphi} d\gamma_n \int e^{(\varphi \Box w)(x)-\langle x_0, x\rangle} d\gamma_n(x) \le 1.$$

The proofs are similar to the ones described above, and are omitted.

Note that property (even τ) (as opposed to the more standard property (τ), say) does not seem to easily tensorize. This is the reason we need the full strength of the high dimensional functional version of Santaló inequality.

7 Appendix: Proofs of convergence theorems for log-concave functions

Proof of Lemma 3.2: For the first part of the lemma, first recall that the convergence is locally uniform on the interior of the support of f (see e.g. Theorem 10.8 in [R]). Hence if $\int f = \infty$, the lemma follows by restricting the integrals to large enough compact sets. Assume that $\int f < \infty$. By log-concavity, $\sup_{|x|>R} f(x) \to 0$ as $R \to \infty$. Pick a point x_0 with $f(x_0) > 0$. We may assume without loss of generality that $x_0 = 0$ and $f(x_0) > 1$. Let R be such that $|x| = R \Rightarrow f(x) < \frac{1}{e}$. Then for n large enough, say $n > n_0$,

$$f_n(0) > 1, \quad |x| = R \Rightarrow f_n(x) < \frac{1}{e}. \tag{14}$$

Notice that on $\{x : |x| > R\}$ we have $f_n \le e^{-\frac{|x|}{R}}$. Indeed,

$$\frac{1}{e} > f_n\left(R\frac{x}{|x|}\right) \ge f_n(0)^{1-\frac{R}{|x|}} f_n(x)^{\frac{R}{|x|}} \ge f_n(x)^{\frac{R}{|x|}}.$$

Of course $\int_{|x|>R} e^{-\frac{|x|}{R}}$ is finite, and by the dominated convergence theorem,

$$\int_{|x|>R} f_n \to \int_{|x|>R} f.$$

In the compact domain $\{x; |x| \le R\}$ the convergence is uniform, and the first part of the lemma follows.

Similarly, since also $\int_{|x|>R} |x|(1/4)^{|x|/R} < \infty$, the dominated convergence theorem also implies the second part of the lemma.

For the third assertion, denote $g_n = -\log f_n$ and $g = -\log f$, convex functions. Then $g_n \to g$ pointwise and locally uniformly on the interior of the support of g. It is enough to show that $\mathcal{L}g_n \to \mathcal{L}g$ on a dense set. Let x be a smooth point of $\mathcal{L}g$. Then there exists a unique supporting hyperplane to $\mathcal{L}g$ at x. Hence $g(y) - \langle x, y\rangle$ has a unique minimum in y, say y_0. Fix $\varepsilon > 0$. By the uniform convergence, for large enough n,

$$|y - y_0| = \varepsilon \quad \Longrightarrow \quad g_n(y) - \langle x, y\rangle > g_n(y_0) - \langle x, y_0\rangle.$$

Hence, the convex function $g_n(y) - \langle x, y\rangle$ has a local minimum in the ball of radius ε around y_0, and for a large enough n,

$$\mathcal{L}g_n(x) = \sup_{|y-y_0|<\varepsilon} \langle x, y\rangle - g_n(y) \stackrel{n\to\infty}{\longrightarrow} \mathcal{L}g(x)$$

as the convergence is uniform on the ball. Since for log-concave functions the set of smooth points is dense (see Theorem 25.5 in [R]), the third part of the lemma follows. □

Proof of Lemma 3.3: Fix $y \in \mathbb{R}^n$. Then for any $y' \in \mathbb{R}^n$,

$$\log f(y') \le -\langle y, y'\rangle - \log f^\circ(y)$$

and hence, if $-\langle y, x_s\rangle + \log f^\circ(y) < s$,

$$\begin{aligned}\widetilde{f}_s(y') = f_s(y' - x_s) &\le \left(1 - \frac{\langle y, y' - x_s\rangle + \log f^\circ(y)}{s}\right)_+^s \\ &= \left(1 + \frac{\langle y, x_s\rangle - \log f^\circ(y)}{s}\right)^s \left[1 - \frac{1}{s}\left\langle \frac{y}{1 + \frac{\langle y, x_s\rangle - \log f^\circ(y)}{s}}, y'\right\rangle\right]_+^s.\end{aligned}$$

We conclude that $\mathcal{L}_s\widetilde{f}_s\left(\frac{y}{1+\frac{\langle y,x_s\rangle - \log f^\circ(y)}{s}}\right) \ge \left(1 + \frac{\langle y,x_s\rangle - \log f^\circ(y)}{s}\right)^{-s} \ge e^{-\langle y,x_s\rangle} f^\circ(y)$. Denote $g_s = \mathcal{L}_s\widetilde{f}_s$ and $y_s = \frac{y}{1+\frac{\langle y,x_s\rangle - \log f^\circ(y)}{s}}$. Then for any y in the support of f°, (since $x_s \to 0$)

$$y_s \stackrel{s\to\infty}{\longrightarrow} y, \quad \liminf_{s\to\infty} g_s(y_s) \ge f^\circ(y). \tag{15}$$

Since g_s are log-concave, by Theorem 10.9 in [R], there exists a subsequence s_n and a log-concave function g, such that $g_{s_n} \to g$ converges locally uniformly on the interior of the support of g. By (15), and since the convergence is locally uniform, $g \geq f^\circ$. Hence, by Lemma 3.2,

$$\liminf_{n\to\infty} \int g_{s_n} = \int g \geq \int f^\circ.$$

We conclude that any increasing sequence s_n contains a subsequence s_{n_k} such that

$$\liminf_{k\to\infty} \int \mathcal{L}_{s_{n_k}} \widetilde{f}_{s_{n_k}} = \liminf_{k\to\infty} \int g_{s_{n_k}} \geq \int f^\circ.$$

This concludes the proof. □

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