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We proved in class the following variant of Slepian's lemma:

Lemma 0.1. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be real centered Gaussian random variables such that*

(i) $\mathbb{E}X_i^2 = \mathbb{E}Y_i^2$ *for all i, and*

(ii) $\mathbb{E}|X_i - X_j|^2 \le \mathbb{E}|Y_i - Y_j|^2$ *for all i and j.*

Then $\mathbb{E}\max X_i \le \mathbb{E}\max Y_i$.

It is known that we condition (i) is actually unneeded in Lemma 0.1. Below we prove this fact, up to a factor of two:

Corollary 0.2. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be real centered Gaussian random variables such that $\mathbb{E}|X_i - X_j|^2 \le \mathbb{E}|Y_i - Y_j|^2$ for all i, j. Then $\mathbb{E}\max X_i \le 2\mathbb{E}\max Y_i$.*

Proof (from the Ledoux-Talagrand book): First, we replace X_i by $X_i - X_1$ and $Y_i - Y_1$ for all i. This replacement does not alter neither the assumptions nor the conclusions of the lemma. Hence we may assume that $X_1 = Y_1 = 0$. In particular, from our assumption,

$$\mathbb{E}X_i^2 \le \mathbb{E}Y_i^2 \qquad (i \ge 1). \tag{1}$$

Next, introduce a standard normal random variable $g \sim N(0,1)$, independent of the X_i's and the Y_i's. For $i \ge 1$ define

$$\begin{cases} \tilde{X}_i = X_i + g\sigma \\ \tilde{Y}_i = Y_i + g\sqrt{\sigma^2 + \mathbb{E}X_i^2 - \mathbb{E}Y_i^2} \end{cases}$$

where $\sigma = \max_i \sqrt{\mathbb{E}Y_i^2}$. Then $\mathbb{E}\tilde{X}_i^2 = \mathbb{E}\tilde{Y}_i^2$ and for all i and j,

$$\mathbb{E}|\tilde{X}_i - \tilde{X}_j|^2 = \mathbb{E}|X_i - X_j|^2 \le \mathbb{E}|Y_i - Y_j|^2 \le \mathbb{E}|\tilde{Y}_i - \tilde{Y}_j|^2.$$

Hence we may invoke Lemma 0.1, and deduce that $\mathbb{E}\max_i \tilde{X}_i \le \mathbb{E}\max_i \tilde{Y}_i$. Since $\mathbb{E}\max_i \tilde{X}_i = \mathbb{E}\max_i X_i$, all that remains is to prove that

$$\mathbb{E}\max_i \tilde{Y}_i \le 2\mathbb{E}\max_i Y_i. \tag{2}$$

In order to prove (2), we note that

$$\mathbb{E}\max_i \tilde{Y}_i \le \mathbb{E}\max_i Y_i + \mathbb{E}g^+ \cdot \max_i \sqrt{\sigma^2 + \mathbb{E}X_i^2 - \mathbb{E}Y_i^2} \le \mathbb{E}\max_i Y_i + \sigma\mathbb{E}g^+,$$

where $x^+ = \max\{x, 0\}$, and where we used (1) in the last passage. Thus, in order to prove (1), it suffices to show that $\sigma\mathbb{E}g^+ \le \mathbb{E}\max_i Y_i$. This holds true because of the following chain of equalities and inequalities:

$$\begin{aligned} \mathbb{E}\max_i Y_i &= \frac{\mathbb{E}\max_i Y_i + \mathbb{E}\max_i(-Y_i)}{2} = \frac{\mathbb{E}\max_i Y_i - \min_i Y_i}{2} = \frac{1}{2}\cdot\mathbb{E}\max_{i,j}|Y_i - Y_j| \\ &\ge \frac{1}{2}\cdot\mathbb{E}\max_i |Y_i| \ge \frac{1}{2}\cdot\max_i \mathbb{E}|Y_i| = \frac{1}{2}\cdot\mathbb{E}|g|\cdot\max_i\sqrt{\mathbb{E}Y_i^2} = \mathbb{E}g^+\cdot\sigma, \end{aligned}$$

and the proof is complete. □