Needle decompositions in Riemannian geometry

Bo'az Klartag

Weizmann Institute of Science

Mathematics colloquium at Tel Aviv Univesity,

December 2020.

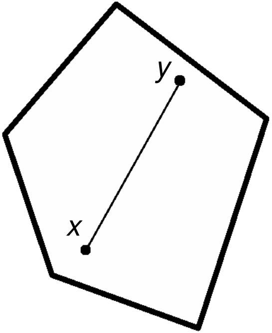
The Poincaré inequality

Theorem (Poincaré, 1890 and 1894)

Let $K \subseteq \mathbb{R}^3$ be convex and open.
Let $f : K \to \mathbb{R}$ be C^1-smooth, with $\int_K f = 0$. Then,

$$\lambda_K \int_K f^2 \leq \int_K |\nabla f|^2$$

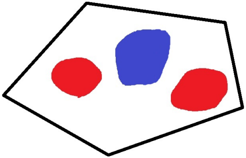
where $\lambda_K \geq (16/9) \cdot Diam^{-2}(K)$.



- In 2D, Poincaré got a better constant, $24/7$.
- Related to Wirtinger's inequality on periodic functions in one dimension (sharp constant, roughly a decade later).
- The largest possible λ_K is the **Poincaré constant** of K.
- Proof: Estimate $\int_{K \times K} |f(x) - f(y)|^2 dxdy$ via segments.

Motivation: The heat equation

- Suppose $K \subseteq \mathbb{R}^3$ with ∂K an *'insulator'*, i.e., heat is not escaping/entering K.
- Write $u_t(x)$ for the temperature at the point $x \in K$ at time $t \geq 0$.



Heat equation (Neumann boundary conditions)

$$\begin{cases} \dot{u}_t = \Delta u_t & \text{in } K \\ \frac{\partial u_t}{\partial n} = 0 & \text{on } \partial K \end{cases}$$

Fourier's law: Heat flux is proportional to the temp. gradient.

Rate of convergence to equilibrium

$$\frac{1}{|K|}\int_K u_0 = 1 \quad \Longrightarrow \quad \|u_t - 1\|_{L^2(K)} \leq e^{-t\lambda_K}\|u_0 - 1\|_{L^2(K)}$$

Higher dimensions

The Poincaré inequality was generalized to all dimensions:

Theorem (Payne-Weinberger, 1960)

Let $K \subseteq \mathbb{R}^n$ be convex and open, let μ be the Lebesgue measure on K. If $f : K \to \mathbb{R}$ is C^1-smooth with $\int_K f d\mu = 0$, then,

$$\frac{\pi^2}{Diam^2(K)} \int_K f^2 d\mu \leq \int_K |\nabla f|^2 d\mu.$$

- The constant π^2 is best possible in every dimension n. E.g.,
$$K = (-\pi/2, \pi/2), \quad f(x) = \sin(x).$$
- In contrast, Poincaré's proof would lead to an exponential dependence on the dimension.
- Not only the Lebesgue measure on K, we may consider any log-concave measure.

Higher dimensions

The Poincaré inequality was generalized to all dimensions:

Theorem (Payne-Weinberger, 1960)

Let $K \subseteq \mathbb{R}^n$ be convex and open, let μ be any **log-concave** measure on K. If $f : K \to \mathbb{R}$ is C^1-smooth with $\int_K f d\mu = 0$, then,

$$\frac{\pi^2}{Diam^2(K)} \int_K f^2 d\mu \leq \int_K |\nabla f|^2 d\mu.$$

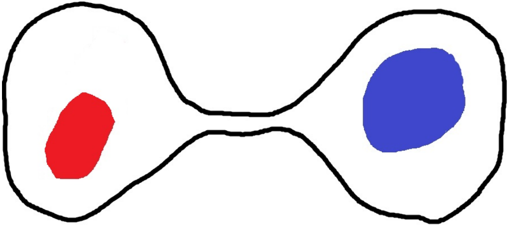
- The constant π^2 is best possible in every dimension n. E.g.,
$$K = (-\pi/2, \pi/2), \quad f(x) = \sin(x).$$
- In contrast, Poincaré's proof would lead to an exponential dependence on the dimension.
- A **log-concave** measure μ on K is a measure with density of the form e^{-H}, where the function H is **convex.**

The role of convexity / log-concavity

- For $\Omega \subseteq \mathbb{R}^n$, the Poincaré coefficient λ_Ω measures the connectivity or conductance of Ω.

Convexity is a strong form of connectedness

Without convexity/log-concavity assumptions:



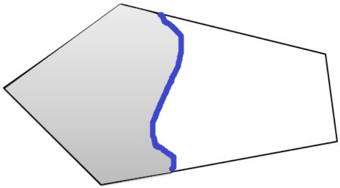
long time to reach equilibrium,
regardless of the diameter

Many other ways to measure connectivity

The isoperimetric constant

For an open set $K \subset \mathbb{R}^n$ define

$$h_K = \inf_{A \subset K} \frac{|\partial A \cap K|}{\min\{|A|, |K \setminus A|\}}$$



- If K is strictly-convex with smooth boundary, the infimum is attained when $|A| = |K|/2$ (Sternberg-Zumbrun, 1999).

Theorem (Cheeger '70, Buser '82, Ledoux '04)

For any open, convex set $K \subseteq \mathbb{R}^n$,

$$\frac{h_K^2}{4} \leq \lambda_K \leq 9h_K^2.$$

- Mixing time of Markov chains, algorithms for estimating volumes of convex bodies (Dyer-Freeze-Kannan '89, . . .)

How to prove dimension-free bounds for convex sets?

- Payne-Weinberger approach: **Hyperplane bisections.** (developed by Gromov-Milman '87, Lovász-Simonovits '93)
- Need to prove, for $K \subset \mathbb{R}^n$, $f : K \to \mathbb{R}$ and μ log-concave:

$$\int_K f d\mu = 0 \quad \Longrightarrow \quad \int_K f^2 d\mu \leq \frac{Diam^2(K)}{\pi^2} \int_K |\nabla f|^2 d\mu.$$

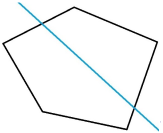
Find a hyperplane $H \subset \mathbb{R}^n$ through barycenter of K such that

$$\int_{K \cap H^+} f d\mu = \int_{K \cap H^-} f d\mu = 0,$$

where H^-, H^+ are the two half-spaces determined by H.

- It suffices to prove, given $\int_{K \cap H^\pm} f d\mu = 0$, that

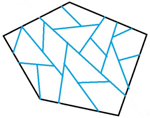
$$\int_{K \cap H^\pm} f^2 d\mu \leq \frac{Diam^2(K \cap H^\pm)}{\pi^2} \int_{K \cap H^\pm} |\nabla f|^2 d\mu.$$



Bisecting again and again

- Repeat bisecting recursively. After ℓ steps, obtain a partition of K into 2^ℓ convex bodies $K_1, \ldots, K_{2^\ell}$ with

$$\int_{K_i} f d\mu = 0 \qquad \text{for } i = 1, \ldots, 2^\ell.$$



The limit object (after induction on dimension):

1. A partition $\{K_\omega\}_{\omega \in \Omega}$ of K into segments (a.k.a "needles").
2. A **disintegration of measure**: prob. measures $\{\mu_\omega\}_{\omega \in \Omega}$ on K, and ν on Ω, with

$$\mu = \int_\Omega \mu_\omega d\nu(\omega)$$

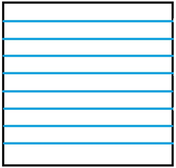
3. ν-Each μ_ω is supported on K_ω with $\int_{K_\omega} f d\mu_\omega = 0$.
4. ν-Each μ_ω is log-concave, by Brunn-Minkowski!

Examples

1. Take $K = [0,1]^2 \subseteq \mathbb{R}^2$ and $f(x,y) = f(x)$. Assume $\int_K f = 0$.

   Here the needles μ_ω are just Lebesgue measures,

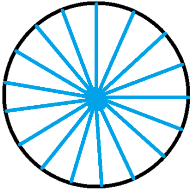
   $$d\mu_\omega(x) = dx.$$



2. Take $K = B(0,1) \subseteq \mathbb{R}^2$ and $f(x,y) = f(\sqrt{x^2+y^2})$ with $\int_K f = 0$. Here the needles μ_ω satisfy

   $$d\mu_\omega(r) = 2r dr.$$

   (which is log-concave)



Reduction to one dimension

The Payne-Weinberger inequality is reduced to a 1D statement:

$$\int_{K_\omega} f d\mu_\omega = 0 \quad \Longrightarrow \quad \int_{K_\omega} f^2 d\mu_\omega \leq \frac{Diam^2(K_\omega)}{\pi^2} \int_{K_\omega} |\nabla f|^2 d\mu_\omega.$$

This is because

1. $\mu = \int_\Omega \mu_\omega d\nu(\omega)$,
2. All μ_ω are log-concave with $\int_{K_\omega} f d\mu_\omega = 0$.

All that remains is to prove:

Lemma (Payne-Weinberger, '60)

Let μ be a log-concave measure, $Supp(\mu) \subseteq [-D, D]$. Then,

$$\int_{-D}^{D} f d\mu = 0 \quad \Longrightarrow \quad \int_{-D}^{D} f^2 d\mu \leq \frac{4D^2}{\pi^2} \int_{-D}^{D} |f'|^2 d\mu.$$

The Kannan-Lovász-Simonovits "localization method"

These needle decompositions have a few applications, such as:

Theorem ("reverse Hölder inequality", Bourgain '91, Bobkov '00, Nazarov-Sodin-Volberg '03, …)

Let $K \subseteq \mathbb{R}^n$ be convex, μ a log-concave prob. measure on K. Let p be any polynomial of degree d in n variables. Then,

$$\|p\|_{L^2(\mu)} \leq C_d \|p\|_{L^1(\mu)}$$

where $C_d > 0$ depends only on d (and not the dimension).

Theorem ("waist of the sphere", Gromov '03, also Almgren '60s)

Let $f : S^n \to \mathbb{R}^k$ be continuous, $k \leq n$. Then for some $x \in \mathbb{R}^k$,

$$|f^{-1}(x) + \varepsilon| \geq |S^{n-k} + \varepsilon| \qquad \text{for all } \varepsilon > 0,$$

where $A + \varepsilon = \{x \in S^n \, ; \, d(x, A) < \varepsilon\}$ and $S^{n-k} \subseteq S^n$.

Bisections work only in symmetric spaces...

What is the analog of the needle decompositions in an abstract **Riemannian manifold** $\mathcal{M}$?

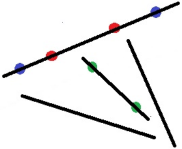
- Bisections are no longer possible.
- Are there other ways to construct partitions into segments?

Monge, 1781

A transportation problem induces a partition into segments.

Let μ and ν be smooth prob. measures in $\mathbb{R}^n$, disjoint supports. A **transportation** is a map $T : \mathbb{R}^n \to \mathbb{R}^n$ with

$$T_*\mu = \nu.$$



- There is a transportation such that the segments $\{(x, T(x))\}_{x \in Supp(\mu)}$ do not intersect (unless overlap).

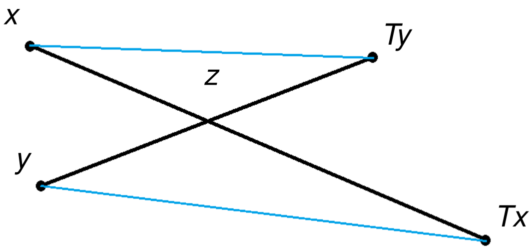
Monge's heuristics

Let μ and ν be smooth measures in $\mathbb{R}^n$, same total mass. Consider a transportation $T : \mathbb{R}^n \to \mathbb{R}^n$ that minimizes the cost

$$\int_{\mathbb{R}^n} |Tx - x| d\mu(x) = \inf_{S_*(\mu)=\nu} \int_{\mathbb{R}^n} |Sx - x| d\mu(x).$$

Use the triangle inequality: Assume by contradiction that

$$(x, Tx) \cap (y, Ty) = \{z\}.$$



The Monge-Kantorovich transportation problem

1. Suppose that $\mathcal{M}$ is an n-dimensional Riemannian manifold. Either complete, or at least **geodesically convex.**
2. A measure μ on $\mathcal{M}$ with a smooth density. (maybe the Riemannian volume measure.)
3. A measurable function $f : \mathcal{M} \to \mathbb{R}$ with $\int_{\mathcal{M}} f d\mu = 0$ (and some mild integrability assumption).

Consider the transportation problem between the two measures

$$d\nu_1 = f^+ d\mu \qquad \text{and} \qquad d\nu_2 = f^- d\mu.$$

We study a transportation $T_* \nu_1 = \nu_2$ of minimal cost

$$c(T) = \int_{\mathcal{M}} d(x, Tx) d\nu_1(x).$$

Structure of the optimal transportation

- Recall that $\int_{\mathcal{M}} f d\mu = 0$. Then an optimal transportation T exists and it induces the following structure:

Theorem ("Resolution of the Monge-Kantorovich problem")

There exists a partition $\{\mathcal{I}_\omega\}_{\omega \in \Omega}$ of $\mathcal{M}$ into **minimizing geodesics** and measures ν on Ω, and $\{\mu_\omega\}_{\omega \in \Omega}$ on $\mathcal{M}$ with

$$\mu = \int_\Omega \mu_\omega d\nu(\omega) \qquad \text{(disintegration of measure)},$$

and for ν-any $\omega \in \Omega$, the measure μ_ω is supported on $\mathcal{I}_\omega$ with $\int_{\mathcal{I}_\omega} f d\mu_\omega = 0$.

- A result of Evans and Gangbo '99, Trudinger and Wang '01, Caffarelli, Feldman and McCann '02, Ambrosio '03, Feldman and McCann '03.
- Like localization, but where is the log-concavity of needles?

Example - the sphere S^n

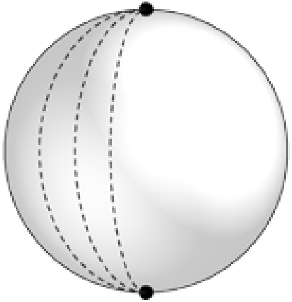
In this example:

- $\mathcal{M} = S^n$
- The measure μ is the Riemannian volume on $S^n \subseteq \mathbb{R}^{n+1}$.
- $f(x_0, \ldots, x_n) = x_n$, clearly $\int_{S^n} f d\mu = 0$.

1. We obtain a partition of S^n into needles which are **meridians.**
2. The density on each needle is proportional to

$$\rho(t) = \sin^{n-1} t \qquad t \in (0, \pi)$$

   in arclength parametrization ("spherical polar coordinates").
3. Note that $\left(\rho^{\frac{1}{n-1}}\right)'' + \rho^{\frac{1}{n-1}} = 0$.



Ricci curvature appears

Assume μ is the Riemannian volume on $\mathcal{M}$, and $\int f d\mu = 0$.

Theorem ("Riemannian needle decomposition", K. '17)

There is a partition $\{\mathcal{I}_\omega\}_{\omega \in \Omega}$ of $\mathcal{M}$ and measures ν on Ω, and $\{\mu_\omega\}_{\omega \in \Omega}$ on $\mathcal{M}$ with $\mu = \int_\Omega \mu_\omega d\nu(\omega)$ such that for any $\omega \in \Omega$,

1. The measure μ_ω is supported on the minimizing geodesic

   $$\mathcal{I}_\omega = \{\gamma_\omega(t)\}_{t \in (a_\omega, b_\omega)} \qquad \text{(arclength parametrization)}$$

   with C^∞-smooth, positive density $\rho = \rho_\omega : (a_\omega, b_\omega) \to \mathbb{R}$.
2. $\displaystyle\int_{\mathcal{I}_\omega} f d\mu_\omega = 0.$
3. Set $\kappa(t) = Ricci(\dot\gamma(t), \dot\gamma(t)), n = \dim(\mathcal{M})$. Then we have

   $$\left(\rho^{\frac{1}{n-1}}\right)'' + \frac{\kappa}{n-1} \cdot \rho^{\frac{1}{n-1}} \leq 0.$$

Remarks on the theorem

- If μ is not the Riemannian measure, replace the dimension n by $N \in (-\infty, 1] \cup [n, +\infty]$ and use the generalized Ricci tensor (Bakry-Émery, '85):

$$Ricci_{\mu,N} = Ricci_{\mathcal{M}} + \mathrm{Hess}\Psi - \frac{\nabla\Psi \otimes \nabla\Psi}{N-n}$$

where $d\mu/d\lambda_{\mathcal{M}} = \exp(-\Psi)$. Also set $Ricci_{\mu} = Ricci_{\mu,\infty}$.

When $Ricci_{\mathcal{M}} \geq 0$, the needle density ρ satisfies

$$\left(\rho^{\frac{1}{n-1}}\right)'' \leq \left(\rho^{\frac{1}{n-1}}\right)'' + \frac{\kappa}{n-1} \cdot \rho^{\frac{1}{n-1}} \leq 0.$$

Thus $\rho^{1/(n-1)}$ is concave and in particular ρ is **log-concave**.

- This recovers the case of $\mathbb{R}^n$, without use of bisections.
- Already generalized to measure-metric spaces (Cavalletti and Mondino) and to Finsler manifolds (Ohta).

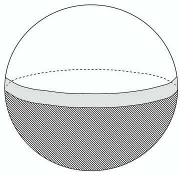
An application: Lévy-Gromov isoperimetric inequality

- Suppose $\mathcal{M}$ is n-dimensional, geodesically-convex, and

$$Ricci_{\mathcal{M}} \geq n-1 \quad (= Ricci_{S^n}).$$

- For a subset $A \subseteq \mathcal{M}$ denote

$$A + \varepsilon = \{x \in \mathcal{M}\,;\, d(x, A) < \varepsilon\},$$



the ε-neighborhood of A.

- Let μ and σ be Riemannian measures on $\mathcal{M}$ and S^n, respectively, normalized to be prob. measures.

Theorem ("Lévy-Gromov isoperimetric inequality")

For any $A \subseteq \mathcal{M}$ and a geodesic ball $B \subseteq S^n$,

$$\mu(A) = \sigma(B) \qquad \Longrightarrow \qquad \forall \varepsilon > 0,\ \mu(A+\varepsilon) \geq \sigma(B+\varepsilon).$$

Proof of Lévy-Gromov's isoperimetric inequality

- Given measurable $A \subseteq \mathcal{M}$ with $\mu(A) = \lambda \in (0,1)$, define

$$f(x) = (1-\lambda)\cdot 1_A(x) - \lambda \cdot 1_{\mathcal{M}\setminus A}(x).$$

- Apply needle decomposition for f to obtain $\mu = \int_\Omega \mu_\omega d\nu(\omega)$, where ν and $\{\mu_\omega\}$ are prob. measures.

Properties of the needle decomposition

1. Set $A_\omega = A \cap \mathcal{I}_\omega$, where $\mathcal{I}_\Omega = Supp(\mu_\omega)$ is a minimizing geodesic. Then,

$$\mu_\omega(A_\omega) = \lambda \qquad\qquad \forall \omega \in \Omega.$$

2. For any $\varepsilon > 0$,

$$\mu(A+\varepsilon) = \int_\Omega \mu_\omega(A+\varepsilon)d\nu(\omega) \geq \int_\Omega \mu_\omega(A_\omega+\varepsilon)d\nu(\omega)$$

with equality when $\mathcal{M} = S^n$ and $A = B$ is a cap in S^n.

Proof of Lévy-Gromov's isoperimetric inequality

- Our needle density ρ is "more concave" than polar spherical coordinates, i.e., needles with density $\sin^{n-1} t$.

One-dimensional lemma

Let $\rho : (a,b) \to \mathbb{R}$ be smooth and positive with

$$\left(\rho^{\frac{1}{n-1}}\right)'' + \rho^{\frac{1}{n-1}} \leq 0. \tag{1}$$

Let $A \subseteq (a,b)$ and $B = [0,t_0] \subseteq [0,\pi]$. Then for any $\varepsilon > 0$,

$$\frac{\int_A \rho}{\int_a^b \rho} = \frac{\int_B \sin^{n-1} t dt}{\int_0^\pi \sin^{n-1} t dt} \implies \frac{\int_{A+\varepsilon} \rho}{\int_a^b \rho} \geq \frac{\int_{B+\varepsilon} \sin^{n-1} t dt}{\int_0^\pi \sin^{n-1} t dt}.$$

- In fact, from (1) the isoperimetric profile I of $(\mathbb{R}, |\cdot|, \rho)$ satisfies

$$\left(I^{\frac{n}{n-1}}\right)'' + n \cdot I^{\frac{1}{n-1}-1} \leq 0.$$

More applications of needle decompositions

Assume that $\mathcal{M}$ is geodesically-convex with non-negative Ricci. Using Needle decompositions we can obtain:

1. Poincaré constant (Li-Yau '80, Yang-Zhong '84):

$$\lambda_{\mathcal{M}} \geq \pi^2 / Diam^2(\mathcal{M})$$

2. Brunn-Minkowski type inequality: For any measurable $A, B \subseteq \mathcal{M}$ and $0 < \lambda < 1$,

$$Vol(\lambda A + (1-\lambda)B) \geq Vol(A)^{\lambda} Vol(B)^{1-\lambda}$$

where $\lambda A + (1-\lambda)B$ consists of all points $\gamma(\lambda)$ where γ is a geodesic with $\gamma(1) \in A, \gamma(0) \in B$.
(Cordero-Erausquin, McCann, Schmuckenschlaeger '01).

3. Log-Sobolev inequalities (Wang '97), reverse Cheeger inequality $\lambda_{\mathcal{M}} \leq c \cdot h_{\mathcal{M}}^2$ (Buser '84), spectral gap and Lipschitz functions (E. Milman '09).

Another application: The 4 functions theorem

Assume $\mathcal{M}$ is geodesically-convex, μ a measure, $Ricci_\mu \geq 0$.

The four functions theorem (Riemannian version of KLS '95)

Let $\alpha, \beta > 0$. Let $f_1, f_2, f_3, f_4 : \mathcal{M} \to [0, +\infty)$ be measurable functions. Assume that for any probability measure η on $\mathcal{M}$ which is a log-concave needle,

$$\left(\int_{\mathcal{M}} f_1 d\eta \right)^{\alpha} \left(\int_{\mathcal{M}} f_2 d\eta \right)^{\beta} \leq \left(\int_{\mathcal{M}} f_3 d\eta \right)^{\alpha} \left(\int_{\mathcal{M}} f_4 d\eta \right)^{\beta}$$

whenever f_1, f_2, f_3, f_4 are η-integrable. Then,

$$\left(\int_{\mathcal{M}} f_1 d\mu \right)^{\alpha} \left(\int_{\mathcal{M}} f_2 d\mu \right)^{\beta} \leq \left(\int_{\mathcal{M}} f_3 d\mu \right)^{\alpha} \left(\int_{\mathcal{M}} f_4 d\mu \right)^{\beta}.$$

- Recall: A **log-concave needle** is a measure, supported on a minimizing geodesic, with a log-concave density in arclength parameterization.

One last application: Dilation inequalities

Definition (Nazarov, Sodin, Volberg '03, Bobkov and Nazarov '08, Fradelizi '09)

For $A \subseteq \mathcal{M}$ and $0 < \varepsilon < 1$, the set $\mathcal{N}_\varepsilon(A)$ contains all $x \in \mathcal{M}$ for which $\exists$ a minimizing geodesic $\gamma : [a, b] \to \mathcal{M}$ with $\gamma(a) = x$ and

$$\lambda_1 \left(\{ t \in [a, b] \, ; \, \gamma(t) \in A \} \right) \geq (1 - \varepsilon) \cdot (b - a),$$

where λ_1 is the Lebesgue measure in the interval $[a, b] \subseteq \mathbb{R}$.

- Thus $\mathcal{N}_\varepsilon(A)$ is a kind of an ε-dilation of the set A.

Theorem (Riemannian version of Bobkov-Nazarov '08)

Assume $\mathcal{M}$ is n-dimensional, geodesically-convex, μ is prob., $Ricci_\mu \geq 0$. Let $A \subseteq \mathcal{M}$ be measurable with $\mu(A) > 0$. Then,

$$\mu(\mathcal{M} \setminus A)^{1/n} \geq (1 - \varepsilon) \cdot \mu(\mathcal{M} \setminus \mathcal{N}_\varepsilon(A))^{1/n} + \varepsilon.$$

Comparison with the quadratic cost

- Given probability measures ν_1, ν_2 on $\mathcal{M}$, consider all transportations $T_*\nu_1 = \nu_2$ with the **quadratic cost**

$$c(T) = \int_{\mathcal{M}} d^2(x, Tx) d\nu_1(x).$$

Theorem (Brenier '87, McCann '95)

When $\mathcal{M} = \mathbb{R}^n$, the map T of minimal quadratic cost has the form

$$T = \nabla \Phi$$

where Φ is a convex function on $\mathbb{R}^n$. (and vice versa)

- Generalization to Riemannian manifolds by McCann '01: The optimal map T has the form $T(x) = \exp_x(\nabla \Phi)$, where $-\Phi$ is a $d^2/2$-concave function.
- This yields some of the aforementioned applications.

Proof of Riemannian needle decomposition theorem

Kantorovich duality (1940s): Let $f \in L^1(\mu)$ with $\int f d\mu = 0$, set $d\nu_1 = f^+ d\mu$ and $d\nu_2 = f^- d\nu$. Then,

$$\inf_{S_*(\nu_1)=\nu_2} \int_{\mathcal{M}} d(Sx, x) d\nu_1(x) = \sup_{\|u\|_{Lip}\leq 1} \left[\int_{\mathcal{M}} uf d\mu \right].$$

- Moreover, let S and u be optimizers. Then,

$$S(x) = y \qquad \Longrightarrow \qquad |u(x) - u(y)| = d(x, y).$$

Definition

A point $y \in \mathcal{M}$ is a **strain point** of u if $\exists x, z \in \mathcal{M}$ with

1. $d(x, z) = d(x, y) + d(y, z)$.
2. $u(y) - u(x) = d(x, y) > 0, \quad u(z) - u(y) = d(y, z) > 0.$

Strain points of a 1-Lipschitz function $u : \mathcal{M} \to \mathbb{R}$

Write $\mathit{Strain}[u] \subseteq \mathcal{M}$ for the collection of all strain points of u.

Proposition

1. $\mu(\mathit{Supp}(f) \setminus \mathit{Strain}[u]) = 0$.
2. The following is an equivalence relation on $\mathit{Strain}[u]$:

$$x \sim y \quad \Longleftrightarrow \quad |u(x) - u(y)| = d(x, y).$$

- The equivalence classes are minimizing geodesics, the **transport rays** from Evans-Gangbo '99. The optimal transport map S acts along transport rays.

Write $T^{\circ}[u]$ for the collection of all such transport rays.

1. Disintegration of measure: $\mu|_{\mathit{Strain}[u]} = \int_{T^{\circ}[u]} \mu_{\mathcal{I}} d\nu(\mathcal{I})$.
2. Feldman-McCann '03: $\int_{\mathcal{I}} f d\mu_{\mathcal{I}} = 0$ for ν-almost any $\mathcal{I}$.

Higher regularity: In *Strain*$[u]$, it's almost $C^{1,1}$

Definition

$Strain_\varepsilon[u]$ consists of points $y \in \mathcal{M}$ for which $\exists x, z \in \mathcal{M}$ with

1. $d(x,z) = d(x,y) + d(y,z)$.
2. $u(y) - u(x) = d(x,y) \geq \varepsilon, \quad u(z) - u(y) = d(y,z) \geq \varepsilon$.

- Clearly, $Strain[u] = \cup_{\varepsilon>0} Strain_\varepsilon[u]$.

Theorem ("$C^{1,1}$-regularity")

Let $\mathcal{M}$ be a geodesically-convex Riemannian manifold.
Let $\varepsilon > 0$ and let $u : \mathcal{M} \to \mathbb{R}$ satisfy $\|u\|_{Lip} \leq 1$.
Then there exists a $C^{1,1}$-function $\tilde{u} : \mathcal{M} \to \mathbb{R}$ with

$$\forall x \in Strain_\varepsilon[u], \qquad \tilde{u}(x) = u(x), \;\; \nabla\tilde{u}(x) = \nabla u(x).$$

Proof: Whitney's extension theorem and a geometric lemma of Feldman and McCann. $\square$

Geodesics orthogonal to level sets of u

Thanks to $C^{1,1}$-regularity:

At almost any point $p \in Strain[u]$ there is a symmetric **second fundamental form** II_p for the hypersurface

$$\{x \in \mathcal{M}\, ;\, u(x) = u(p)\},$$

which is the Hessian of u, restricted to the tangent space.

- The transport rays are geodesics orthogonal to a level set of u. This resembles a standard measure disintegration in Riemannian geometry (going back to Paul Levy, 1919).

Theorem ("Normal decomposition of Riemannian volume")

Write ρ for the density of $\mu_{\mathcal{I}}$ with respect to arclength. Then:

$$\frac{d}{dt} \log \rho(t) = Tr[II], \quad \frac{d^2}{dt^2} \log \rho(t) = -Tr[(II)^2] - Ric(\nabla u, \nabla u).$$

Uniqueness of maximizer

- Concavity of the needle density follows from $Tr[(II)^2] \geq 0$.
- Works nicely with a non-Riemannian volume measure μ, as long as its density satisfies Bakry-Émery concavity.

Corollary

Assume $Supp(f)$ has a full μ-measure. Let $u_1, u_2 : \mathcal{M} \to \mathbb{R}$ be 1-Lip. functions, maximizers of the Kantorovich problem. Then

$$u_1 - u_2 \equiv Const.$$

Proof: Also $(u_1 + u_2)/2$ is a maximizer. Thus $Strain[u_i]$ has full measure, as well as $Strain[(u_1 + u_2)/2]$. Hence for a.e. $x \in \mathcal{M}$,

$$|\nabla u_1(x)| = |\nabla u_2(x)| = \left| \frac{\nabla u_1(x) + \nabla u_2(x)}{2} \right| = 1.$$

Therefore $\nabla u_1 = \nabla u_2$ almost everywhere in $\mathcal{M}$. □

A comment on curved needles and nodal sets

- Suppose that $\mathcal{M}$ is a Riemannian manifold, say compact. Let $f : \mathcal{M} \to \mathbb{R}$ be a (non-constant) Laplace eigenfunction,

$$\Delta f = -\lambda f, \qquad (\lambda > 0).$$

- Consider the vector field $V = \nabla f$ and view its integral curves as **curved needles**, with natural parametrization.

Disintegration of the Riemannian measure μ w.r.t the partition

Decompose $\mu = \int_\Omega \mu_\omega d\nu(\omega)$ where μ_ω is supported on an integral curve $\mathcal{I}_\omega$ with density ρ_ω.

Theorem

For ν-a.e. $\omega \in \Omega$, the density $\rho = \rho_\omega$ is **log-concave***. Moreover, its maximum is on the nodal set $\{f = 0\}$, if reached by needle.*

- Trivial to prove once stated, no curvature assumptions.

The end

Thank you!

One of the images (sphere with meridians) was taken from `www2.rdrop.com/~half`