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The Raymond and Beverley Sackler

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Concentration of Functions Beyond Lévy's Inequality

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by

Erez Buchweitz

This Thesis was prepared under the supervision of

Prof. Bo'az Klartag

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הפקולטה למדעים מדוייקים ע"ש ריימונד ובברלי סאקלר
בית הספר למדעי המתמטיקה

ריכוז של פונקציות מעבר לאי־שוויון Lévy

חיבור זה מוגש כחלק מהדרישות לקבלת התואר "מוסמך למדעים"
באוניברסיטת תל אביב

מאת
ארז בוכוייץ

העבודה הוכנה בהדרכתו של
פרופ' בועז קלרטג

תשרי תשע"ז

עבור פונקציה ממשית המוגדרת על ספירת היחידה $S^{n-1} = \{x \in \mathbb{R}^n :\ x_1^2 + ... + x_n^2 = 1\}$ ומקיימת תנאי ליפשיץ, אי־שוויון ריכוז המידה הקלאסי של P. Lévy מ־1919 מבטיח כי על 99% משטח הספירה הפונקציה מקבלת ערכים שאינם רחוקים יותר מ־$\frac{5}{\sqrt{n}}$ מממוצע הפונקציה. כלומר, פונקציות כאלה מרוכזות מאוד סביב ממוצען, וריכוז זה אף הולך וגובר ככל שהמימד גדל.

בעבודה זו אנו נתקלים במשפחה טבעית של פונקציות על הספירה, אשר ריכוז כל אחת מהן סביב ממוצעה הדוק הרבה יותר ממה שניתן היה לצפות על פי התיאוריה הקלאסית. במובן זה, התופעה המתבטאת בדוגמאות אלה היא עדינה מכדי שפרמטר הליפשיץ של פונקציה יוכל לתפוס אותה. הפונקציות נבנות מתוך משפחה של התפלגויות הסתברות על $\mathbb{R}^n$ בעלות תכונה מסוימת ובנרמול מסוים. עבור התפלגות על $\mathbb{R}^n$, אנו בוחנים את מרכז המסה של חצי מרחב על פי אותה התפלגות. מכיוון שחצי מרחב דרך הראשית מוגדר על פי אחד משני וקטורי יחידה המאונכים לו, מתקבלת פונקציה של הספירה ־ כיוון חצי המרחב.

העבודה מחולקת לפרקים. לאחר הקדמה, בפרק השני אנו מתארים את התנאים שעל התפלגות לקיים על מנת שהפונקציה המצומדת לה תיהנה מריכוז חזק במיוחד. אחד התנאים הוא נרמול אשר דומה לנרמול האיזוטרופי אך שונה ממנו. התנאי השני אינהרנטי להתפלגות. בשלב זה של העבודה אנו מוכיחים כי מתקיים ריכוז חזק יותר מאשר מנובא על ידי התוצאה של Lévy, אך כזה שחל באופן צר בלבד ־ אנו מראים כי שונות הפונקציה הספירית (המומנט השני) קטנה משמעותית מאשר היינו מצפים. לעומת זאת, התוצאה הקלאסית רחבה יותר ומספקת חסמים על כל המומנטים $p \geq 1$. בהמשך העבודה אנחנו מרחיבים תוצאה זו לכלול מומנטים נוספים.

בפרקים השלישי והרביעי אנו נותנים דוגמאות למשפחות התפלגויות אשר מקיימות את התנאים אותם ניסחנו בפרק השני. בפרק השלישי אנו דנים בהתפלגות הדיסקרטית אשר משייכת הסתברות שווה לכל אחד מקודקודי הקוביה $\{-1,1\}^n$. התפלגות זו מקיימת את הריכוז החזק, ולמעשה אין צורך בכל 2^n קודקודי הקוביה ־ מספיק לקחת רק מעט יותר מ־n^2 קודקודים אקראיים כדי שהתנאים יתקיימו בהסתברות גבוהה. בפרק הרביעי אנחנו דנים במשפחה החשובה של התפלגויות לוג־קעורות. אנו מוכיחים כי כל התפלגות לוג־קעורה מקיימת את התנאים הנדרשים, עד כדי מרכוז ומתיחה. לשם כך אנחנו מנתחים בהרחבה את הנרמול אשר הופיע כתנאי בפרק השני ומראים שהוא תמיד קיים ויחיד תחת הנחות חלשות, ושבמקרה הלוג־קעור הוא אף לא רחוק מהנרמול האיזוטרופי.

בפרק החמישי אנו ממשיכים לדון בהתפלגויות לוג־קעורות, ומראים שבמקרה זה ניתן להרחיב את החסם המשופר שהשגנו על המומנט השני כך שיכלול גם כל שאר המומנטים $p \geq 1$, ובכך להשיג שיפור משמעותי על החסמים הנובעים מאי־השוויון של Lévy גם למומנטים נוספים ומכאן גם לחסם משופר על זנב התפלגות ערכי הפונקציה. בפרק השישי והאחרון אנחנו מציגים כיוון שעוד לא הבשיל במלואו, ושעשוי להוביל לריכוז חזק אף יותר עבור התפלגויות זוגיות.

Contents

Chapter 1

Introduction

1.1 Overview

The starting point of our discussion is a remark made by R. Eldan and B. Klartag in [6]. They observed that given a symmetric random vector $X \in \mathbb{R}^n$, under some assumptions, the function

$$\theta \ \mapsto \ \mathbb{E}\,|\langle X, \theta \rangle|$$

for $\theta \in S^{n-1} = \{x \in \mathbb{R}^n : \ x_1^2 + ... + x_n^2 = 1\}$ might have variance more concentrated than naïvely expected.

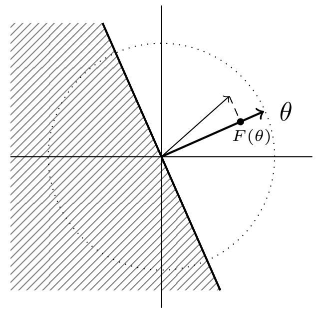
We undertake a slightly different approach; Given a Borel probability measure μ on $\mathbb{R}^n$, we study

$$F(\theta) \ = \ \int_{\mathbb{R}^n} \langle x, \theta \rangle_+ \, d\mu(x)$$

for $\theta \in S^{n-1}$. Here, $t_+ = \max\{t, 0\}$. Borrowing from Eldan & Klartag's perspective, $F(\theta) + F(-\theta)$ is the marginal first moment of μ in the direction θ. One can also perceive $F(\theta)$ as the θ component of the (unnormalized) center of mass of μ on the half-space in the direction of θ, as illustrated by the following representation:

$$F(\theta) \ = \ \left\langle \int_{H_\theta} x \ d\mu \ , \ \theta \right\rangle$$

where $H_\theta = \{x \in \mathbb{R}^n : \ \langle x, \theta \rangle > 0\}$ is the open half-space in direction of θ.



A third outlook on F is as encapsulating some measure of symmetry. When μ is spherically symmetric, F is constant. We may say that the more F is fluctuant, the less "symmetric" μ is.

In order to analyze F, we will assume the following normalization: μ is centered and

$$\int_{\mathbb{R}^n} x \otimes x \, \frac{d\mu}{|x|} \;=\; \frac{\alpha}{\sqrt{n}} Id \tag{1.1}$$

for some $\alpha > 0$. Here, $x \otimes x$ is the $n \times n$ matrix given by $(x \otimes x)_{ij} = x_i x_j$. Note the resemblance to the isotropic normalization: $\int_{\mathbb{R}^n} x \otimes x \, d\mu \;=\; Id$.

We are thus interested in studying the concentration of F. How concentrated can we expect F to be? The classical theory promotes us to describe functions on the sphere in terms of their Lipschitz semi-norms:

$$\|f\|_{\mathrm{Lip}} \;=\; \sup_{\theta,\eta \in S^{n-1}} \frac{|f(\theta) - f(\eta)|}{|\theta - \eta|}.$$

In our case it can be shown that $\|F\|_{\mathrm{Lip}} \leq 3\alpha$ whatever μ is, so long as it abides by the normalization (1.1). Then Lévy's inequality guarantees the tail bound

$$\mathbb{P}\left[|F - \mathbb{E}F| > t\right] \;\leq\; 2e^{-(n-1)t^2/18\alpha^2} \tag{1.2}$$

for any $t > 0$. Here, $\mathbb{E}F$ is the mean of F and the probability $\mathbb{P}$ over the values of F are both taken with respect to σ_{n-1}, the normalized surface measure on S^{n-1}. In terms of the variance, inequality (1.2) entails

$$\mathrm{Var}(F) \;=\; \int_{S^{n-1}} (F - \mathbb{E}F)^2 \, d\sigma_{n-1} \;\leq\; \frac{36\alpha^2}{n-1}.$$

It is important to note that $\|F\|_{\mathrm{Lip}}$ cannot be assumed to be smaller than α times a universal constant in general. Thus the above bound is qualitatively the sharpest bound the Lipschitz paradigm can produce.

Our aim in this work is to improve the concentration bounds provided by Lévy's inequality, for F.

This thesis is organized as follows.
In Chapter 2, we show that under our normalization and an extra assumption, in fact

$$\mathrm{Var}(F) \ \leq \ \frac{C(\alpha^2+\beta)}{n^2}.$$

Here, α is the parameter from the normalization (1.1), β is another parameter of the measure μ and $C > 0$ is a universal constant. In all cases that we encounter β is of the order of magnitude of a universal constant (as is α). Thus we significantly improve the variance bound implied by Lévy's inequality.

The main question our work leaves unanswered is whether this tighter concentration, which necessarily stems from qualities the Lipschitz parameter is too coarse to utilize, is a specific property of this family of functions alone or whether it is a manifestation of some deep overarching phenomenon in concentration of measure.

In Chapter 3 we discuss as an example the discrete cube $\{-1,1\}^n$. We show that for the discrete measure distributed uniformly on $\{-1,1\}^n$, it holds that $\mathrm{Var}(F) \leq \frac{C}{n^2}$ with $C > 0$ some a universal constant. Furthermore, $\|F\|_{\mathrm{Lip}}$ is greater than a universal constant independently of the dimension n. We then prove that the result still holds, with high probability, when we take only a tiny random subset of the cube ($n^{2+\alpha}$ points, $\alpha > 0$).

In Chapter 4 we discuss the case of a log-concave measure. To this end we introduce an affine position of a probability measure which we call the L^p-isotropic position, where $p = 2$ corresponds to the isotropic position and $p = 1$ corresponds to our normalization. We present conditions on p, μ for the existence and uniqueness of an L^p-isotropic position. Specifically, the L^1-isotropic position exists and is unique up to orthogonal transformations provided that μ upholds some mild high-dimensionality and integrability conditions. Furthermore we prove a useful property; when μ is log-concave the L^1-isotropic position is "not too far" from the isotropic position in some specific sense of proximity. We conclude with the corollary that there exists a universal constant C such that any log-concave probability measure has an affine position in which $\mathrm{Var}(F) \leq \frac{C}{n^2}$ (and the first moment of the measure is of order $\sqrt{n}$, to avoid trivialities).

In Chapter 5, we maintain with the log-concave case and show that the variance bound can be extended into an exponential tail bound, in the form of

$$\mathbb{P}\left[|F - \mathbb{E}F| > t\right] \ \leq \ 2e^{-cnt} \tag{1.3}$$

where $c > 0$ is a universal constant. The bound (1.3) is much stronger asymptotically than the bound (1.2) obtained by Lévy's inequality. To illustrate this, we note that the tail bound (1.3) entails the moment bounds

$$\|F\|_p \ = \ \left(\int_{S^{n-1}} |F - \mathbb{E}F|^p \ d\sigma_{n-1}\right)^{\frac{1}{p}} \ \leq \ C\frac{p}{n}$$

for any $p \geq 1$. Compare these to the moment bounds implied by inequality (1.2):

$$\|F\|_p \ = \ \left(\int_{S^{n-1}} |F - \mathbb{E}F|^p \ d\sigma_{n-1}\right)^{\frac{1}{p}} \ \leq \ C\frac{\sqrt{p}}{\sqrt{n}}.$$

In Chapter 6, we present preliminary results of an approach that may yield an even stronger concentration for some special case of even log-concave measures.

The proofs of some minor claims were omitted from the body of each chapter as they were deemed not important enough to interfere with the flow of the discussion. They appear in the corresponding appendices situated at the end of each chapter. The diligent reader may take the time to review those claims he/she finds interesting.

1.2 Notation

Throughout this thesis, the letters $c, C, c', C', \tilde{c}, \tilde{C}, ...$ stand for positive universal constants whose values differ between occurrences. We usually denote by lower-case letters constants that are assumed to be sufficiently small, and by upper-case letters constants that are assumed to be sufficiently large.

The Euclidean norm in any dimension is denoted as $|\cdot|$, and the Euclidean inner product is $\langle \cdot, \cdot \rangle$. We sometimes add the subscript $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ to signify the space in which the inner product is taken.

The unit sphere in $\mathbb{R}^n$ is denoted as $S^{n-1} = \{x \in \mathbb{R}^n : \; |x| = 1\}$. We denote by σ_{n-1} the uniform surface probability measure on S^{n-1}, alternatively the Haar measure or the unique rotation invariant measure on S^{n-1}. We sometimes omit specifying the dimension and write simply σ when there is no risk of confusion.

When $\theta \in S^{n-1}$ we denote by $\theta^{\perp} = \{x \in \mathbb{R}^n : \; \langle x, \theta \rangle = 0\}$ the hyperplane orthogonal to θ, and by $H_\theta = \{x \in \mathbb{R}^n : \; \langle x, \theta \rangle > 0\}$ the open half-space in the direction of θ. We sometimes allow the notation $x^{\perp}$ or H_x when $x \in \mathbb{R}^n$ (not necessarily $|x| = 1$).

We use the symbols $\mathbb{E}, \mathbb{P}$ as shorthand for expectation and probability, respectively, when there is no doubt with respect to what space and measure they are taken. For example, if $f : S^{n-1} \to \mathbb{R}$ is a function, then

$$\mathbb{E} f \; = \; \int_{S^{n-1}} f(\theta) \, d\sigma_{n-1}(\theta) \quad ; \quad \mathbb{P}\left[|f| > t\right] \; = \; \sigma_{n-1}\left(\left\{\theta \in S^{n-1} : \; |f(\theta)| > t\right\}\right).$$

A measure on $\mathbb{R}^n$ which is absolutely continuous with respect to the Lebesgue measure will be abbreviated as a Leb-a.c. measure. The symbol $x \otimes x$ when $x \in \mathbb{R}^n$ denotes an $n \times n$ matrix whose entries are $(x \otimes x)_{ij} = x_i x_j$. We use the standard notation GL_n, SL_n, O_n for the $n \times n$ matrix groups of invertible, determinant-one and orthogonal matrices, respectively.

Chapter 2

Variance Bound

2.1 Introduction

We open by formally defining the function that is the subject of this work.

Definition 2.1.1. Let μ be a Borel probability measure on $\mathbb{R}^n$ with finite first moment, i.e. $\int_{\mathbb{R}^n} |x| \, d\mu < \infty$. Define the function $F_\mu : S^{n-1} \to [0, \infty)$ by

$$F_\mu(\theta) \;=\; \int_{\mathbb{R}^n} \langle x, \theta \rangle_+ \; d\mu(x).$$

When there is no risk of confusion we abbreviate $F_\mu = F$. The assumption of the finiteness of the first moment is sufficient to ensure that F_μ assumes finite values. The main result of this chapter is the following.

Proposition 2.1.2. *Let μ be a centered Borel probability measure on $\mathbb{R}^n$. Assume that for some $\alpha, \beta \in (0, \infty)$,*

(1) $$\int_{\mathbb{R}^n} x \otimes x \, \frac{d\mu}{|x|} \;=\; \frac{\alpha}{\sqrt{n}} Id$$

(2) $$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\langle x, y \rangle^4}{|x|^3 |y|^3} \, d\mu(x) \, d\mu(y) \;\le\; \frac{\beta}{n}$$

Then, $$Var(F_\mu) \;\le\; \frac{C\,(\alpha^2 + \beta)}{n^2}$$

where $C > 0$ is a universal constant.

For clarification, the variance

$$\mathrm{Var}(F_\mu) \;=\; \int_{S^{n-1}} \left(F_\mu(\theta) - \mathbb{E}F_\mu\right)^2 \, d\sigma_{n-1}(\theta)$$

is taken with respect to σ_{n-1}, and $\mathbb{E}F_\mu = \int_{S^{n-1}} F_\mu(\theta)\, d\sigma_{n-1}(\theta)$ is the expectation.

For now we regard assumptions (1) and (2) as technical conditions that μ must satisfy. The normalization (1) will be investigated in Chapter 4, while the integral in (2) will revisit us in Chapter 6. Of course, it must be asked what measures satisfy these conditions, if any. Important cases and examples are given in Chapters 3 and 4.

Two quick corollaries of property (1) must be conceded as they will come in handy in future discussions.

Claim 2.1.3. *Let μ be a Borel probability measure satisfying (1). Then:*

- $$\int_{\mathbb{R}^n} \langle x, \xi\rangle^2 \, \frac{d\mu(x)}{|x|} \;=\; \frac{\alpha}{\sqrt{n}}|\xi|^2 \qquad ; \qquad \forall\, \xi \in \mathbb{R}^n$$
- $$\int_{\mathbb{R}^n} |x| \, d\mu \;=\; \sqrt{n}\alpha.$$

Proof. Denote $M = \int_{\mathbb{R}^n} x \otimes x \, \frac{d\mu}{|x|}$. As the entries of M are $M_{ij} = \int_{\mathbb{R}^n} x_i x_j \, \frac{d\mu}{|x|}$, simple linear algebra shows

$$\int_{\mathbb{R}^n} \langle x, \xi\rangle^2 \, \frac{d\mu(x)}{|x|} \;=\; \langle M\xi, \xi\rangle \quad ; \quad \int_{\mathbb{R}^n} |x| \, d\mu \;=\; \int_{\mathbb{R}^n} |x|^2 \, \frac{d\mu(x)}{|x|} \;=\; \mathrm{Tr}M.$$

□

To appreciate the bound implied by Proposition 2.1.2, compare it to Lévy's concentration inequality for Lipschitz functions (Theorem 1.7.9 in [4]):

Theorem 2.1.4 (Lévy's inequality, 19')**.** *Let $f : S^{n-1} \to \mathbb{R}$ be a Lipschitz function. Then,*

$$\mathbb{P}\left[|f - \mathbb{E}f| > t\right] \;\leq\; 2e^{-(n-1)t^2/2\|f\|_{Lip}^2}$$

for any $t > 0$.

The tail bound in Lévy's inequality implies the variance bound $\mathrm{Var}(f) \leq \frac{4\|f\|_{\mathrm{Lip}}^2}{n-1}$ (for a proof, see Appendix 2.5.1). In our case, it holds that

$$\|F_\mu\|_{\mathrm{Lip}} \leq 3\alpha \tag{2.1}$$

whenever μ satisfies property (1) with parameter α (for a proof, see Appendix 2.5.2). Thus, Lévy's inequality implies

$$\mathrm{Var}(F_\mu) \ \leq \ \frac{36\alpha^2}{n-1}. \tag{2.2}$$

Notice that we provide only an upper bound (2.1) on the Lipschitz parameter. If $\|F_\mu\|_{\mathrm{Lip}}$ is in fact smaller, Lévy's inequality can be used to procure a tighter bound. In other words, we ask whether the bound (2.2) is the best bound Lévy's inequality can produce for F_μ. While for some measures $\|F_\mu\|_{\mathrm{Lip}}$ may indeed be small (e.g. for γ_n the standard Gaussian measure on $\mathbb{R}^n$, $\|F_{\gamma_n}\|_{\mathrm{Lip}} = 0$), one cannot assume in general that $\|F_\mu\|_{\mathrm{Lip}}$ is anything less than α times a universal constant. In Chapter 3 we present an example that demonstrates this. Consequently the bound (2.2) is qualitatively the best bound Lévy's inequality can provide, and our result (Proposition 2.1.2) is indeed a considerable improvement, when β is not too large. In all the cases we present, β is of the order of magnitude of a universal constant.

When μ is Leb-a.c., Proposition 2.1.2 is a straightforward corollary of the following proposition, which is also of independent interest to us.

Proposition 2.1.5. *Let μ be a Leb-a.c. probability measure on $\mathbb{R}^n$ with finite first moment. Then F_μ is C^1-smooth.*
If, in addition, μ is centered and satisfies assumptions (1) and (2) of Proposition 2.1.2 for $\alpha, \beta \in (0, \infty)$, then

$$\mathbb{E}\left[|\nabla_S F|^2\right] \ = \ \int_{S^{n-1}} |\nabla_S F(\theta)|^2 \ d\sigma_{n-1}(\theta) \ \leq \ \frac{C\left(\alpha^2+\beta\right)}{n}$$

where $C > 0$ is a universal constant.

We say that a function $f : S^{n-1} \to \mathbb{R}$ is C^1-*smooth (differentiable)* if some extension of f to $\mathbb{R}^n$ (or some open neighborhood of S^{n-1}) is C^1-smooth (differentiable). The *spherical gradient* $\nabla_S f(\theta) \in \mathbb{R}^n$ of such a function is defined by

$$\nabla_S f(\theta) \;=\; P_{\theta^\perp} \nabla f(\theta). \tag{2.3}$$

where $P_{\theta^\perp} : \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the hyperplane perpendicular to θ and ∇f is the gradient of an extension of f to a neighborhood of the sphere. Alternatively, $\nabla_S f$ is naturally defined as the intrinsic gradient of f when we consider S^{n-1} as a manifold embedded in $\mathbb{R}^n$. We refer to Appendix A in [2] for a detailed introduction of the notion of the spherical gradient.

Note that the constants C differ between Propositions 2.1.2 and 2.1.5.

The reduction from Proposition 2.1.2 to Proposition 2.1.5 is via the Poincaré inequality on the sphere. We thus provide two proofs to Proposition 2.1.2, one that relies on Proposition 2.1.5 and holds for the Leb-a.c. case, and a direct proof for the general setting. The direct proof and the proof of Proposition 2.1.5 rely on the same argument and are very similar to each other.

Next are a few lemmas that will be required in the discussion to follow, in order to solve some integrals of some "*double-anchor*" homogeneous functions on the sphere. By "double-anchor" we mean functions $f : S^{n-1} \to \mathbb{R}$ of the form $f(\theta) = \tilde{f}(\langle \theta, \eta \rangle, \langle \theta, \xi \rangle)$ for some fixed points $\eta, \xi \in S^{n-1}$. Informally, the chain of reasoning in the lemmas will be:

$$\int_{S^{n-1}} d\sigma_{n-1} \overset{\text{polar integration}}{\curvearrowright} \int_{\mathbb{R}^n} d\gamma_n \overset{\text{projection}}{\curvearrowright} \int_{\mathbb{R}^2} d\gamma_2 \overset{\text{polar integration}}{\curvearrowright} \int_{S^1} d\sigma_1$$

where γ_n denotes the standard Gaussian measure on $\mathbb{R}^n$.

2.2 Integrating "double-anchor" functions

First is a polar integration formula for homogeneous functions. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *p-homogeneous* if $f(tx) = t^p f(x)$ for any $t > 0$, $x \in \mathbb{R}^n$.

Lemma 2.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be γ_n-measurable and p-homogeneous. Then:*

$$\int_{\mathbb{R}^n} f(x)\; d\gamma_n(x) \;= C_{n,p} \int_{S^{n-1}} f(\theta)\; d\sigma_{n-1}(\theta)$$

where $C_{n,p} = n \cdot 2^{\frac{p}{2}-1} \cdot \frac{\Gamma(\frac{n+p}{2})}{\Gamma(\frac{n}{2}+1)}$.

Proof.

$$\int_{\mathbb{R}^n} f(x)\ d\gamma_n(x) \ = \ (2\pi)^{-n/2} \int_{\mathbb{R}^n} |x|^p\ f(x/\,|x|)\ e^{-\frac{|x|^2}{2}}\ dx$$

By polar integration, this is

$$= \ n \cdot \kappa_n \cdot (2\pi)^{-n/2} \cdot \int_0^\infty r^{n+p-1}\ e^{-\frac{r^2}{2}}\ dr \cdot \int_{S^{n-1}} f(\theta)\ d\sigma_{n-1}(\theta)$$

where $\kappa_n = \pi^{n/2}/\Gamma(\frac{n}{2}+1)$ is the volume of the unit ball in dimension n. The integral over $[0,\infty)$ equals $2^{\frac{n+p}{2}-1}\Gamma(\frac{n+p}{2})$ via the change of variable $s = r^2/2$. Thus we continue

$$= \ n \cdot \kappa_n \cdot (2\pi)^{-n/2} \cdot 2^{\frac{n+p}{2}-1} \cdot \Gamma\left(\frac{n+p}{2}\right) \cdot \int_{S^{n-1}} f(\theta)\ d\sigma_{n-1}(\theta),$$

and we see that the lemma is proven, with

$$C_{n,p} \ = \ n \cdot 2^{\frac{p}{2}-1} \cdot \frac{\Gamma(\frac{n+p}{2})}{\Gamma(\frac{n}{2}+1)}.$$

□

Next, we use Lemma 2.2.1 to reduce an integral over S^{n-1} to an integral over $[0, 2\pi]$.

Lemma 2.2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue-measurable, p-homogeneous and let $\eta, \xi \in S^{n-1}$ be linearly independent. Then:*

$$\int_{S^{n-1}} f(\langle\theta,\eta\rangle)\ f(\langle\theta,\xi\rangle)\ d\sigma(\theta) \ = \ \frac{C_{n,2p}^{-1}\ C_{2,2p}}{2\pi} \int_0^{2\pi} f\left(\cos t\right) f\left(\cos(t-\rho)\right)\ dt,$$

where $\rho = \rho(\eta,\xi) = \arccos\langle\eta,\xi\rangle \in [0,\pi]$ is the angle between η and ξ.

Proof. Apply Lemma 2.2.1 to the $2p$-homogeneous $f(\langle\cdot,\eta\rangle)f(\langle\cdot,\xi\rangle)$. We get

$$\int_{S^{n-1}} f\left(\langle\theta,\eta\rangle\right)\ f\left(\langle\theta,\xi\rangle\right)\ d\sigma(\theta) \ = \ C_{n,2p}^{-1} \int_{\mathbb{R}^n} f\left(\langle x,\eta\rangle\right)\ f\left(\langle x,\xi\rangle\right)\ d\gamma_n(x)$$

Let $P : \mathbb{R}^n \to \mathbb{R}^2$ be the orthogonal projection onto the plane *span* $\{\eta,\xi\}$ which takes $\eta \mapsto (1,0)$ and $\xi \mapsto (\cos\rho, \sin\rho)$ where $\rho \in (0,\pi)$, $\cos\rho = \langle\eta,\xi\rangle$.

A property of the standard Gaussian distribution is that its marginals are lower dimensional standard Gaussian distributions. In our case $P_*\gamma_n = \gamma_2$ and we continue,

$$= C_{n,2p}^{-1} \int_{\mathbb{R}^2} f\left(\langle x, (1,0)\rangle\right) \, f\left(\langle x, (\cos\rho, \sin\rho)\rangle\right) \, d\gamma_2(x)$$

We use Lemma 2.2.1 again,

$$\begin{aligned}
&= \frac{C_{n,2p}^{-1}\, C_{2,2p}}{2\pi} \int_0^{2\pi} f\left(\langle (\cos t, \sin t), (1,0)\rangle\right) \, f\left(\langle (\cos t, \sin t), (\cos\rho, \sin\rho)\rangle\right) \, dt \\
&= \frac{C_{n,2p}^{-1}\, C_{2,2p}}{2\pi} \int_0^{2\pi} f\left(\cos t\right) f\left(\cos(t-\rho)\right) \, dt
\end{aligned}$$

as $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$. □

The last two lemmas of this section are calculations of specific integrals that will appear in the proofs of Propositions 2.1.2 and 2.1.5.

Lemma 2.2.3. *For any* $x, y \in \mathbb{R}^n$,

$$\int_{S^{n-1}} \mathbb{1}_{\langle x,\theta\rangle \geq 0} \, \mathbb{1}_{\langle y,\theta\rangle \geq 0} \, d\sigma(\theta) \; = \; \frac{\pi - \arccos\left\langle \frac{x}{|x|}, \frac{y}{|y|}\right\rangle}{2\pi}$$

where $\arccos\left\langle \frac{x}{|x|}, \frac{y}{|y|}\right\rangle \in [0, \pi]$.

Here, $\theta \mapsto \mathbb{1}_{\langle x,\theta\rangle \geq 0}$ is the indicator function, assuming value one if $\langle x, \theta\rangle \geq 0$ and zero otherwise.

Proof. Without loss of generality we may assume $|x|, |y| = 1$. If $x = y$ then both the left and right hand sides are equal one half, and if $x = -y$ both are equal zero. We may now proceed under the assumption that $x, y \in S^{n-1}$ are linearly independent. Observing that $C_{d,0} = 1$ for all d we have by Lemma 2.2.2:

$$\begin{aligned}
\int_{S^{n-1}} \mathbb{1}_{\langle x,\theta\rangle \geq 0} \, \mathbb{1}_{\langle y,\theta\rangle \geq 0} \, d\sigma(\theta) \; &= \; \frac{1}{2\pi} \int_0^{2\pi} \mathbb{1}_{\cos t \geq 0} \, \mathbb{1}_{\cos(t-\rho) \geq 0} \, dt \; = \\
&= \; \frac{1}{2\pi} \int_0^{2\pi} \mathbb{1}_{t \in (-\frac{\pi}{2}+\rho, \frac{\pi}{2})} dt \; = \; \frac{\pi - \rho}{2\pi}
\end{aligned}$$

where $\rho = \arccos\left\langle \frac{x}{|x|}, \frac{y}{|y|}\right\rangle \in [0, \pi]$. □

Lemma 2.2.4. *For any* $\eta, \xi \in S^{n-1}$,

$$\int_{S^{n-1}} \langle \theta, \eta \rangle_+ \langle \theta, \xi \rangle_+ \; d\sigma(\theta) \;=\; \frac{(\pi - \arccos \langle \eta, \xi \rangle) \langle \eta, \xi \rangle + \sqrt{1 - \langle \eta, \xi \rangle^2}}{2\pi n}$$

where $\arccos \langle \eta, \xi \rangle \in [0, \pi]$.

Proof. If $\eta = \xi$ then by Lemma 2.2.1 the left-hand side is

$$\begin{aligned} \int_{S^{n-1}} \langle \theta, \eta \rangle_+ \langle \theta, \xi \rangle_+ d\sigma(\theta) \;&=\; \int_{S^{n-1}} (\langle \theta, \eta \rangle_+)^2 \; d\sigma(\theta) \\ &=\; \frac{1}{2} \int_{S^{n-1}} \theta_1^2 \; d\sigma(\theta) \;=\; \frac{1}{2n} \int_{S^{n-1}} |\theta|^2 \, d\sigma(\theta) \;=\; \frac{1}{2n} \end{aligned}$$

by the rotation invariance of σ, equal to the right-hand side. If $\eta = -\xi$, both equal zero. We may now proceed under the assumption that η, ξ are linearly independent. Observing that $C_{n,2} = n, \; C_{2,2} = 2$ we have by Lemma 2.2.2,

$$\begin{aligned} \int_{S^{n-1}} \langle \theta, \eta \rangle_+ \langle \theta, \xi \rangle_+ \; d\sigma(\theta) \;&=\; \frac{1}{\pi n} \int_0^{2\pi} \cos(t)_+ \cos(t - \rho)_+ \; dt \;= \\ = \frac{1}{\pi n} \int_{-\pi/2+\rho}^{\pi/2} \cos t \cos(t - \rho) \; dt \;&=\; \frac{1}{2\pi n} \int_{-\pi/2+\rho}^{\pi/2} (\cos \rho + \cos(2t - \rho)) \; dt \;= \\ &=\; \frac{1}{2\pi n} \left((\pi - \rho) \cos \rho + \sin \rho \right) \end{aligned}$$

where $\cos \rho = \langle \eta, \xi \rangle$.

□

We are now ready to prove the main propositions of this chapter.

2.3 Absolutely continuous setting

As we noted before, when μ is Leb-a.c. Proposition 2.1.2 can be deduced immediately from Proposition 2.1.5 based on the well known Poincaré inequality on the sphere.

Theorem 2.3.1 (Poincaré inequality)**.** *Let $f : S^{n-1} \to \mathbb{R}$ be C^1-smooth. Then,*

$$Var(f) \;\le\; \frac{1}{n-1}\int_{S^{n-1}} |\nabla_S f|^2 \, d\sigma_{n-1}.$$

The Poincaré inequality for C^2 functions can easily be derived using spherical harmonics (see, e.g. the argument in [2]), whereas C^1 functions can be approximated by C^2 functions in an appropriate Sobolev space (see [8]) to obtain the above version of the inequality. We thus turn to proving Proposition 2.1.5. Let us first calculate the derivative of F_μ.

Lemma 2.3.2. *Let μ be a Borel probability measure on $\mathbb{R}^n$ with finite first moment, and let $\theta \in S^{n-1}$. If $\mu(\theta^\perp) = 0$ then F_μ is differentiable at θ and*

$$\nabla_S F_\mu(\theta) \;=\; \int_{H_\theta} P_{\theta^\perp} x \; d\mu(x).$$

Here, the integral is defined over the half-space $H_\theta = \{x \in \mathbb{R}^n : \; \langle x, \theta\rangle > 0\}$, $\theta^\perp = \{x \in \mathbb{R}^n : \langle x, \theta \rangle = 0\}$ is the hyperplane orthogonal to θ and $P_{\theta^\perp} : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto $\theta^\perp$.

If μ is Leb-a.c., it associates measure zero to any hyperplane and the condition is fulfilled for all $\theta \in S^{n-1}$. Moreover, f is C^1-smooth.

Corollary 2.3.3. *Let μ be a Leb-a.c. probability measure on $\mathbb{R}^n$ with finite first moment. Then F_μ is C^1-smooth and*

$$\nabla_S F_\mu(\theta) \;=\; \int_{H_\theta} P_{\theta^\perp} x \; d\mu(x)$$

for any $\theta \in S^{n-1}$.

For the purpose of this section, we extend F_μ 1-homogeneously to the entire $\mathbb{R}^n$. That is, we set

$$F_\mu(y) \;=\; |y|\, F_\mu(y/\,|y|) \;=\; \int_{\mathbb{R}^n} \langle x, y\rangle_+ \; d\mu(x) \qquad ; \quad y \in \mathbb{R}^n.$$

Henceforth we treat F_μ as a function over $\mathbb{R}^n$. Assuming F_μ is differentiable at $\theta \in S^{n-1}$, the spherical gradient is $\nabla_S F_\mu(\theta) = P_{\theta^\perp} \nabla F(\theta)$. This will allow us to work with the more comfortable ∇F rather than $\nabla_S F$.

Proof of Lemma 2.3.2. We show that

$$F(\theta + h) \;=\; F(\theta) + \Big\langle \int_{H_\theta} x \; d\mu,\; h \Big\rangle + o\,(|h|) \qquad \text{as } |h| \to 0$$

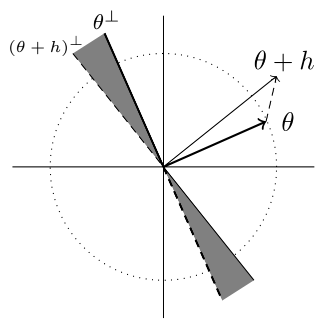
and this will prove that $\nabla f(\theta) = \int_{H_\theta} x \; d\mu$ as we require. Let $h \in \mathbb{R}^n$, and we calculate:

$$\begin{aligned}
&\left| F(\theta + h) - F(\theta) - \int_{H_\theta} \langle x, h \rangle \;\; d\mu(x) \right| \\
&\qquad = \; \left| \int_{H_{\theta+h}} \langle x, \theta + h \rangle \;\; d\mu(x) - \int_{H_\theta} \langle x, \theta + h \rangle \;\; d\mu(x) \right| \\
&\qquad = \; \int_{H_{\theta+h} \Delta H_\theta} |\langle x, \theta + h \rangle| \;\; d\mu(x) \;\; \overset{\circledast}{\leq} \;\; 2 \int_{H_{\theta+h} \Delta H_\theta} |\langle x, h \rangle| \;\; d\mu(x) \\
&\qquad \leq \; 2\,|h| \int_{H_{\theta+h} \Delta H_\theta} |x| \;\; d\mu \;\; = \;\; 2\,|h| \int_{\mathbb{R}^n} |x| \;\; \mathbb{1}_{H_{\theta+h} \Delta H_\theta}(x) \; d\mu(x).
\end{aligned}$$

with the transition $\circledast$ because

$$x \in H_{\theta+h} \setminus H_\theta \quad \Longleftrightarrow \quad -\langle x, h \rangle < \langle x, \theta \rangle < 0 \quad \Longleftrightarrow \quad -x \in H_\theta \setminus H_{\theta+h}.$$

Here, $\mathbb{1}_{H_{\theta+h} \Delta H_\theta}(x)$ is the indicator function assuming value one if $x \in H_{\theta+h} \Delta H_\theta$ and zero otherwise.



To conclude the proof, we need to show that the last integral above converges to 0 as $h \to 0$. See that the integrand $x \mapsto |x|\,\mathbb{1}_{H_{\theta+h} \Delta H_\theta}(x)$ is bounded by the convergent first moment function $x \mapsto |x|$. Hence by the dominated convergence theorem, we are left to show pointwise convergence a.e.; that is, $|x|\,\mathbb{1}_{H_{\theta+h} \Delta H_\theta}(x) \to 0$ as $h \to 0$ for μ-a.e. $x \in \mathbb{R}^n$. Indeed, assume $x \notin \theta^\perp$. When $|h| < |\langle x, \theta \rangle| \,/ 2\,|x|$ we have

$$|\langle x, \theta + h \rangle - \langle x, \theta \rangle| \;=\; |\langle x, h \rangle| \;\leq\; |x|\,|h| \;\leq\; \frac{|\langle x, \theta \rangle|}{2},$$

therefore sign $\langle x, \theta + h \rangle = \text{sign} \langle x, \theta \rangle$ hence $x \in H_{\theta+h} \iff x \in H_\theta$ and $\mathbb{1}_{H_{\theta+h} \Delta H_\theta}(x) = 0$. To conclude the proof, by the assumption of the lemma $\mu(\theta^\perp) = 0$. □

Now to prove Corollary 2.3.3.

Proof of Corollary 2.3.3. By Lemma 2.3.2, F is differentiable on S^{n-1}. Fix $\theta \in S^{n-1}$, and we show that

$$\nabla_S F(\eta) \longrightarrow \nabla_S F(\theta) \qquad \text{as } \eta \to \theta,\ \eta \in S^{n-1}.$$

Indeed,

$$\nabla_S F(\eta) = P_{\eta^\perp} \int_{\mathbb{R}^n} x \mathbb{1}_{H_\eta}(x)\, d\mu(x).$$

When $\eta \to \theta$ we have $P_{\eta^\perp} \to P_{\theta^\perp}$ and $x\mathbb{1}_{H_\eta}(x) \to x\mathbb{1}_{H_\theta}(x)$ for μ-a.e. $x \in \mathbb{R}^n$ as μ is absolutely continuous. We apply the dominated convergence theorem to the above integral coordinate-wise. Each coordinate function $x_i \mathbb{1}_{H_\eta}(x)$ is bounded by the integrable first moment function $x \mapsto |x|$, hence by dominated convergence we have $\nabla_S F(\eta) \to \nabla_S F(\theta)$. □

We now prove Proposition 2.1.5.

Proof of Proposition 2.1.5.
We require to bound $\int_{S^{n-1}} |\nabla_S F|^2 \, d\sigma_{n-1}$. As $\nabla_S F(\theta) = P_{\theta^\perp} \nabla F(\theta)$ for $\theta \in S^{n-1}$ we have by the Pythagorean theorem that

$$|\nabla_S F(\theta)|^2 = |\nabla F(\theta)|^2 - \langle \nabla F(\theta), \theta \rangle^2 . \tag{2.4}$$

Note that $\langle \nabla F(\theta), \theta \rangle$ equals $F(\theta)$. This is true in general for any 1-homogeneous function, and also evident from the formula for $\nabla F(\theta)$ given in Corollary 2.3.3. We tend to each of the components in (2.4) separately. Rewrite ∇F as obtained in Corollary 2.3.3, in the following form:

$$\nabla F(\theta) = \int_{\mathbb{R}^n} x \ \mathbb{1}_{\langle x, \theta \rangle > 0} \, d\mu(x)$$

Now,

$$\int_{S^{n-1}} |\nabla F(\theta)|^2 \, d\sigma(\theta) \; = \; \int_{S^{n-1}} \langle \nabla F(\theta), \nabla F(\theta) \rangle \; d\sigma(\theta) \; =$$

$$= \int_{S^{n-1}} \Big\langle \int_{\mathbb{R}^n} x \; \mathbb{1}_{\langle x,\theta\rangle \geq 0} \; d\mu(x) \; , \; \int_{\mathbb{R}^n} y \; \mathbb{1}_{\langle y,\theta\rangle \geq 0} \; d\mu(y) \Big\rangle d\sigma(\theta) \; =$$

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle x, y \rangle \;\; \mathbb{1}_{\langle x,\theta\rangle \geq 0} \; \mathbb{1}_{\langle y,\theta\rangle \geq 0} \; d\mu(x) \; d\mu(y) \; d\sigma(\theta)$$

We use Fubini's theorem to switch the order of integration

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle x, y \rangle \; \left(\int_{S^{n-1}} \mathbb{1}_{\langle x,\theta\rangle \geq 0} \; \mathbb{1}_{\langle y,\theta\rangle \geq 0} \; d\sigma(\theta) \right) \; d\mu(x) \; d\mu(y)$$

then Lemma 2.2.3

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle x, y \rangle \; \frac{\pi - \arccos \left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle}{2\pi} \; d\mu(x) \; d\mu(y)$$

Expanding the function $\tau \mapsto \frac{\pi - \arccos \tau}{2\pi}$, $\tau \in [-1, 1]$, $\arccos \tau \in [0, \pi]$ into a power series around 0 we see that $\frac{\pi - \arccos \tau}{2\pi} = \frac{1}{4} + \frac{1}{2\pi}\tau + a \, |\tau|^3$, where $|a|$ is bounded by a constant (for a proof, see Appendix 2.5.3). Hence we may continue,

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{\langle x, y \rangle}{4} \; + \; \frac{\langle x, y \rangle^2}{2\pi |x||y|} \; + \; a \frac{\langle x, y \rangle^4}{|x|^3 |y|^3} \right) \; d\mu(x) \; d\mu(y).$$

As to the other component in the decomposition (2.4),

$$\int_{S^{n-1}} \langle \nabla F(\theta), \theta \rangle^2 \; d\sigma(\theta) \; = \; \int_{S^{n-1}} F(\theta)^2 \, d\sigma_{n-1}(\theta)$$

$$= \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \langle x, \theta \rangle_+ \; d\mu(x) \right) \; \left(\int_{\mathbb{R}^n} \langle y, \theta \rangle_+ \; d\mu(y) \right) \; d\sigma(\theta)$$

By Fubini's theorem

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x||y| \left(\int_{S^{n-1}} \left\langle \theta, \frac{x}{|x|} \right\rangle_+ \left\langle \theta, \frac{y}{|y|} \right\rangle_+ \, d\sigma(\theta) \right) \; d\mu(x) \; d\mu(y)$$

and by Lemma 2.2.4,

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x||y| \left(\frac{\left(\pi - \arccos\left\langle \frac{x}{|x|}, \frac{y}{|y|}\right\rangle\right)\left\langle \frac{x}{|x|}, \frac{y}{|y|}\right\rangle + \sqrt{1-\left\langle \frac{x}{|x|}, \frac{y}{|y|}\right\rangle^2}}{2\pi n} \right) \, d\mu(x)\, d\mu(y).$$

Expanding the function $\tau \mapsto \frac{(\pi - \arccos\tau)\tau + \sqrt{1-\tau^2}}{2\pi}$, $\tau \in [-1,1]$, $\arccos\tau \in [0,\pi]$ into a power series around 0 we see that $\frac{(\pi-\arccos\tau)\tau+\sqrt{1-\tau^2}}{2\pi} = \frac{1}{2\pi} + \frac{1}{4}\tau + b\tau^2$ where $|b|$ is bounded by a constant (for a proof, see Appendix 2.5.4). Hence we may continue,

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\frac{|x||y|}{2\pi n} \; + \; \frac{\langle x,y\rangle}{4n} \; + \; b\frac{\langle x,y\rangle^2}{n|x||y|} \right) \, d\mu(x)\, d\mu(y)$$

We arrive at:

$$\int_{S^{n-1}} |\nabla_S F|^2 \, d\sigma \; = \; \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Bigg(\left(\frac{1}{4} - \frac{1}{4n} \right) \langle x,y\rangle \; +$$

$$+ \; \frac{\langle x,y\rangle^2}{2\pi|x||y|} \; - \; \frac{|x||y|}{2\pi n} \; - \; b\frac{\langle x,y\rangle^2}{n|x||y|} \; + \; a\frac{\langle x,y\rangle^4}{|x|^3|y|^3} \Bigg) \, d\mu(x)\, d\mu(y).$$

The first component vanishes with integration because μ is centered;

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle x,y\rangle \; d\mu(x)\, d\mu(y) \; = \; \left| \int_{\mathbb{R}^n} x \; d\mu \right|^2 \; = \; 0.$$

The second and third components cancel each other out with integration, due to assumption (1); The second is

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle x,y\rangle^2}{|x||y|} \, d\mu(x)\, d\mu(y) \; = \; \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \langle x,y\rangle^2 \; \frac{d\mu(x)}{|x|} \right) \frac{d\mu(y)}{|y|} \; =$$

$$= \; \int_{\mathbb{R}^n} \frac{\alpha}{\sqrt{n}} |y|^2 \, \frac{d\mu(y)}{|y|} \; = \; \frac{\alpha}{\sqrt{n}} \cdot n \cdot \frac{\alpha}{\sqrt{n}} \; = \; \alpha^2$$

as we saw in Claim 2.1.3. The third is

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|x||y|}{n} \, d\mu(x)\, d\mu(y) \; = \; \frac{1}{n} \left(\int_{\mathbb{R}^n} |x| \; d\mu(x) \right)^2 \; = \; \frac{1}{n} \left(n \cdot \frac{\alpha}{\sqrt{n}} \right)^2 \; = \alpha^2$$

as well. In fact, the second and third components cancel each other out *exactly* if and only if $\int_{\mathbb{R}^n} x \otimes x \, d\mu/|x|$ is a scalar matrix (see Appendix 2.5.5). The fourth component integrates to at most α^2/n times a universal constant by the calculation above. The fifth component integrates to at most β/n times a universal constant due to assumption (2). $\square$

2.4 General setting

In the general setting, F_μ may not be differentiable. Yet, applying a similar argument to $\int_{S^{n-1}} (F_\mu - \mathbb{E}F_\mu)^2 \, d\sigma_{n-1}$ rather than to $\int_{S^{n-1}} |\nabla_S F_\mu|^2 \, d\sigma_{n-1}$ yields the required result.

Direct proof of Proposition 2.1.2. We start with $\mathrm{Var}(F) = \int_{S^{n-1}} F^2 \, d\sigma_{n-1} - \left(\int_{S^{n-1}} F \, d\sigma\right)^2$. The first component we have already calculated as part of the proof of Proposition 2.1.5. It is

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x||y| \left(\frac{\left(\pi - \arccos\left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle\right)\left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle + \sqrt{1 - \left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle^2}}{2\pi n} \right) \, d\mu(x) \, d\mu(y).$$

Expanding $\tau \mapsto \frac{(\pi - \arccos\tau)\tau + \sqrt{1-\tau^2}}{2\pi}$ into a Taylor series around 0 we see that $\frac{(\pi - \arccos\tau)\tau + \sqrt{1-\tau^2}}{2\pi} = \frac{1}{2\pi} + \frac{1}{4}\tau + \frac{1}{4\pi}\tau^2 + c\tau^4$ where $|c|$ is bounded by a constant (for a proof, see Appendix 2.5.6). Hence we may continue,

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|x||y|}{2\pi n} + \frac{\langle x, y\rangle}{4n} + \frac{\langle x, y\rangle^2}{4\pi n|x||y|} + c\frac{\langle x, y\rangle^4}{n|x|^3|y|^3} \right) \, d\mu(x) \, d\mu(y).$$

As for the second part, by Fubini's theorem

$$\int_{S^{n-1}} F \, d\sigma_{n-1} = \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} \langle x, \theta\rangle_+ \, d\sigma_{n-1}(\theta) \right) \, d\mu(x).$$

Use Lemma 2.2.1 to obtain

$$\int_{S^{n-1}} \langle x, \theta\rangle_+ \, d\sigma_{n-1}(\theta) = C_{n,1}^{-1} \int_{\mathbb{R}^n} \langle x, z\rangle_+ \, d\gamma_n(z)$$

and by orthogonally projecting onto the line *span* $\{x\}$,

$$= \frac{C_{n,1}^{-1}}{\sqrt{2\pi}} |x| \int_{\mathbb{R}} t_+ e^{-t^2/2} \, dt = \frac{C_{n,1}^{-1}}{\sqrt{2\pi}} |x| \int_0^\infty t e^{-t^2/2} \, dt = \frac{C_{n,1}^{-1}}{\sqrt{2\pi}} |x|.$$

Therefore,

$$\left(\int_{S^{n-1}} F \, d\sigma_{n-1}\right)^2 = \frac{C_{n,1}^{-2}}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x||y| \, d\mu(x) \, d\mu(y).$$

It holds that $C_{n,1}^{-2} = \frac{1}{n} + \frac{1}{2n^2} + d\frac{1}{n^3}$ where $|d|$ is bounded by a constant (for a proof, see Appendix 2.5.7). We have attained:

$$\left(\int_{S^{n-1}} F \, d\sigma_{n-1}\right)^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|x||y|}{2\pi n} + \frac{|x||y|}{4\pi n^2} + d\frac{|x||y|}{n^3}\right) d\mu(x) \, d\mu(y).$$

Combining the two parts,

$$\begin{aligned} \mathrm{Var}(F) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Bigg(& \frac{\langle x, y\rangle}{4n} + \frac{\langle x, y\rangle^2}{4\pi n|x||y|} - \frac{|x||y|}{4\pi n^2} \\ & + c\frac{\langle x, y\rangle^4}{n|x|^3|y|^3} - d\frac{|x||y|}{n^3}\Bigg) d\mu(x) \, d\mu(y). \end{aligned}$$

We conclude as in the proof of Proposition 2.1.5. The first component vanishes with integration as μ is centered. The second and third components cancel each other out due to assumption (1). The fourth component integrates to at most β/n^2 times a universal constant due to assumption (2). The fifth components integrates to at most α^2/n^2 times a universal constant. □

2.5 Appendix

2.5.1

Claim 2.5.1. *Let $f : S^{n-1} \to \mathbb{R}$, $a > 0$ and assume that*

$$\mathbb{P}\left[|f - \mathbb{E}f| > t\right] \le 2e^{-(n-1)t^2/2a^2}$$

for any $t > 0$. Then,

$$\mathit{Var}(f) \le \frac{4a^2}{n-1}.$$

Proof. Write $g = f - \mathbb{E}f$. Then $\mathrm{Var}(f) = \mathrm{Var}(g)$ and $\mathbb{P}\left[|f - \mathbb{E}f| > t\right] = \mathbb{P}\left[|g| > t\right]$. By Fubini's theorem,

$$\frac{n-1}{4a^2}\mathrm{Var}(g) \;=\; \mathbb{E}\left[\left(\sqrt{\frac{n-1}{4a^2}}g\right)^2\right] \;=\; \mathbb{E}\left[\int_0^{\sqrt{\frac{n-1}{4a^2}}|g|} 2t\ dt\right] \;=$$

$$=\; \mathbb{E}\left[\int_0^\infty 2t \cdot \mathbb{1}_{\sqrt{\frac{n-1}{4a^2}}|g|>t}\ dt\right] \;=\; \int_0^\infty 2t \cdot \mathbb{E}\left[\mathbb{1}_{\sqrt{\frac{n-1}{4a^2}}|g|>t}\right] dt \;=$$

$$=\; \int_0^\infty 2t \cdot \mathbb{P}\left[|g| > t\sqrt{\frac{4a^2}{n-1}}\right] dt \;\le\; \int_0^\infty 4te^{-2t^2}\ dt \;=\; \left[-e^{-2t^2}\right]_0^\infty \;=\; 1.$$

□

2.5.2

Claim 2.5.2. *Let μ be a Borel probability measure on $\mathbb{R}^n$ such that*

$$\int_{S^{n-1}} x \otimes x\ \frac{d\mu}{|x|} \;=\; \frac{\alpha}{\sqrt{n}} Id. \tag{2.5}$$

Then $\|F_\mu\|_{Lip} \le 3\alpha$.

In the proof we use the following lemma.

Lemma 2.5.3. *Let $A \subseteq \mathbb{R}^n$ be a Borel set and let $\xi \in \mathbb{R}^n$. Then,*

$$\left|\int_A \langle x, \xi\rangle\ d\mu(x)\right| \;\le\; \alpha|\xi|.$$

Proof.

$$\left|\int_A \langle x, \xi\rangle\ d\mu(x)\right| \;\le\; \int_{\mathbb{R}^n} \frac{|\langle x, \xi\rangle|}{\sqrt{|x|}}\sqrt{|x|}\ d\mu(x) \;\le$$

$$\le\; \sqrt{\int_{\mathbb{R}^n} \langle x, \xi\rangle^2\ \frac{d\mu(x)}{|x|}}\ \sqrt{\int_{\mathbb{R}^n} |x|\ d\mu(x)} \;=\; \sqrt{\frac{\alpha}{\sqrt{n}}|\xi|^2} \cdot \sqrt{\sqrt{n}\alpha} \;=\; \alpha|\xi|$$

based on the calculation given in Claim 2.1.3. We also used the Cauchy-Schwartz inequality.

□

Proof of claim. Let $\theta, \eta \in S^{n-1}$. We want to prove that

$$|F(\theta) - F(\eta)| \;\le\; 3\alpha|\theta - \eta|.$$

We handle first the case that $\langle \theta, \eta \rangle \ge 0$. We have:

$$|F(\theta) - F(\eta)| \;=\; \Big| \int_{H_\theta} \langle x, \theta \rangle \;\; d\mu(x) \;-\; \int_{H_\eta} \langle x, \eta \rangle \;\; d\mu(x) \Big| \;=$$

$$= \;\Big| \int_{H_\theta \cap H_\eta} \langle x, \theta - \eta \rangle \;\; d\mu(x) \;+\; \int_{H_\theta \setminus H_\eta} \langle x, \theta \rangle \;\; d\mu(x) \;-\; \int_{H_\eta \setminus H_\theta} \langle x, \eta \rangle \;\; d\mu(x) \Big|$$

$$\le \;\Big| \int_{H_\theta \cap H_\eta} \langle x, \theta - \eta \rangle \;\; d\mu(x) \Big| \;+\; \int_{H_\theta \setminus H_\eta} \langle x, \theta \rangle \;\; d\mu(x) \;+\; \int_{H_\eta \setminus H_\theta} \langle x, \eta \rangle \;\; d\mu(x).$$

For the first integral, we apply the lemma:

$$\Big| \int_{H_\theta \cap H_\eta} \langle x, \theta - \eta \rangle \;\; d\mu(x) \Big| \;\le\; \alpha|\theta - \eta|.$$

For the second integral, note that for any $x \in H_\theta \setminus H_\eta$, since $\langle \theta, \eta \rangle \ge 0$ and $\langle x, \eta \rangle < 0$ it holds that $0 < \langle x, \theta \rangle \le \langle x, \theta - \langle \theta, \eta \rangle \eta \rangle$. We apply the lemma again:

$$\int_{H_\theta \setminus H_\eta} \langle x, \theta \rangle \;\; d\mu(x) \;\le\; \int_{H_\theta \setminus H_\eta} \langle x, \theta - \langle \theta, \eta \rangle \eta \rangle \;\; d\mu(x) \;\le\; \alpha|\theta - \langle \theta, \eta \rangle \eta|$$

and see that $|\theta - \langle \theta, \eta \rangle \eta| = \min_{t \in \mathbb{R}} |\theta - t\eta| \le |\theta - \eta|$.
The third integral is treated in a similar manner.

Turning to the case where $\langle \theta, \eta \rangle < 0$, on the one hand $|\theta - \eta| \ge \sqrt{2}$. On the other hand, $|F(\theta) - F(\eta)| \le F(\theta) + F(\eta) \le 2\alpha$ since for any $\xi \in S^{n-1}$ we have

$$F(\xi) \;=\; \int_{\langle x, \xi \rangle > 0} \langle x, \xi \rangle \;\; d\mu(x) \;\le\; \alpha|\xi| \;=\; \alpha.$$

We again applied the lemma. Hence, $|F(\theta) - F(\eta)| \le 2\alpha \le \sqrt{2}\alpha\, |\theta - \eta|$.

□

2.5.3

Claim 2.5.4. *There is a constant* $C > 0$ *such that:*

$$\left| \frac{\pi - \arccos\tau}{2\pi} \;-\; \frac{1}{4} \;-\; \frac{1}{2\pi}\tau \right| \;\leq\; C\left|\tau^3\right|$$

for any $\tau \in [-1, 1]$. *Here,* $\arccos\tau \in [0, \pi]$.

Proof. Denote $G(\tau) = \arccos\tau$. Then

$$G'(\tau) \;=\; -\frac{1}{\sqrt{1-\tau^2}} = -\left(1-\tau^2\right)^{-\frac{1}{2}},$$

$$G''(\tau) \;=\; -\left(-\frac{1}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{3}{2}} \;=\; -\tau\left(1-\tau^2\right)^{-\frac{3}{2}},$$

$$G'''(\tau) \;=\; -\left(1-\tau^2\right)^{-\frac{3}{2}} - \tau\left(-\frac{3}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{5}{2}} \;=$$

$$=\; -\left(1-\tau^2\right)^{-\frac{3}{2}} - 3\tau^2\left(1-\tau^2\right)^{-\frac{5}{2}}.$$

Hence,

$$G(0) = \frac{\pi}{2} \quad ; \quad G'(0) = -1 \quad ; \quad G''(0) = 0.$$

Denote $g(\tau) = \frac{\pi - \arccos\tau}{2\pi} - \frac{1}{4} - \frac{1}{2\pi}\tau$. Then,

$$g(0) = 0 \quad ; \quad g'(0) = 0 \quad ; \quad g''(0) = 0.$$

By Lagrange's formula for the residual of the Taylor series, $g(\tau) = \frac{1}{6}g'''(\tilde{\tau})\tau^3$ where $|\tilde{\tau}| < |\tau|$. g''' is continuous on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ therefore bounded. On $\left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$ the function $\frac{|g(\tau)|}{|\tau^3|}$ is continuous and thus bounded as well. ☐

2.5.4

Claim 2.5.5. *There is a constant* $C > 0$ *such that:*

$$\left| \frac{(\pi - \arccos\tau)\tau + \sqrt{1-\tau^2}}{2\pi} \;-\; \frac{1}{2\pi} \;-\; \frac{1}{4}\tau \right| \;\leq\; C\tau^2$$

for any $\tau \in [-1, 1]$. *Here,* $\arccos\tau \in [0, \pi]$.

Proof. Denote $H(\tau) = \sqrt{1-\tau^2}$. Then

$$H'(\tau) \;=\; \left(\frac{1}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{1}{2}} \;=\; -\tau\left(1-\tau^2\right)^{-\frac{1}{2}},$$

$$H''(\tau) \;=\; -\left(1-\tau^2\right)^{-\frac{1}{2}} - \tau\left(-\frac{1}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{3}{2}} \;=$$

$$=\; -\left(1-\tau^2\right)^{-\frac{1}{2}} - \tau^2\left(1-\tau^2\right)^{-\frac{3}{2}}.$$

Hence,

$$H(0) \;=\; 1 \quad ; \quad H'(0) = 0.$$

Adopting the notation $G(\tau) = \arccos\tau$ from Appendix 2.5.3, we denote

$$\begin{aligned} h(\tau) \;&=\; \frac{(\pi - \arccos\tau)\,\tau + \sqrt{1-\tau^2}}{2\pi} - \frac{1}{2\pi} - \frac{1}{4}\tau \;= \\ &=\; \frac{(\pi - G(\tau))\,\tau + H(\tau)}{2\pi} - \frac{1}{2\pi} - \frac{1}{4}\tau. \end{aligned}$$

Then,

$$h'(\tau) \;=\; \frac{\pi - G(\tau) - G'(\tau)\tau + H'(\tau)}{2\pi} - \frac{1}{4},$$

$$h''(\tau) \;=\; \frac{-2G'(\tau) - G''(\tau)\tau + H''(\tau)}{2\pi}.$$

Recalling the computations done in Appendix 2.5.3, we have:

$$h(0) = 0 \quad ; \quad h'(0) = 0.$$

By Lagrange's formula for the residual of the Taylor series, $h(\tau) = \frac{1}{2}h''(\tilde{\tau})\tau^2$ where $|\tilde{\tau}| < |\tau|$. h'' is continuous on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ therefore bounded. On $\left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$ the function $\frac{|h(\tau)|}{\tau^2}$ is continuous and thus bounded as well. □

2.5.5

In the proof of Proposition 2.1.5, the following expression arises:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle x, y\rangle^2}{|x||y|}\, d\mu(x)\, d\mu(y) \;-\; \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|x||y|}{n}\, d\mu(x)\, d\mu(y), \tag{2.6}$$

which we require to be 0. In this side note we take interest in the connection of this expression with the matrix $M = \int_{\mathbb{R}^n} x \otimes x \ \frac{d\mu(x)}{|x|}$.

The left-hand side integral equals $\|M\|_{\text{HS}}^2$. This can be observed through taking one integral at a time:

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \langle x, y \rangle^2 \ \frac{d\mu(x)}{|x|} \right) \ \frac{d\mu(y)}{|y|} \ = \ \int_{\mathbb{R}^n} \langle My, y \rangle \ \frac{d\mu(y)}{|y|} \ =$$

$$= \ \int_{\mathbb{R}^n} \langle M, (y \otimes y) \rangle_{\mathbb{R}^{n^2}} \ \frac{d\mu(y)}{|y|} \ = \ \langle M, M \rangle_{\mathbb{R}^{n^2}} \ = \ \|M\|_{\text{HS}}^2 \, .$$

The right-hand side integral equals $\frac{1}{n}(\text{Tr}M)^2$.
It holds true for M an $n \times n$ matrix that

$$\|M\|_{\text{HS}}^2 \ = \ \frac{1}{n}(\text{Tr}M)^2 \qquad \Longleftrightarrow \qquad M \text{ is a scalar matrix.}$$

This is just the Cauchy-Schwartz inequality on $\mathbb{R}^{n^2}$:

$$(\text{Tr}M)^2 \ = \ \langle Id, M \rangle_{\mathbb{R}^{n^2}}^2 \ \leq \ \|Id\|_{\mathbb{R}^{n^2}}^2 \|M\|_{\mathbb{R}^{n^2}}^2 \ = \ n \, \|M\|_{\text{HS}}^2 \, ,$$

with equality if and only if M is parallel to Id in $\mathbb{R}^{n^2}$.

2.5.6

Claim 2.5.6. *There is a constant $C > 0$ such that:*

$$\left| \frac{(\pi - \arccos \tau)\tau + \sqrt{1 - \tau^2}}{2\pi} \ - \ \frac{1}{2\pi} \ - \ \frac{1}{4}\tau \ - \ \frac{1}{4\pi}\tau^2 \right| \ \leq \ C\tau^4$$

for any $\tau \in [-1, 1]$. Here, $\arccos \tau \in [0, \pi]$.

Proof. Denote $H(\tau) = \sqrt{1 - \tau^2}$ as in Appendix 2.5.4. We have calculated:

$$H'(\tau) \ = \ -\tau \left(1 - \tau^2\right)^{-\frac{1}{2}} \quad ; \quad H''(\tau) \ = \ - \left(1 - \tau^2\right)^{-\frac{1}{2}} - \tau^2 \left(1 - \tau^2\right)^{-\frac{3}{2}} .$$

We need the next derivatives,

$$H'''(\tau) \ = \ - \left(-\frac{1}{2} \right) (-2\tau) \left(1 - \tau^2\right)^{-\frac{3}{2}} \ - \ 2\tau \left(1 - \tau^2\right)^{-\frac{3}{2}}$$

$$- \tau^2 \left(-\frac{3}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{5}{2}} \;=\; -3\tau\left(1-\tau^2\right)^{-\frac{3}{2}} \;-\; 3\tau^3\left(1-\tau^2\right)^{-\frac{5}{2}},$$

$$H''''(\tau) \;=\; -3\left(1-\tau^2\right)^{-\frac{3}{2}} \;-\; 3\tau\left(-\frac{3}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{5}{2}}$$

$$9\tau^2\left(1-\tau^2\right)^{-\frac{5}{2}} \;-\; 3\tau^3\left(-\frac{5}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{7}{2}} \;=$$

$$=\; -3\left(1-\tau^2\right)^{-\frac{3}{2}} \;-\; 18\tau^2\left(1-\tau^2\right) - \frac{5}{2} \;-\; 15\tau^4\left(1-\tau^2\right)^{-\frac{7}{2}}.$$

Hence,

$$H(0) \;=\; 1 \quad ; \quad H'(0) = 0 \quad ; \quad H''(0) = -1 \quad ; \quad H'''(0) = 0.$$

Denote $G(\tau) = \arccos\tau$ as in Appendix 2.5.3. We have calculated:

$$G'(\tau) \;=\; -\left(1-\tau^2\right)^{-\frac{1}{2}} \quad ; \quad G''(\tau) \;=\; -\tau\left(1-\tau^2\right)^{-\frac{3}{2}}$$

$$G'''(\tau) \;=\; -\left(1-\tau^2\right)^{-\frac{3}{2}} - 3\tau^2\left(1-\tau^2\right)^{-\frac{5}{2}}.$$

We need the next derivative,

$$G''''(\tau) \;=\; -\left(-\frac{3}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{3}{2}} \;-\; 6\tau\left(1-\tau^2\right)^{-\frac{5}{2}}$$

$$-\; 3\tau^2\left(-\frac{5}{2}\right)(-2\tau)\left(1-\tau^2\right)^{-\frac{7}{2}} \;=$$

$$=\; -3\tau\left(1-\tau^2\right)^{-\frac{3}{2}} \;-\; 6\tau\left(1-\tau^2\right)^{-\frac{5}{2}} \;-\; 15\tau^3\left(1-\tau^2\right)^{-\frac{7}{2}}.$$

Hence,

$$G(0) = \frac{\pi}{2} \quad ; \quad G'(0) = -1 \quad ; \quad G''(0) = 0 \quad ; \quad G'''(0) = -1.$$

Denote

$$h(\tau) \;=\; \frac{(\pi - \arccos\tau)\,\tau + \sqrt{1-\tau^2}}{2\pi} - \frac{1}{2\pi} - \frac{1}{4}\tau - \frac{1}{4\pi}\tau^2 \;=$$
$$=\; \frac{(\pi - G(\tau))\,\tau + H(\tau)}{2\pi} - \frac{1}{2\pi} - \frac{1}{4}\tau - \frac{1}{4\pi}\tau^2.$$

Then,

$$\begin{aligned}
h'(\tau) &= \frac{\pi - G(\tau) - G'(\tau)\tau + H'(\tau)}{2\pi} - \frac{1}{4} - \frac{1}{2\pi}\tau, \\
h''(\tau) &= \frac{-2G'(\tau) - G''(\tau)\tau + H''(\tau)}{2\pi} - \frac{1}{2\pi}, \\
h'''(\tau) &= \frac{-3G''(\tau) - G'''(\tau)\tau + H'''(\tau)}{2\pi}, \\
h''''(\tau) &= \frac{-4G'''(\tau) - G''''(\tau)\tau + H''''(\tau)}{2\pi}.
\end{aligned}$$

Thus:

$$h(0) = 0 \quad ; \quad h'(0) = 0 \quad ; \quad h''(0) = 0 \quad ; \quad h'''(0) = 0.$$

By Langrage's formula for the residual of the Taylor series, $h(\tau) = \frac{1}{24}h''''(\tilde{\tau})\tau^2$ where $|\tilde{\tau}| < |\tau|$. h'''' is continuous on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ therefore bounded. On $\left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$ the function $\frac{|h(\tau)|}{\tau^4}$ is continuous and thus bounded as well. □

2.5.7

Claim 2.5.7. *There is a constant $C > 0$ such that:*

$$\left| C_{n,1}^{-2} - \frac{1}{n} - \frac{1}{2n^2} \right| \leq \frac{C}{n^3}^2.$$

Proof. A calculation shows that

$$C_{n,1}^{-2} = \frac{\Gamma(\frac{n}{2})^2}{2\Gamma(\frac{n+1}{2})}.$$

It follows from a well-known result (see [15]) that

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \frac{\sqrt{2}}{\sqrt{n}} + \frac{1}{\sqrt{8}n^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{n^{\frac{5}{2}}}\right),$$

from which the required result is easily obtained. □

Chapter 3

The Discrete Cube

In this chapter we discuss the discrete cube $\{-1,1\}^n$. We start by showing that our variance bound (Proposition 2.1.2) holds in the rather simple case of the discrete measure distributed evenly on the discrete cube. The major part of this chapter is then devoted to proving that the result still holds, with high probability, when we consider a measure distributed evenly on a relatively tiny amount of random points from $\{-1,1\}^n$.

3.1 The discrete cube

The discrete cube is the set $\{-1,1\}^n \subseteq \mathbb{R}^n$ of the vertices of the cube $[-1,1]^n$. We denote it by E^n when the notation $\{-1,1\}^n$ is too long. Note that $|\{-1,1\}^n| = 2^n$. We consider the uniform probability measure on the discrete cube, defined as

$$\mu_n = 2^{-n} \sum_{x \in E^n} \delta_x$$

where δ_x is the delta measure assigning measure 1 to the single atom $x \in \mathbb{R}^n$. In other words, μ_n is the joint distribution $X = (X_1, ..., X_n)$ where the random variables $X_1, ..., X_n$ are independent coin flips; $X_i \sim \mathrm{Unif}(\{-1,1\}), \ i = 1, ..., n$. They admit $\mathbb{E}X_i = 0$ and $\mathbb{E}[X_i X_j] = \delta_{ij}$ for all i, j.

We now show that μ_n meets the conditions of Proposition 2.1.2, with $\alpha = 1, \ \beta = 3$.

Claim 3.1.1. *μ_n is centered and $\int_{\mathbb{R}^n} x \otimes x \ \frac{d\mu_n}{|x|} = \frac{1}{\sqrt{n}} Id$.*

Proof. First, μ_n is centered; $\mathbb{E}X = (\mathbb{E}X_1, ..., \mathbb{E}X_n) = (0, ..., 0)$. Second,

$$\int_{\mathbb{R}^n} x_i x_j \, \frac{d\mu_n}{|x|} \; = \; \mathbb{E}_{X_1,...,X_n}\left[\frac{X_i X_j}{|X|}\right] \; = \; \frac{1}{\sqrt{n}}\mathbb{E}_{X_1,...,X_n}\left[X_i X_j\right] \; = \; \frac{\delta_{ij}}{\sqrt{n}}.$$

□

Condition (2) is satisfied with $\beta = 3$ as a direct corollary of Khinchine's inequality (see, e.g. [12]):

Theorem 3.1.2 (Khinchine's inequality)**.** *For any $1 \leq p < \infty$ there exist constants $A_p, B_p > 0$ such that for any $n \in \mathbb{N}$ and any $a \in \mathbb{R}^n$,*

$$A_p|a| \; \leq \; \mathbb{E}\left[|\langle a, X\rangle|^p\right]^{1/p} \; \leq \; B_p|a|$$

where $X \sim Unif(\{-1, 1\}^n)$.

Here, the notation $X \sim \text{Unif}\,(\{-1, 1\}^n)$ means that X is a random variable distributing uniformly on $\{-1, 1\}^n$. It follows that

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle x, y\rangle^4}{|x|^3|y|^3}\, d\mu_n(x)\, d\mu_n(y) \; &= \; \frac{2^{-n}}{n^3}\sum_{y\in E^n}\left(2^{-n}\sum_{x\in E^n}\langle x, y\rangle^4\right) \\ &\leq \; \frac{2^{-n}}{n^3}\sum_{y\in E^n}\left(B_4^4|y|^4\right) \; = \; \frac{B_4^4}{n}. \end{aligned}$$

Haagerup [9] found the optimal constants B_p, $p > 0$. In our case, $\beta = B_4^4 = 3$. Let us explicitly state the result we obtain.

Corollary 3.1.3. *Let μ_n be the discrete measure distributed uniformly on $\{-1, 1\}^n \subseteq \mathbb{R}^n$. Then,*

$$Var(F_{\mu_n}) \; \leq \; \frac{4C}{n^2}$$

where C is the universal constant from Proposition 2.1.2.

It is important mentioning that $\|F_{\mu_n}\|_{\text{Lip}} \geq 1/\sqrt{8}$ in any dimension n, thus guaranteeing that Proposition 2.1.2 indeed improves Lévy's inequality for μ_n, asymptotically. In the proof of this, we shall want to use the next simple lemma tying the Lipschitz semi-norm to the spherical gradient. The lemma's proof appears in Appendix 3.3.1.

Lemma 3.1.4. *Let $f : S^{n-1} \to \mathbb{R}$ be differentiable at $\theta \in S^{n-1}$. Then $\|f\|_{Lip} \geq |\nabla_S f(\theta)|$.*

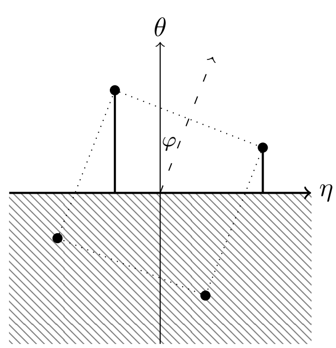
Claim 3.1.5. $\|F_{\mu_n}\|_{Lip} \geq 1/\sqrt{8}$.

Proof. Fix $\theta = (\cos\varphi, \sin\varphi, 0, ..., 0)$ for $\varphi \in (0, \frac{\pi}{4})$ and evaluate $|\nabla_S F_{\mu_n}(\theta)|$. First we need to verify that F_{μ_n} is indeed differentiable at θ. By Lemma 2.3.2 a sufficient condition is that $\mu_n(\theta^\perp) = 0$ where $\theta^\perp \subseteq \mathbb{R}^n$ is the hyperplane orthogonal to θ. Indeed, if $x \perp \theta$ for $x \in E^n$ it must be that $|\cos\varphi| = |\sin\varphi|$ which does not hold for any $\varphi \in (0, \frac{\pi}{4})$. Therefore we may write

$$\nabla_S F_{\mu_n}(\theta) \ = \ \int_{\langle x,\theta\rangle > 0} P_{\theta^\perp} x \ d\mu(x) \ = \ 2^{-n} \sum_{x \in E^n \cap H_\theta} P_{\theta^\perp} x.$$

Note that for $x \in E^n$, $\langle x, \theta\rangle > 0 \iff x_1 = 1$ because $\cos\varphi > \sin\varphi$. Thus for $\eta = (\sin\varphi, -\cos\varphi, 0, ..., 0) \perp \theta$ we have

$$\langle \nabla_S F_{\mu_n}(\theta), \eta\rangle \ = \ 2^{-n} \sum_{x \in E^n, \ x_1 = 1} \langle x, \eta\rangle \ = \ 2^{-n} \sum_{x \in E^n, \ x_1 = 1} (\sin\varphi + x_2 \cos\varphi) \ =$$
$$= \ \frac{1}{4}\left((\sin\varphi + \cos\varphi) + (\sin\varphi - \cos\varphi)\right) \ = \ \frac{\sin\varphi}{2}.$$



On the other hand, $\langle \nabla_S F_{\mu_n}(\theta), \eta\rangle \leq |\nabla_S F_{\mu_n}(\theta)||\eta| = |\nabla_S F_{\mu_n}(\theta)|$ (in fact $\eta \| \nabla_S F_{\mu_n}(\theta)$). Hence by Lemma 3.1.4,

$$\|F_{\mu_n}\|_{\text{Lip}} \ \geq \ |\nabla_S F_{\mu_n}(\theta)| \ = \ \frac{\sin\varphi}{2} \qquad \forall \ \varphi \in (0, \pi/4).$$

□

We furthermore note that applying a similar argument to the continuous cube $[-1,1]^n$ (that is, the uniform measure on it) yields a similar result. This may be considered as an absolutely continuous example in which the Lipschitz constant is bounded away from zero over all dimensions.

3.2 Tiny subsets of the discrete cube

In this section we prove:

Proposition 3.2.1. *Assume $n \geq 8$ and let $N = n^{2+\alpha}$, $\alpha > 0$. Let $X_1, ..., X_N \sim$ Unif$(\{-1,1\}^n)$ be independent. Denote by μ the probability measure evenly distributed among the atoms $X_1, ..., X_N$ (with repetitions), i.e.*

$$\mu \; = \; \frac{1}{N} \sum_{k=1}^{N} \delta_{X_k}.$$

Then with probability at least $1 - \gamma(n, \alpha)$,

$$Var(F_\mu) \; \leq \; \frac{C}{n^2}.$$

Here, $C > 0$ is an universal constant and

$$\gamma(n, \alpha) \; = \; n(n+1)e^{-n^\alpha} \; + \; e^{-n^{1+\alpha/2}/16e}.$$

In particular, when $\alpha > 0$ is fixed we have $\gamma(n, \alpha) \to 0$ as $n \to \infty$.

As we are dealing with random subsets of $\{-1,1\}^n$, we cannot expect the measure μ to have, for example, center of mass *exactly* at the origin, as required by Proposition 2.1.2. We will therefore need an alternative version of Proposition 2.1.2, one that will allow for the measure μ to satisfy the required conditions just approximately.

Proposition 3.2.2. *Let μ be a Borel probability measure on $\mathbb{R}^n$. Assume that for some $\beta, \gamma, \delta, \varepsilon \in (0, \infty)$*

(*i*) $$\Big| \int_{\mathbb{R}^n} x \; d\mu \Big| \; \leq \; \frac{\varepsilon}{\sqrt{n}}.$$

(ii) $$\left| \|M_\mu\|_{HS}^2 - \frac{1}{n}(\mathit{Tr}M_\mu)^2 \right| \leq \frac{\delta}{n} \qquad \textit{and} \qquad \mathit{Tr}M_\mu \leq \gamma\sqrt{n}$$

$$\textit{for} \quad M_\mu = \int_{\mathbb{R}^n} x \otimes x \ \frac{d\mu(x)}{|x|}.$$

(iii) $$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle x,y\rangle^4}{|x|^3|y|^3} \, d\mu(x)\, d\mu(y) \ \leq \ \frac{\beta}{n}.$$

Then, $$\mathit{Var}(F_\mu) \ \leq \ \frac{C(\varepsilon^2+\delta+\beta+\gamma^2)}{n^2}$$

where $C > 0$ is a universal constant.

Condition (i) requires the center of mass to be sufficiently close to the origin. To understand the connection between condition (ii) and condition (1) in Proposition 2.1.2, see Appendix 2.5.5. Condition (iii) is just (2).

Careful reading will reveal that the direct proof of Proposition 3.2.2 (Section 2.4) allows for the relaxed conditions above as it is.

Proof of Proposition 3.2.2. Repeat the argument in the direct proof of Proposition 3.2.2 (Section 2.4). In the end we get:

$$\begin{aligned}\mathrm{Var}(F_\mu) \ = \ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Bigg(& \frac{\langle x,y\rangle}{4n} \ + \ \frac{\langle x,y\rangle^2}{4\pi n|x||y|} \ - \ \frac{|x||y|}{4\pi n^2} \\ & + \ c\frac{\langle x,y\rangle^4}{n|x|^3|y|^3} \ + \ d\frac{|x||y|}{n^3} \Bigg) \, d\mu(x)\, d\mu(y)\end{aligned}$$

where $|c|, |d|$ are bounded by a universal constant.

The first integral equals $\left|\int_{\mathbb{R}^n} x \ d\mu\right|^2/4n \leq \varepsilon^2/4n^2$, due to assumption (i).
The second integral is $\|M_\mu\|_{\mathrm{HS}}^2/4\pi n$ and the third is $(\mathrm{Tr}M_\mu)^2/4\pi n$ (for a proof see Appendix 2.5.5). By assumption (ii) their difference is at most $\delta/4\pi n^2$.
The fourth integral is at most β/n^2 times a universal constant, by assumption (iii).
The fifth integral is $(\mathrm{Tr}M_\mu)^2/n^3 = \gamma^2/n^2$ times a universal constant, by assumption (ii). □

In the proof of Proposition 3.2.1 we will be dealing with sums of random signs. It is well known (see, e.g. [13]) that an average of independent random signs admits a tight Gaussian concentration:

$$\mathbb{P}\left[\frac{1}{N}\left|\sum_{k=1}^{N}\varepsilon_k\right| > t\right] \;\leq\; 2e^{-Nt^2} \tag{3.1}$$

for any $t > 0$, where $\varepsilon_1, ..., \varepsilon_N \sim \mathrm{Unif}\left(\{-1, 1\}\right)$ are independent.

The proof of Proposition 3.2.1 will come as a consequence of the succeeding discussion. We adopt the following setting; assume $X_1, ..., X_N \sim \mathrm{Unif}\left(\{-1, 1\}^n\right)$ are independent, $N \in \mathbb{N}$. Define

$$\mu \;=\; \frac{1}{N}\sum_{k=1}^{N}\delta_{X_k}$$

the discrete measure whose mass distributes evenly among the atoms $X_1, ..., X_N$. It is important to note that the coordinates of $X_1, ..., X_N$ constitute $N \cdot n$ independent random signs:

$$X_{ki} \sim \mathrm{Unif}\left(\{-1, 1\}\right) \qquad ; \qquad k = 1, ..., N,\; i = 1, ..., n.$$

We estimate the probability with which μ meets conditions (i),(ii),(iii).

Condition (i).

Claim 3.2.3. *With probability at least* $1 - 2ne^{-N/n^2}$,

$$\Big|\int_{\mathbb{R}^n} x \; d\mu\Big| \leq \frac{1}{\sqrt{n}}.$$

Proof. For $t > 0$,

$$\begin{aligned}
\mathbb{P}\Big[\Big|\int_{\mathbb{R}^n} x\ d\mu\Big| > t\Big] &\le \mathbb{P}\Big[\exists i : \Big|\int_{\mathbb{R}^n} x_i\ d\mu\Big| > t/\sqrt{n}\Big] \\
&\le \sum_{i=1}^{n} \mathbb{P}\Big[\Big|\int_{\mathbb{R}^n} x_i\ d\mu\Big| > t/\sqrt{n}\Big] \\
&= \sum_{i=1}^{n} \mathbb{P}\Big[\frac{1}{N}\Big|\sum_{k=1}^{N} X_{ki}\Big| > t/\sqrt{n}\Big]
\end{aligned}$$

where for each i, $X_{1i}, ..., X_{Ni}$ are independent random signs. Thus from inequality (3.1) we have for all $t > 0$:

$$\mathbb{P}\Big[\Big|\int_{\mathbb{R}^n} x\ d\mu\Big| > t\Big] \le 2ne^{-Nt^2/n}.$$

□

Condition (ii).
Write $M = \int_{\mathbb{R}^n} x \otimes x\ \frac{d\mu}{|x|}$. Notice that the diagonal elements of M are all equal $\frac{1}{\sqrt{n}}$;

$$M_{ii} = \int_{\mathbb{R}^n} x_i^2\ \frac{d\mu}{|x|} = \frac{1}{\sqrt{n}}\frac{1}{N}\sum_{k=1}^{N} X_{ki}^2 = \frac{1}{\sqrt{n}}.$$

Thus assumption (ii) boils down to the sum of the off-diagonal elements of M squared being small enough. By the symmetry of M it suffices to bound the lower triangular half, that is, to bound

$$\sum_{1\le i<j\le n} M_{ij}^2 = \sum_{1\le i<j\le n} \left(\frac{1}{N}\sum_{k=1}^{N} \frac{X_{ki}X_{kj}}{\sqrt{n}}\right)^2.$$

Claim 3.2.4. *With probability at least* $1 - n(n-1)e^{-N/n(n-1)}$,

$$\Big|\ \|M\|_{HS}^2 - \frac{1}{n}(TrM)^2\Big| \le \frac{1}{n}.$$

Proof. For $t > 0$,

$$\begin{aligned}
\mathbb{P}\Big[\sum_{1\le i<j\le n} \Big(\frac{1}{N} \sum_{k=1}^{N} \frac{X_{ki}X_{kj}}{\sqrt{n}} \Big)^2 > t \Big] & \\
& \le \ \mathbb{P}\left[\exists i < j : \ \left| \frac{1}{N} \sum_{k=1}^{N} \frac{X_{ki}X_{kj}}{\sqrt{n}} \right| > \sqrt{\frac{2t}{n(n-1)}} \ \right] \\
& \le \sum_{1\le i<j\le n} \mathbb{P}\left[\ \left| \frac{1}{N} \sum_{k=1}^{N} X_{ki}X_{kj} \right| > \sqrt{\frac{2t}{n-1}} \ \right] \\
& \le \ n(n-1)e^{-2Nt/(n-1)}
\end{aligned}$$

by inequality (3.1) and the fact that $Y_k = X_{ki}X_{kj}$, $k = 1, ..., N$ are a set of N independent random signs, for any i, j such that $i < j$. □

Condition (iii).

Observe that assumption (iii) would be met if we had:

$$\int_{\mathbb{R}^n} \langle x, a\rangle^4 \ d\mu(x) \ \le \ \beta|a|^4,$$

at least for any $a \in \{-1, 1\}^n$; Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle x, y\rangle^4}{|x|^3|y|^3} \ d\mu(x) \ d\mu(y) \ &= \ \frac{1}{n^3}\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \langle x, y\rangle^4 \ d\mu(x) \right) d\mu(y) \\
&\le \ \frac{1}{n^3}\int_{\mathbb{R}^n} \beta|y|^4 \ d\mu(y) \ = \ \frac{\beta}{n}.
\end{aligned}$$

In other words, it suffices to show

$$\frac{1}{N}\sum_{k=1}^{N} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_i X_{ki} \right)^4 \ \le \ \beta$$

for any $a \in \{-1, 1\}^n$. Notice that switching a with any other $\xi \in \{-1, 1\}^n$ does not change the problem; we have $a_i X_{ki} \sim \mathrm{Unif}\left(\{-1, 1\}^n\right)$ regardless of the value of $a_i \in \{-1, 1\}$. Thus it suffices to take $a = (1, ..., 1)$. Our proof will be based on the following theorem, proven in [14].

Theorem 3.2.5 (Schmuckenschläeger). *Let $0 < \alpha < 1$ and let $Y_1, ..., Y_N$ be non-negative i.i.d. random variables with $A = \mathbb{E}e^{Y^\alpha} < \infty$. Then for*

$$t \geq \sqrt{A}\; 2^{\frac{2}{\alpha}+1}\; \Gamma\big(1 + 1/\alpha\big) \quad ; \quad N \geq \frac{4}{A}\Big(\alpha^{\frac{1}{\alpha}}(1-\alpha)\Gamma\big(1+1/\alpha\big)\Big)^{-2}$$

we have

$$\mathbb{P}\left[\frac{1}{N}\sum_{k=1}^{N} Y_k > t\right] \;\leq\; \exp\left[-\left(\frac{1-\alpha}{2^{\frac{1}{\alpha}+2}}\; Nt\right)^{\alpha}\right].$$

Claim 3.2.6. *If $N \geq 2^6$ then with probability at least $1 - e^{-\sqrt{N}/16e}$,*

$$\frac{1}{N}\sum_{k=1}^{N}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_{ki}\right)^4 \;\leq\; 2^7.$$

Proof.

Define
$$Y_k \;=\; \frac{1}{n^2}\left(\sum_{i=1}^{n} X_{ki}\right)^4 \qquad ; \quad k = 1, ..., N.$$

Abbreviate $Y = Y_1 = \frac{1}{n^2}\left(\sum_{i=1}^{n}\varepsilon_i\right)^4$, $\varepsilon = X_1$. First we bound the moments of Y, using Khinchine's inequality (Theorem 3.1.2). For $p \geq 1$,

$$\mathbb{E}[Y^p]^{\frac{1}{p}} \;=\; \frac{1}{n^2}\left(\mathbb{E}\,|\langle(1,...,1),\varepsilon\rangle|^{4p}\right)^{\frac{1}{4p}\cdot 4} \;\leq\; B_{4p}^4.$$

The best constants B_p are known [9]. For us it is enough to know that $B_p \leq \sqrt{8p}$ when $p \geq 1$ (see Appendix 3.3.2). We shall apply Theorem 3.2.5 to $Z = \frac{Y}{2^{10}e^2}$, $\alpha = \frac{1}{2}$. We use the moments to bound $\mathbb{E}e^{\sqrt{Z}}$:

$$\begin{aligned}
\mathbb{E}\; e^{\sqrt{Z}} \;&=\; \mathbb{E}\; e^{\frac{\sqrt{Y}}{2^5 e}} \;=\; \mathbb{E}\left[\sum_{p=0}^{\infty}\frac{(2^5e)^{-p}}{p!}Y^{\frac{p}{2}}\right] \;=\; \sum_{p=0}^{\infty}\frac{(2^5e)^{-p}}{p!}\mathbb{E}\left[Y^{\frac{p}{2}}\right] \;\leq \\
&\leq\; \sum_{p=0}^{\infty}\frac{(2^5e)^{-p}}{p!}B_{2p}^{2p} \;\leq\; \sum_{p=0}^{\infty}\frac{(2^5e)^{-p}}{p!}2^{4p}p^p \;=\; \sum_{p=0}^{\infty}2^{-p}\frac{p^p}{e^p p!} \;\leq \\
&\leq\; \sum_{p=0}^{\infty}2^{-p} \;=\; \frac{1}{1-\frac{1}{2}} \;=\; 2
\end{aligned}$$

because $p^p \le e^p p!$ (Stirling's approximation). On the other hand, $\mathbb{E}e^{\sqrt{Z}} \ge \mathbb{E}e^0 = 1$. We may thus apply Theorem 3.2.5 to $Z = \frac{Y}{2^{10}e^2}$, $\alpha = \frac{1}{2}$ to obtain, for

$$N \ \ge \ 2^6 \quad ; \quad t \ \ge \ 2^{6+\frac{1}{2}}$$

that

$$\mathbb{P}\left[\frac{1}{N}\sum_{k=1}^{N} Y_k > t\right] \ \le \ e^{-\sqrt{Nt/2^{15}e^2}}.$$

□

3.3 Appendix

3.3.1

Proof of Lemma 3.1.4. The spherical Taylor expansion states that:

$$f(\theta') - f(\theta) \ = \ \langle \nabla_S f(\theta), \theta' - \theta \rangle \ + \ o(|\theta' - \theta|)$$

as $\theta' \to \theta$, $\theta' \in S^{n-1}$. Write $v = \nabla_S f(\theta)/|\nabla_S f(\theta)| \in S^{n-1}$. For $\varepsilon > 0$ take $\theta_\varepsilon = \frac{\theta+\varepsilon v}{|\theta+\varepsilon v|} = \frac{1}{\sqrt{1+\varepsilon^2}}(\theta + \varepsilon v) \in S^{n-1}$. Then

$$f(\theta_\varepsilon) - f(\theta) \ = \ \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}|v| \ + \ o(\varepsilon).$$

Dividing both sides by ε, the right hand side tends to $|v| = |\nabla_S f(\theta)|$ as $\varepsilon \to 0$, as required. □

3.3.2

Haagerup [9] found the best constants in the Khinchine inequality $B_p = \sqrt{2}\left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\right)^{\frac{1}{p}}$. We state the following as an independent claim, for future reference.

Claim 3.3.1. *$\Gamma\left(\frac{p+1}{2}\right)^{1/p} \le \ 2\sqrt{p}$ for $p \ge 1$.*

Proof. We use the well known upper bound for the gamma function (see, e.g. [1]):

$$\Gamma(x) \ \le \ \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}e^{\frac{1}{12x}} \quad ; \quad x > 0.$$

We have,

$$\Gamma\left(\frac{p+1}{2}\right)^{\frac{1}{p}} \;\leq\; \left(\sqrt{2\pi}\left(\frac{p+1}{2}\right)^{\frac{p}{2}} e^{-\frac{p+1}{2}} e^{\frac{1}{6(p+1)}}\right)^{\frac{1}{p}} \;=$$
$$=\; 2^{\frac{1}{2p}} \cdot \pi^{\frac{1}{2p}} \cdot \sqrt{p+1} \cdot \frac{1}{\sqrt{2}} \cdot e^{-\frac{1}{2}-\frac{1}{2p}} \cdot e^{\frac{1}{6p(p+1)}}$$

As $p \geq 1$ this is

$$\leq\; \sqrt{2} \cdot \sqrt{\pi} \cdot \sqrt{p+p} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{e}} \cdot 1 \cdot e^{\frac{1}{12}} \;=\; \sqrt{2\pi} e^{-5/12} \sqrt{p} \;\leq\; 2\sqrt{p}.$$

□

Thus,

$$B_p \;=\; \sqrt{2}\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}\right)^{\frac{1}{p}} \;\leq\; \frac{\sqrt{2}}{\pi^{\frac{1}{2p}}} 2\sqrt{p} \;\leq\; \sqrt{8p}.$$

Chapter 4

The Log-Concave Case

4.1 Introduction

In this chapter we study the normalization (1) from Proposition 2.1.2, in a more general form.

Definition 4.1.1. Let μ be a centered Borel probability measure on $\mathbb{R}^n$, $p > 0$. Define the *L^p-covariance matrix* of μ by

$$\mathrm{Cov}_p(\mu) \ := \int_{\mathbb{R}^n} x \otimes x \ |x|^{p-2} \ d\mu(x).$$

Definition 4.1.2. Let μ be a centered Borel probability measure on $\mathbb{R}^n$, $p > 0$ such that $Z_{p,\mu} := \int_{\mathbb{R}^n} |x|^{p-2} \ d\mu(x)$ is finite and nonzero. We say that μ is *L^p-isotropic* if $\mathrm{Cov}_p(\mu) = Z_{p,\mu} Id$.
That is, if

$$\int_{\mathbb{R}^n} x \otimes x \ \frac{|x|^{p-2}}{Z_{p,\mu}} \ d\mu(x) \ = \ Id.$$

We see that $p = 1$ corresponds to our normalization (1) from Proposition 2.1.2 with $\alpha = Z_{1,\mu}$:

$$\mathrm{Cov}_1(\mu) \ = \int_{\mathbb{R}^n} x \otimes x \ \frac{d\mu}{|x|} \ = \ Z_{1,\mu} Id.$$

Moreover, $\mathrm{Cov}_2(\mu)$ is the covariance matrix of μ and the L^2-isotropic condition is the *isotropic* condition:

$$\mathrm{Cov}(\mu) \;=\; \int_{\mathbb{R}^n} x \otimes x \; d\mu \;=\; Id$$

in addition to μ being centered.

We will present conditions for the existence and uniqueness of an L^p-isotropic position of a given probability measure. Let us introduce the notion of a position of a measure.

Positions of a measure. Assume $\mathbb{R}^n$ is endowed with a Borel probability measure μ. Any measurable transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ induces a (new) Borel probability measure $T_*\mu$ on $\mathbb{R}^n$, defined by

$$T_*\mu(A) \;=\; \mu\left(T^{-1}\left(A\right)\right) \qquad ; \quad A \subseteq \mathbb{R}^n \text{ Borel.}$$

(We sometimes allow $T : \mathbb{R}^n \to \mathbb{R}^m$). $T_*\mu$ is called the *push-forward measure* or *image* of μ under T. A special case is when T is an invertible linear transformation, i.e. $T \in GL_n$. In this case we call $T_*\mu$ a *position* (or a *linear position*) of μ. For example, it is known (see, e.g. [4]) that any non-degenerate centered measure on $\mathbb{R}^n$ has an isotropic position.

We note two useful properties of push-forward measures. First, $(TS)_*\mu = T_*S_*\mu$. Second is the change of variable formula:

$$\int_{\mathbb{R}^n} f(x) \; dT_*\mu \;=\; \int_{\mathbb{R}^n} f(Tx) \; d\mu$$

for any measurable $f : \mathbb{R}^n \to \mathbb{R}$.

As a consequence of the investigation of the L^p-isotropic position, we will obtain the main result of this chapter.

Proposition 4.1.3. *Let μ be an L^1-isotropic log-concave probability measure on $\mathbb{R}^n$, $n \geq \tilde{C}$. Then*

$$\frac{c}{\sqrt{n}} \;\leq\; Z_{1,\mu} \;\leq\; \frac{C}{\sqrt{n}} \quad \text{and} \quad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle x,y\rangle^4}{|x|^3|y|^3} \; d\mu(x) \; d\mu(y) \;\leq\; \frac{C'}{n}.$$

Here, $c, C, C', \tilde{C} > 0$ are universal constants.

Corollary 4.1.4. *For any centered Leb-a.c. log-concave probability measure μ on $\mathbb{R}^n$, $n \geq C'$, there is a position in which μ is L^1-isotropic and satisfies*

$$Var(F_\mu) \ \leq \ \frac{C}{n^2}.$$

Here, $C, C' > 0$ are universal constants.

To prove Proposition 4.1.3 we will require an additional property of the L^p-isotropic position, which is specific to log-concave measures; That it is "not far" from the isotropic position, in some sense of proximity that will be explained in due course. We conclude the introduction with a brief review of log-concave measures.

Log-concave measures.

Definition 4.1.5. A Borel measure μ on $\mathbb{R}^n$ is said to be *log-concave* if

$$\mu(\lambda A + (1-\lambda)B) \ \geq \ \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for any compact sets $A, B \subseteq \mathbb{R}^n$ and any $\lambda \in [0,1]$.

Here, $A+B = \{a+b : \ a \in A, \ b \in B\}$ is the Minkowski sum. Any log-concave measure not supported on any hyperplane has a density with respect to the Lebesgue measure, and that density is a log-concave function.

Definition 4.1.6. A function $f : \mathbb{R}^n \to [0, \infty)$ is said to be *log-concave* if $\log f$ is concave. Equivalently, if

$$f(\lambda x + (1-\lambda)y) \ \geq \ f(x)^\lambda f(y)^{1-\lambda}$$

for any $x, y \in \mathbb{R}^n$ and any $\lambda \in [0,1]$.

We thus restrict our future discussion of log-concave measures to measures with a density, as any log-concave measure which is not absolutely continuous has a density on some lower-dimensional subspace.

To complement the statement from before, any log-concave function is the density of a log-concave measure.

The significance of log-concave measures is due to them naturally appearing in various mathematical subjects, as well as the fact that many questions regarding convex bodies can be equivalently formulated in the language of log-concave measures. We recommend [4] for a thorough presentation of log-concave measures, as well as reference for the properties of log-concave measures we mentioned above.

4.2 Existence of position

Proposition 4.2.1. *Let μ be a Borel probability measure on $\mathbb{R}^n$, $p > 0$. Assume that both $\int_{\mathbb{R}^n} |x|^{p-2}\, d\mu$ and $\int_{\mathbb{R}^n} |x|^p\, d\mu$ are finite and nonzero and that the support of μ is not contained in any hyperplane. Then there exists an L^p-isotropic position of μ.*

We start the discussion toward the proof of Proposition 4.2.1 with an immediate observation; when the L^p-covariance matrix is scalar we can, by dilation, obtain an L^p-isotropic position.

Lemma 4.2.2. *Let μ be a centered Borel probability measure on $\mathbb{R}^n$, $p > 0$, such that $Z_{p,\mu}$ finite and nonzero. Assume that $Cov_p(\mu)$ is scalar. Then $T_*\mu$ is L^p-isotropic for*

$$T \;=\; \sqrt{\frac{nZ_{p,\mu}}{TrCov_p(\mu)}} \cdot Id \;\in\; GL_n.$$

Proof. Denote by $\kappa > 0$ the diagonal value in $\mathrm{Cov}_p(\mu)$ and note that $\kappa = \frac{1}{n}\mathrm{TrCov}_p(\mu)$. Write $a = \sqrt{Z_{p,\mu}/\kappa}$ and take $T = aId$. The following calculation we state as an independent claim, for future reference.

Claim 4.2.3. *Let $T = aId$, $a > 0$. Then $Z_{p,T_*\mu} \;=\; a^{p-2}Z_{p,\mu}$.*

Proof.

$$Z_{p,T_*\mu} \;=\; \int_{\mathbb{R}^n} |x|^{p-2}\; dT_*\mu \;=\; \int_{\mathbb{R}^n} |Tx|^{p-2}\; d\mu \;=\; a^{p-2}Z_{p,\mu}.$$

□

We may calculate:

$$\frac{\mathrm{Cov}_p(T_*\mu)}{Z_{p,T_*\mu}} \;=\; \int_{\mathbb{R}^n} x\otimes x\;\frac{|x|^{p-2}}{Z_{p,T_*\mu}}\;dT_*\mu \;=\; \int_{\mathbb{R}^n} (Tx)\otimes(Tx)\;\frac{|Tx|^{p-2}}{Z_{p,T_*\mu}}\;d\mu$$

$$=\; \int_{\mathbb{R}^n} a^2\; x\otimes x\;\frac{a^{p-2}\,|x|^{p-2}}{a^{p-2}Z_{p,\mu}}\;d\mu \;=\; \frac{a^2}{Z_{p,\mu}}\mathrm{Cov}_p(\mu) \;=\; \frac{a^2}{Z_{p,\mu}}\cdot\kappa Id \;=\; Id.$$

□

Thus proving the existence of an L^p-isotropic position boils down to finding a position in which $\mathrm{Cov}_p(\mu)$ is scalar. Positions such as these have been known to arise from extremal problems (see [7]). In our case, the position arises as a solution to the minimization problem of finding

$$\arg\min_{S\in SL_n}\int_{\mathbb{R}^n}|Sx|^p\;d\mu.$$

Lemma 4.2.4. *Let μ be a Borel probability measure on $\mathbb{R}^n$, $p>0$, such that $\int_{\mathbb{R}^n}|x|^p\,d\mu$ is finite and nonzero. Assume that for any $S\in SL_n$ we have*

$$\int_{\mathbb{R}^n}|x|^p\;\;d\mu \;\le\; \int_{\mathbb{R}^n}|Sx|^p\;\;d\mu.$$

Then $Cov_p(\mu)$ is scalar.

The case $p=2$ is well-known. That is, the isotropic position arises from the problem of minimizing $\int_{\mathbb{R}^n}|Sx|^2\;d\mu$ over all $S\in SL_n$. The proof of Lemma 4.2.4 is a straightforward generalization of a well-known variational argument for the isotropic case (see [7]), and is given in Appendix 4.6.1.

Continuing, proving the existence of an L^p-isotropic position reduces to showing the existence of a minimum of $\int_{\mathbb{R}^n}|Sx|^p\,d\mu$ over all $S\in SL_n$.

Lemma 4.2.5. *Let μ be a Borel probability measure on $\mathbb{R}^n$, $p>0$, such that $\int_{\mathbb{R}^n}|x|^p\,d\mu$ is finite and the support of μ is not contained in any hyperplane. Then the infimum*

$$\inf_{S\in SL_n}\int_{\mathbb{R}^n}|Sx|^p\;\;d\mu$$

is attained by some $S\in SL_n$.

Proof.

Step 1. $$\inf_{\theta \in S^{n-1}} \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p \, d\mu(x) \; > \; 0.$$

Proof. Define $f : S^{n-1} \to [0, \infty)$ by

$$f(\theta) \; = \; \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p \, d\mu(x).$$

The integrand $x \mapsto |\langle x, \theta \rangle|^p$ is dominated by the integrable p'th moment function $x \mapsto |x|^p$, therefore f is continuous by the dominated convergence theorem. Now, S^{n-1} is compact hence the infimum of f is attained by some $\theta \in S^{n-1}$. It is left to see that $f(\theta) > 0$ as otherwise μ would be supported on the hyperplane $\theta^{\perp}$. □

Step 2. There exists $M_{\mu,p} > 0$ such that for any $T \in GL(n)$,

$$\|T\|_{\text{op}} \; \leq \; M_{\mu,p} \left(\int_{\mathbb{R}^n} |Tx|^p \, d\mu(x) \right)^{1/p}.$$

Proof. Let $T \in GL(n)$, and let $v_1, ..., v_n \in \mathbb{R}^n$ be such that $(Tx)_i = \langle v_i, x \rangle$ for $x \in \mathbb{R}^n$. Denote by i_0 an index for which $v_{i_0} = \max_i |v_i|$ and observe that

$$\|T\|_{\text{op}}^2 \; = \; \sup_{\theta \in S^{n-1}} |T\theta|^2 \; = \; \sup_{\theta \in S^{n-1}} \sum_{i=1}^{n} \langle v_i, \theta \rangle^2 \; \leq \; \sum_{i=1}^{n} |v_i|^2 \; \leq \; n |v_{i_0}|^2.$$

On the other hand, $|Tx|^2 \; = \; \sum_i \langle v_i, x \rangle^2 \; \geq \; \langle v_{i_0}, x \rangle^2$. Write $\delta_{\mu,p}$ for the positive infimum obtained in Step 1. We thus get:

$$\int_{\mathbb{R}^n} |Tx|^p \, d\mu(x) \; \geq \; \int_{\mathbb{R}^n} |\langle v_{i_0}, x \rangle|^p \, d\mu(x) \; \geq \; \delta_{\mu,p} |v_{i_0}|^p \; \geq \; n^{-p/2} \delta_{\mu,p} \, \|T\|_{\text{op}}^p \, .$$

□

Step 3. The infimum $\inf \int_{\mathbb{R}^n} |Sx|^p \, d\mu$ over all $S \in SL_n$ is attained by some $S \in SL_n$.

Proof. It suffices to take the infimum over the set

$$A \; = \; \left\{ S \in SL(n) : \; \int_{\mathbb{R}^n} |Sx|^p \, d\mu \leq R \right\}$$

for $R = \int_{\mathbb{R}^n} |x|^p \, d\mu$. The set A is closed as a subset of $M_n \simeq \mathbb{R}^{n^2}$, and also bounded in the operator norm; Indeed, for any $S \in A$ we have

$$\|S\|_{\text{op}} \;\leq\; M_{\mu,p} \left(\int_{\mathbb{R}^n} |Sx|^p \; d\mu \right)^{\frac{1}{p}} \;\leq\; M_{\mu,p} R^{1/p}$$

where $M_{\mu,p}$ is the constant from Step 2. Thus A is compact.
It remains to be convinced that the function $S \mapsto \int_{\mathbb{R}^n} |Sx|^p \; d\mu$ is continuous; Observe that $|Sx|^p \;\leq \|S\|_{\text{op}}^p \, |x|^p \leq M_{\mu,p}^p R |x|^p$ which is an integrable function, and so continuity follows from the dominated convergence theorem. □

We have thus completed the proof of Lemma 4.2.5. □

Proof of Proposition 4.2.1. By Lemma 4.2.5, let $S \in SL_n$ be such that

$$\int_{\mathbb{R}^n} |Sx|^p \, d\mu \;\leq\; \int_{\mathbb{R}^n} |\tilde{S}x|^p d\mu$$

for any $\tilde{S} \in SL_n$. By Lemma 4.2.4, $\text{Cov}_p(S_*\mu)$ is scalar as

$$\int_{\mathbb{R}^n} |Sx|^p \, d\mu \;=\; \int_{\mathbb{R}^n} |x|^p \, dS_*\mu.$$

Note that $S_*\mu$ is still centered. Thus by Lemma 4.2.2, $T_* S_* \mu = (TS)_* \mu$ is L^p-isotropic for some $T \in GL_n$. □

4.3 Uniqueness of position

Proposition 4.3.1. *Let μ be a centered Borel probability measure on $\mathbb{R}^n$, $0 < p < 4$. If an L^p-isotropic position exists for μ then it is unique up to O_n.*

Note the added assumption; $p < 4$. By unique up to O_n we mean that if μ is L^p-isotropic then $U_*\mu$ is L^p-isotropic for any $U \in O_n$ and that no other $T_*\mu$ is L^p-isotropic, $T \in GL_n$.

Before we go about proving Proposition 4.3.1, we emphasize an important property of the L^p-isotropic position implied by uniqueness.

Corollary 4.3.2. *Let μ be a centered Borel probability measure on $\mathbb{R}^n$, $0 < p < 4$. Assume that μ is L^p-isotropic . Then,*

$$\int_{\mathbb{R}^n} |x|^p \, d\mu \ \leq \ \int_{\mathbb{R}^n} |Sx|^p \, d\mu$$

for any $S \in SL_n$.

Proof. Let μ be L^p-isotropic. We recount the steps of the proof of the existence of the L^p-isotropic position (Proposition 4.2.1), to get another L^p-isotropic position; By Lemma 4.2.5 we may take $S \in SL_n$ such that

$$\int_{\mathbb{R}^n} |Sx|^p \ d\mu \ \leq \ \int_{\mathbb{R}^n} |\tilde{S}x|^p \ d\mu$$

for any $\tilde{S} \in SL_n$. By Lemma 4.2.4, $\mathrm{Cov}_p(S_*\mu)$ is scalar. By Lemma 4.2.2, there is a dilation $T = aS$ such that $T_*\mu$ is L^p-isotropic . Note that all the assumptions required for the lemmas we have used are implied by the definition of the L^p-isotropic position. Observe that $T_*\mu$ retains the minimization property of $S_*\mu$:

$$\int_{\mathbb{R}^n} |x|^p \ dT_*\mu \ \leq \ \int_{\mathbb{R}^n} |\tilde{S}x|^p \ dT_*\mu$$

for any $\tilde{S} \in SL_n$. Indeed, let $\tilde{S} \in SL_n$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} |\tilde{S}x|^p \ dT_*\mu \ &= \ \int_{\mathbb{R}^n} |\tilde{S}Tx|^p \ d\mu \ = \ a^p \int_{\mathbb{R}^n} |\tilde{S}Sx|^p \ d\mu \ \geq \\ &\geq \ a^p \int_{\mathbb{R}^n} |Sx|^p \ d\mu \ = \ \int_{\mathbb{R}^n} |Tx|^p \ d\mu \ = \ \int_{\mathbb{R}^n} |x|^p \ dT_*\mu. \end{aligned}$$

As both μ and $T_*\mu$ are L^p-isotropic, we have by uniqueness (Proposition 4.3.1) that $T \in O_n$. Finally, for a given $\tilde{S} \in SL_n$ define $S' = T\tilde{S}T^{-1} \in SL_n$ and we have:

$$\int |\tilde{S}x|^p \, d\mu \ = \ \int |T^{-1}S'Tx|^p \, d\mu \ = \ \int |S'x|^p \, dT_*\mu \ \geq \ \int |x|^p \, dT_*\mu \ = \ \int |x| \, d\mu.$$

□

We turn to prove Proposition 4.3.1. The next lemma shows that the L^p-isotropic property is preserved under orthogonal transformations.

Lemma 4.3.3. *Let μ be a Borel probability measure on $\mathbb{R}^n$, $p > 0$. Assume that μ is L^p-isotropic. Then $U_*\mu$ is L^p-isotropic for any $U \in O_n$.*

Proof. The following calculation we state as an independent claim, for future reference.

Claim 4.3.4. *If $U \in O_n$ and $Cov_p(\mu)$ is scalar then $Cov_p(U_*\mu) = Cov_p(\mu)$.*

Proof. Note that $x \otimes x = xx^t$ when $x \in \mathbb{R}^n$ is regarded as a column vector. Writing $\mathrm{Cov}_p(\mu) = \kappa Id$, we have

$$\mathrm{Cov}_p(U_*\mu) = \int_{\mathbb{R}^n} xx^t \; |x|^{p-2} \, dU_*\mu = \int_{\mathbb{R}^n} (Ux)(Ux)^t \; |Ux|^{p-2} \, d\mu =$$
$$= U\left(\int_{\mathbb{R}^n} xx^t \; |x|^{p-2} \; d\mu\right) U^t = U\kappa Id U^t = \kappa UU^t = \kappa Id.$$

□

By the claim, and since $U \in O_n$,

$$\mathrm{Cov}_p(U_*\mu) = \mathrm{Cov}_p(\mu) = Z_{p,\mu} Id = Z_{p,U_*\mu} Id.$$

□

We thus wish to prove the converse; That any two L^p-isotropic positions are orthogonal images of each other.

Lemma 4.3.5. *Let μ be a Borel probability measure on $\mathbb{R}^n$, $0 < p < 4$. Assume that both $Cov_p(\mu)$ and $Cov_p(S_*\mu)$ are scalar, $S \in SL_n$. Then $S \in O_n$.*

Proof. We assume that S is diagonal with positive eigenvalues and prove that in this case necessarily $S = Id$. This suffices, as by multiplying by an orthogonal transformation, we may assume that S is positive definite. We may then choose a basis for $\mathbb{R}^n$ in which $S = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ with $\lambda_1 \geq ... \geq \lambda_n > 0$. For a detailed proof of this reduction, see Appendix 4.6.2.

Write $\mathrm{Cov}_p(\mu) = \kappa Id$, $\mathrm{Cov}_p(S_*\mu) = \kappa_s Id$. First, assume that $0 < p \leq 2$. On the one hand,

$$\kappa_s = \int_{\mathbb{R}^n} x_1^2 \, |x|^{p-2} \; dS_*\mu(x) = \lambda_1^2 \int_{\mathbb{R}^n} x_1^2 \, |Sx|^{p-2} \; d\mu(x) \geq$$
$$\geq \lambda_1^2 \; \|S\|_{\mathrm{op}}^{p-2} \int_{\mathbb{R}^n} x_1^2 \, |x|^{p-2} \; d\mu(x) = \kappa \lambda_1^p$$

as $|Sx| \leq \|S\|_{\text{op}}\,|x|$ and $\|S\|_{\text{op}} = \lambda_1$ and $p-2 \leq 0$. On the other hand,

$$\begin{aligned}\kappa_s \;&=\; \int_{\mathbb{R}^n} x_n^2\; |x|^{p-2}\; dS_*\mu(x) \;=\; \lambda_n^2 \int_{\mathbb{R}^n} x_n^2\; |Sx|^{p-2}\; d\mu(x) \;\leq \\ &\leq\; \frac{\lambda_n^2}{\|S^{-1}\|_{\text{op}}^{p-2}} \int_{\mathbb{R}^n} x_n^2\; |x|^{p-2}\; d\mu(x) \;=\; \kappa\lambda_n^p\end{aligned}$$

as $|Sx| \geq |x| \,/\, \|S^{-1}\|_{\text{op}}$ and $\|S^{-1}\|_{\text{op}} = \lambda_n^{-1}$. We got that $\lambda_1 \leq \lambda_n$, therefore S is scalar. Since $S \in SL(n)$ we see that necessarily $S = Id$.
Now assume that $2 < p < 4$. Since $p > 2$, on the one hand

$$\begin{aligned}\kappa_s \;&=\; \int_{\mathbb{R}^n} x_n^2\; |x|^{p-2}\; dS_*\mu(x) \;=\; \lambda_n^2 \int_{\mathbb{R}^n} x_n^2\; |Sx|^{p-2}\; d\mu(x) \;\leq \\ &\leq\; \lambda_n^2\;\; \|S\|_{\text{op}}^{p-2} \int_{\mathbb{R}^n} x_n^2\; |x|^{p-2}\; d\mu(x) \;=\; \kappa\lambda_n^2\lambda_1^{p-2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\kappa_s \;&=\; \int_{\mathbb{R}^n} x_1^2\; |x|^{p-2}\; dS_*\mu(x) \;=\; \lambda_1^2 \int_{\mathbb{R}^n} x_1^2\; |Sx|^{p-2}\; d\mu(x) \;\geq \\ &\geq\; \frac{\lambda_1^2}{\|S^{-1}\|_{\text{op}}^{p-2}} \int_{\mathbb{R}^n} x_1^2\; |x|^{p-2}\; d\mu(x) \;=\; \kappa\lambda_1^2\lambda_n^{p-2}.\end{aligned}$$

We got $\lambda_1^2\lambda_n^{p-2} \leq \lambda_n^2\lambda_1^{p-2}$. Since $p < 4$, this implies $\lambda_1 \leq \lambda_n$ and so $S = Id$ again. □

We are now in position to prove Proposition 4.3.1.

Proof of Proposition 4.3.1. Assume that μ is L^p-isotropic, $0 < p < 4$. Lemma 4.3.3 shows that $U_*\mu$ is also L^p-isotropic, for any $U \in O_n$. Conversely, assume that $T_*\mu$ is L^p-isotropic, $T \in GL_n$. We wish to show that $T \in O_n$.

If $\det T < 0$, we may replace T with UT for $U \in O_n$ such that $\det(UT) > 0$; By Lemma 4.3.3, $(UT)_*\mu$ is still L^p-isotropic and showing $UT \in O_n$ implies $T \in O_n$. We may thus assume, without loss of generality, that $\det T > 0$.

Take $a > 0$ such that $S := aT \in SL_n$. S is a dilation of T and $\mathrm{Cov}_p(T_*\mu)$ is scalar, therefore $\mathrm{Cov}_p(S_*\mu)$ is scalar as well. Lemma 4.3.5 applied to μ and $S_*\mu$ implies that $S \in O_n$. It remains to show that $a = 1$. On the one hand,

$$Z_{p,T_*\mu} Id \;=\; \mathrm{Cov}_p(T_*\mu) \;=\; a^p \mathrm{Cov}_p(S_*\mu) \;=\; a^p \mathrm{Cov}_p(\mu) \;=\; a^p Z_{p,\mu} Id$$

as $S \in O_n$ and $\mathrm{Cov}_p(\mu)$ is scalar. Hence, $Z_{p,T_*\mu} = a^p Z_{p,\mu}$. On the other hand,

$$Z_{p,T_*\mu} \;=\; a^{p-2} \int_{\mathbb{R}^n} |x|^{p-2} \, dS_*\mu \;=\; a^{p-2} \int_{\mathbb{R}^n} |x|^{p-2} \, d\mu \;=\; a^{p-2} Z_{p,\mu}$$

as $S \in O_n$. Thus $a^2 = 1$, and recall $a > 0$. □

4.4 Proximity to the isotropic position

To prove Proposition 4.1.3, we will require another property of the L^p-isotropic position, stated as follows.

Proposition 4.4.1. *Let μ be a centered Leb-a.c. log-concave probability measure on $\mathbb{R}^n$, $n \geq C'$ and let $1 \leq p \leq 2$. Assume that μ is isotropic. If $T \in GL(n)$ is such that $T_*\mu$ is L^p-isotropic, then*

$$c \;\leq\; \left\| T^{-1} \right\|_{op}, \; \|T\|_{op} \;\leq\; C$$

where $c, C, C' > 0$ are universal constants.

Note the added assumption; $1 \leq p \leq 2$. The proof of Proposition 4.4.1 will rely heavily on the following well-known properties of log-concave measures.

Theorem 4.4.2 (Reverse Hölder inequality)**.** *Let μ be a Leb-a.c. log-concave probability measure on $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a semi-norm. Then for any $1 \leq p < q$, we have*

$$\left(\int_{\mathbb{R}^n} |f|^p \; d\mu \right)^{\frac{1}{p}} \;\leq\; \left(\int_{\mathbb{R}^n} |f|^q \; d\mu \right)^{\frac{1}{q}} \;\leq\; C \frac{q}{p} \left(\int_{\mathbb{R}^n} |f|^p \; d\mu \right)^{\frac{1}{p}}$$

where $C > 0$ is a universal constant.

Theorem 4.4.3 (Paouris)**.** *Let μ be a centered isotropic Leb-a.c. log-concave probability measure on $\mathbb{R}^n$. The for any integer $1 \leq k \leq c\sqrt{n}$ we have*

$$\left(\int_{\mathbb{R}^n} |x|^k \ d\mu\right)^{\frac{1}{k}} \simeq \left(\int_{\mathbb{R}^n} |x|^{-k} \ d\mu\right)^{-\frac{1}{k}}$$

where $c > 0$ is a universal constant.

Theorems 4.4.2 and 4.4.3 may be found in [4].

From here on, when we write $A \simeq B$ we mean that there are universal constants $c, C > 0$ such that $cA \leq B \leq CA$. Similarly, when we write $A \lesssim B$ we mean that there is a universal constant $C > 0$ such that $A \leq CB$.

Lemma 4.4.4. *In the setting of Proposition 4.4.1, $\|T^{-1}\|_{op} \leq C$.*

Proof. Without loss of generality we may assume that $\det T > 0$ (otherwise replace T with UT for some $U \in O_n$). Write $T = aS$ with $\det S = 1$, $a > 0$. Then $\mathrm{Cov}_p(S_*\mu)$ is scalar because $\mathrm{Cov}_p(T_*\mu)$ is. Denote by κ the value on the diagonal of $\mathrm{Cov}_p(S_*\mu)$, and note that $\kappa = \frac{1}{n}\mathrm{TrCov}_p(S_*\mu)$. In Lemma 4.2.2 we saw that it must be that $a = \sqrt{Z_{p,S_*\mu}/\kappa}$.

Claim 4.4.5. $Z_{p,T_*\mu} \gtrsim n^{\frac{p-2}{2}}$.

Proof. As $T_*\mu$ is L^p-isotropic, $\mathrm{Cov}_p(T_*\mu) = Z_{p,T_*\mu} Id$. Therefore,

$$Z_{p,T_*\mu} = \frac{1}{n}\mathrm{TrCov}_p(T_*\mu) = \frac{1}{n}\int_{\mathbb{R}^n} |x|^p \ dT_*\mu = \frac{a^p}{n}\int_{\mathbb{R}^n} |x|^p \, dS_*\mu =$$

$$= \frac{a^p}{n}\mathrm{TrCov}_p(S_*\mu) = a^p\kappa = Z_{p,S_*\mu}^{\frac{p}{2}}\kappa^{1-\frac{p}{2}}.$$

We now show that $Z_{p,S_*\mu} \gtrsim n^{\frac{p-2}{p}}\kappa^{\frac{p-2}{p}}$ and this would conclude the proof of the claim.

$$Z_{p,S_*\mu} = \int_{\mathbb{R}^n} |Sx|^{p-2} \ d\mu \overset{\circledast}{\geq} \left(\int_{\mathbb{R}^n} |Sx| \ d\mu\right)^{p-2} \geq$$

$$\overset{\circledast\circledast}{\geq} C^{2-p}p^{2-p}\left(\int_{\mathbb{R}^n} |Sx|^p \ d\mu\right)^{\frac{p-2}{p}} \overset{\circledast\circledast\circledast}{\geq} \min\{1, C\}\left(\mathrm{TrCov}_p(S_*\mu)\right)^{\frac{p-2}{p}} \geq$$

$$\geq \; C' n^{\frac{p-2}{p}} \kappa^{\frac{p-2}{p}}.$$

The transition ⊛ is due to Jensen's inequality (recall $p \leq 2$). The transition ⊛⊛ is due to the reverse-Hölder property (Theorem 4.4.2). In transition ⊛ ⊛ ⊛ we used the assumption $1 \leq p \leq 2$. □

Fix any $\theta \in S^{n-1}$. Since $T_*\mu$ is L^p-isotropic and by the Cauchy-Schwarz inequality,

$$1 = \int_{\mathbb{R}^n} \langle x, \theta \rangle^2 \; \frac{|x|^{p-2}}{Z_{p,T_*\mu}} \; dT_*\mu \leq \frac{1}{Z_{p,T_*\mu}} \sqrt{\int_{\mathbb{R}^n} \langle x, \theta \rangle^4 \; dT_*\mu} \sqrt{\int_{\mathbb{R}^n} |x|^{2(p-2)} \, dT_*\mu}.$$

We estimate the two integrals under the roots. By the reverse Hölder property (Proposition 4.4.2),

$$\sqrt{\int_{\mathbb{R}^n} \langle x, \theta \rangle^4 \; dT_*\mu} \; \leq \; 4C^2 \int_{\mathbb{R}^n} \langle x, \theta \rangle^2 \; dT_*\mu \; = \; 4C^2 \int_{\mathbb{R}^n} \langle x, T^*\theta \rangle^2 \; d\mu \; = \; 4C^2 \, |T^*\theta|^2$$

with the last transition due to μ being isotropic. As for the other square root,

$$\sqrt{\int_{\mathbb{R}^n} |x|^{2(p-2)} \; dT_*\mu} \; = \; \sqrt{\int_{\mathbb{R}^n} |Tx|^{2(p-2)} \; d\mu} \; \leq$$

$$\overset{⊛}{\leq} \; \left\|T^{-1}\right\|_{\text{op}}^{2-p} \sqrt{\int_{\mathbb{R}^n} |x|^{2(p-2)} \; d\mu} \; \overset{⊛⊛}{\leq} \; \left\|T^{-1}\right\|_{\text{op}}^{2-p} \left(\int_{\mathbb{R}^n} |x|^{-2} \; d\mu \right)^{1-\frac{p}{2}} \; \leq$$

$$\overset{⊛⊛⊛}{\leq} \; C' \left\|T^{-1}\right\|_{\text{op}}^{2-p} \left(\int_{\mathbb{R}^n} |x|^2 \; d\mu \right)^{\frac{p}{2}-1} \; \overset{⊛⊛⊛⊛}{=} \; C' \left\|T^{-1}\right\|_{\text{op}}^{2-p} n^{\frac{p}{2}-1}.$$

Transition ⊛ is valid as $|Tx| \geq |x| \left\|T^{-1}\right\|_{\text{op}}^{-1}$ and $p \leq 2$. Transition ⊛⊛ is due to Jensen's inequality and that $1 \leq p \leq 2$. Transition ⊛ ⊛ ⊛ is due to Paouris' theorem (Theorem 4.4.3). The constant C' from Paouris' theorem is universal, as we applied the theorem solely to $k = 2$. Also, this requires n to be larger than a universal constant. Transition ⊛ ⊛ ⊛⊛ is due to μ being isotropic. Plugging the bounds on both roots into the original calculation, we get that

$$1 \; \lesssim \; \frac{1}{Z_{p,T_*\mu}} \, |T^*\theta|^2 \left\|T^{-1}\right\|_{\text{op}}^{2-p} n^{\frac{p}{2}-1} \; \overset{⊛}{\lesssim} \; \frac{1}{n^{\frac{p}{2}-1}} \, |T^*\theta|^2 \left\|T^{-1}\right\|_{\text{op}}^{2-p} n^{\frac{p}{2}-1}$$

with the transition ⊛ due lower bound on $Z_{p,T_*\mu}$ obtained in Claim 4.4.5. Thus,

$$1 \lesssim |T^*\theta|^2 \left\|T^{-1}\right\|_{\mathrm{op}}^{2-p}$$

and recall that is for any $\theta \in S^{n-1}$. Therefore the above holds also for the infimum over all $\theta \in S^{n-1}$:

$$\left\|T^{-1}\right\|_{\mathrm{op}}^{p-2} \lesssim \inf_{\theta\in S^{n-1}} |T^*\theta|^2 = \frac{1}{\|T^{-*}\|_{\mathrm{op}}^2} = \frac{1}{\|T^{-1}\|_{\mathrm{op}}^2}$$

and thus,

$$\left\|T^{-1}\right\|_{\mathrm{op}}^{p} \lesssim 1.$$

□

Lemma 4.4.6. *In the setting of Proposition 4.4.1,* $\|T\|_{op} \leq C$.

Proof. Step 1 is a simple general result in probability theory.
Step 1. Let $Z : \Omega \to \mathbb{R}$ be a non-negative random variable, and let $A \subseteq \Omega$ be some event, $M > 0$ such that $\sqrt{\mathbb{E}\left[Z^2\right]} \leq M \cdot \mathbb{E}Z$ and $\mathbb{P}\left[A\right] \geq 1 - \frac{1}{4M^2}$. Then $\mathbb{E}\left[Z\mathbb{1}_A\right] \geq \frac{1}{2}\mathbb{E}Z$.

Proof. Using the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[Z\mathbb{1}_{A^c}\right] \leq \sqrt{\mathbb{E}\left[Z^2\right]} \cdot \sqrt{\mathbb{P}\left[A^c\right]} \leq M \cdot \mathbb{E}Z \cdot \frac{1}{2M} = \frac{1}{2}\mathbb{E}Z.$$

□

Step 2. For any $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} \langle x,\theta\rangle^2 \frac{|x|^{p-2}}{Z_{p,\mu}} \, d\mu(x) \gtrsim 1.$$

Proof. By the reverse-Hölder property (Theorem 4.4.2), let $M > 0$ be a universal constant for which $\|f\|_4 \leq M \cdot \|f\|_2$ for any semi-norm f. Chebyshev's inequality implies, for any $R > 0$:

$$\mu\left(\left\{|x| \geq R\sqrt{n}\right\}\right) \leq \frac{1}{R\sqrt{n}} \int_{\mathbb{R}^n} |x| \, d\mu \overset{\circledast}{\leq} \frac{1}{R\sqrt{n}} \sqrt{\int_{\mathbb{R}^n} |x|^2 \, d\mu} \overset{\circledast\circledast}{=} \frac{1}{R}.$$

where the transition $\circledast$ is due to the Cauchy-Schwartz inequality and the transition $\circledast\circledast$ due to μ being isotropic. We take $R = 4M^2$. Then

$$\mu\left(\left\{|x| < R\sqrt{n}\right\}\right) \;\geq\; 1 - \frac{1}{4M^2}$$

and we can apply Step 1 to get

$$\begin{aligned}\int_{\mathbb{R}^n} \langle x,\theta\rangle^2 \;\frac{|x|^{p-2}}{Z_{p,\mu}}\; d\mu(x) \;&\geq\; \int_{\left\{|x|<R\sqrt{n}\right\}} \langle x,\theta\rangle^2 \;\frac{|x|^{p-2}}{Z_{p,\mu}}\; d\mu(x) \;\geq \\ &\geq\; \frac{R^{p-2}n^{\frac{p-2}{2}}}{Z_{p,\mu}} \int_{\left\{|x|<R\sqrt{n}\right\}} \langle x,\theta\rangle^2 \; d\mu(x) \;\geq \\ &\geq\; \frac{R^{p-2}n^{\frac{p-2}{2}}}{Z_{p,\mu}} \cdot \frac{1}{2}\int_{\mathbb{R}^n} \langle x,\theta\rangle^2 \; d\mu(x) \;=\; \frac{R^{p-2}n^{\frac{p-2}{2}}}{2Z_{p,\mu}}.\end{aligned}$$

To conclude the proof of Step 2 see that,

$$Z_{p,\mu} \;=\; \int_{\mathbb{R}^n} |x|^{p-2}\; d\mu \;\overset{\circledast}{\leq}\; \left(\int_{\mathbb{R}^n} \frac{d\mu}{|x|^2}\right)^{\frac{2-p}{2}} \;\overset{\circledast\circledast}{\leq}\; C\left(\int_{\mathbb{R}^n} |x|^2\; d\mu\right)^{\frac{p-2}{2}} \;=\; Cn^{\frac{p-2}{2}}.$$

The transition $\circledast$ is due to Jensen's inequality, as $0 \leq 2-p \leq 1$. The transition $\circledast\circledast$ is due to Paouris' theorem (Theorem 4.4.3). Again, the constant C obtained from Paouris' inequality is universal, as we applied the theorem solely to $k = 2$. Also, this requires n to be larger than a universal constant. $\square$

Step 3. $\|T\|_{\mathrm{op}} \lesssim 1$.

Proof. Fix any $\theta \in S^{n-1}$. Then,

$$1 \;=\; \int_{\mathbb{R}^n} \langle x,\theta\rangle^2 \;\frac{|x|^{p-2}}{Z_{p,T_*\mu}}\; dT_*\mu(x) \;=\; \int_{\mathbb{R}^n} \langle x,T^*\theta\rangle^2 \;\frac{|Tx|^{p-2}}{Z_{p,T_*\mu}}\; d\mu(x) \;\geq$$

$$\geq\; \frac{Z_{p,\mu}\,\|T\|_{\mathrm{op}}^{p-2}}{Z_{p,T_*\mu}} \int_{\mathbb{R}^n} \langle x,T_*\theta\rangle^2 \;\frac{|x|^{p-2}}{Z_{p,\mu}}\; d\mu \;\overset{\circledast}{\gtrsim}\; \frac{Z_{p,\mu}\,\|T\|_{\mathrm{op}}^{p-2}}{Z_{p,T_*\mu}}\,|T^*\theta|^2$$

with the transition ⊛ due to Step 2. Taking supremum over all $\theta \in S^{n-1}$ we get:

$$\frac{Z_{p,T_*\mu} \|T\|_{\mathrm{op}}^{2-p}}{Z_{p,\mu}} \gtrsim \sup_{\theta \in S^{n-1}} |T^*\theta|^2 = \|T^*\|_{\mathrm{op}}^2 = \|T\|_{\mathrm{op}}^2 .$$

Hence,

$$\|T\|_{\mathrm{op}}^p \lesssim \frac{Z_{p,T_*\mu}}{Z_{p,\mu}}.$$

What remains is to observe that

$$Z_{p,T_*\mu} = \int_{\mathbb{R}^n} |Tx|^{p-2} \, d\mu \leq \left\|T^{-1}\right\|_{\mathrm{op}}^{2-p} \int_{\mathbb{R}^n} |x|^{p-2} \, d\mu =$$

$$= \left\|T^{-1}\right\|_{\mathrm{op}}^{2-p} Z_{p,\mu} \overset{\circledast}{\lesssim} Z_{p,\mu}$$

where the last transition ⊛ due to Lemma 4.4.4 and since $1 \leq p \leq 2$.

□

To conclude the proof of Proposition 4.4.1, it remains to notice that

$$1 \lesssim \frac{1}{\|T^{-1}\|_{\mathrm{op}}} \leq \|T\|_{\mathrm{op}} \lesssim 1$$

as $\frac{|x|}{\|T^{-1}\|_{\mathrm{op}}} \leq |Tx| \leq \|T\|_{\mathrm{op}} |x|$ and by Lemmas 4.4.4, 4.4.6. □

4.5 The log-concave case

We now prove the main results of this section.

Proof of Proposition 4.1.3. Let μ be an L^1-isotropic log-concave probability measure on $\mathbb{R}^n$. Take $T \in GL_n$ such that $T_*\mu$ is isotropic. By Proposition 4.4.1 we have

$$c \leq \|T\|_{\mathrm{op}}, \left\|T^{-1}\right\|_{\mathrm{op}} \leq C.$$

Therefore,

$$|x| \lesssim \frac{|x|}{\|T^{-1}\|_{\mathrm{op}}} \leq |Tx| \leq \|T\|_{\mathrm{op}} |x| \lesssim |x| ,$$

that is $|Tx| \simeq |x|$. We can now see that $Z_{1,\mu} \simeq \frac{1}{\sqrt{n}}$:

$$Z_{1,\mu} = \int_{\mathbb{R}^n} \frac{d\mu}{|x|} = \int_{\mathbb{R}^n} \frac{dT_*\mu}{|T^{-1}x|} \simeq \int_{\mathbb{R}^n} \frac{dT_*\mu}{|x|} \overset{\circledast}{\simeq} \left(\int_{\mathbb{R}^n} |x| \; dT_*\mu \right)^{-1} \simeq$$

$$\overset{\circledast\circledast}{\simeq} \left(\int_{\mathbb{R}^n} |x|^2 \; dT_*\mu \right)^{-\frac{1}{2}} = \frac{1}{\sqrt{n}}.$$

Transition $\circledast$ is due to Paouris' theorem (Theorem 4.4.3) and transition $\circledast\circledast$ is due to the reverse-Hölder property (Theorem 4.4.2). Second,

$$\int_{\mathbb{R}^n} \frac{\langle x, y \rangle^4}{|x|^3} \; d\mu(x) = \int_{\mathbb{R}^n} \frac{\langle x, T^{-*}y \rangle^4}{|T^{-1}x|^3} \; dT_*\mu(x) \le$$

$$\overset{\circledast}{\le} \sqrt{\int_{\mathbb{R}^n} \langle x, T^{-*}y \rangle^8 \; dT_*\mu(x)} \sqrt{\int_{\mathbb{R}^n} \frac{dT_*\mu(x)}{|T^{-1}x|^6}} \simeq$$

$$\overset{\circledast\circledast}{\simeq} \left|T^{-*}y\right|^4 \sqrt{\int_{\mathbb{R}^n} \frac{dT_*\mu}{|x|^6}} \overset{\circledast\circledast\circledast}{\simeq} \frac{1}{n^{\frac{3}{2}}} |T^{-*}y|^4 \simeq \frac{1}{n^{\frac{3}{2}}} |y|^4.$$

Transition $\circledast$ is due to the Cauchy-Schwarz inequality, transition $\circledast\circledast$ is due to the reverse Hölder property and transition $\circledast\circledast\circledast$ is due to Paouris' theorem. Therefore,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\langle x, y \rangle^4}{|x|^3 |y|^3} \; d\mu(x) \; d\mu(y) \lesssim \frac{1}{n^{\frac{3}{2}}} \int_{\mathbb{R}^n} |y| \; d\mu(y) = \frac{1}{n^{\frac{3}{2}}} \cdot n Z_{1,\mu} \simeq \frac{1}{n}.$$

□

Proof of Corollary 4.1.4. Let μ be a centered Leb-a.c. log-concave probability measure on $\mathbb{R}^n$. We wish to apply the existence Proposition 4.2.1 with $p = 1$. The density of μ is bounded by an exponential function $Ae^{-B|x|}$ for some $A, B > 0$ (see Lemma 2.2.1 in [4]), hence the first moment of μ is finite, and also the first negative moment is finite when $n > 1$. Both moments are nonzero as μ is absolutely continuous. By Proposition 4.2.1 there exists an L^1-isotropic position of μ.

Now assume that μ is L^1-isotropic. By Proposition 4.1.3, the conditions of Proposition 2.1.2 hold with $\alpha = Z_{1,\mu} \le C$, $\beta \le C'$ where C, C' are universal constants. Consequently,

$$\mathrm{Var}(F_\mu) \le \frac{C}{n^2}$$

where C is a universal constant.

□

4.6 Appendix

4.6.1

Proof of Lemma 4.2.4. Fix any nonzero matrix $A \in M_n$. See that the matrix $Id + \varepsilon A$ is invertible when $0 \leq \varepsilon < \|A\|_{\text{op}}^{-1}$; For any $\theta \in S^{n-1}$,

$$|(Id + \varepsilon A)\theta| \;=\; |\theta + \varepsilon A\theta| \;\geq\; |\theta| - |\varepsilon|\,|A\theta| \;\geq\; 1 - |\varepsilon|\,\|A\|_{\text{op}} > 0.$$

Thus, $\dfrac{Id + \varepsilon A}{\det(Id + \varepsilon A)^{1/n}} \in SL(n)$. Consider the following function:

$$f(\varepsilon) \;:=\; \int_{\mathbb{R}^n} \frac{|(Id + \varepsilon A)x|^p}{|\det(Id + \varepsilon A)|^{p/n}}\, d\mu(x)$$

defined for $0 \leq \varepsilon < \frac{1}{100}\|A\|_{\text{op}}^{-1}$. We expand the integrand into a power series around $\varepsilon = 0$. The numerator is:

$$|(Id{+}\varepsilon A)x|^p \;=\; \left(|x|^2{+}2\varepsilon\,\langle Ax, x\rangle{+}\varepsilon^2|Ax|^2\right)^{\frac{p}{2}} \;=\; |x|^p\left(1 + \varepsilon p\frac{\langle Ax, x\rangle}{|x|^2} + \mathcal{O}\left(\varepsilon^2\right)\right).$$

For the proof of the last transition, see Claim 4.6.1 in this appendix. The denominator is:

$$|\det\left(Id + \varepsilon A\right)|^{-\frac{p}{n}} \;=\; 1 - \varepsilon\frac{p}{n}TrA + \mathcal{O}\big(\varepsilon^2\big).$$

For the proof, see Claim 4.6.2 in this appendix. Combining we get:

$$f(\varepsilon) \;=\; \int_{\mathbb{R}^n} |x|^p\left(1 + \varepsilon p\frac{\langle Ax, x\rangle}{|x|^2} + \mathcal{O}\big(\varepsilon^2\big)\right)\left(1 - \varepsilon\frac{p}{n}TrA + \mathcal{O}\big(\varepsilon^2\big)\right)\, d\mu(x)$$

$$=\; \int_{\mathbb{R}^n}\left(|x|^p \;+\; \varepsilon p\,\langle Ax, x\rangle\,|x|^{p-2} \;-\; \varepsilon\frac{p}{n}TrA|x|^p \;+\; |x|^p\mathcal{O}\big(\varepsilon^2\big)\right)\, d\mu(x).$$

Note that $\int_{\mathbb{R}^n} |x|^p \mathcal{O}(\varepsilon^2)\, d\mu(x) = \mathcal{O}(\varepsilon^2)$ as $\int_{\mathbb{R}^n} |x|^p\, d\mu$ is finite. We now use assumption that Id minimizes $\int_{\mathbb{R}^n} |Sx|^p\, d\mu$ over all $S \in SL_n$, which implies $f(\varepsilon) - f(0) \geq 0$. Therefore,

$$0 \;\leq\; \lim_{\varepsilon \to 0^+} \frac{f(\varepsilon) - f(0)}{\varepsilon} \;=\; p \int_{\mathbb{R}^n} \left(\langle Ax, x \rangle\, |x|^{p-2} \;-\; \frac{1}{n} TrA|x|^p \right) d\mu(x).$$

By repeating the same process with $-A$ instead of A we get the opposite direction inequality, thus an equality:

$$\int_{\mathbb{R}^n} \langle Ax, x \rangle \;\; |x|^{p-2}\; d\mu(x) \;\; = \;\; \left(\frac{1}{n} \int_{\mathbb{R}^n} |x|^p \; d\mu(x) \right) \mathrm{Tr}A.$$

To conclude the proof, observe that $\langle Ax, x \rangle = \langle A, x \otimes x \rangle_{\mathbb{R}^{n^2}}$ and that $\mathrm{Tr}A = \langle A, Id \rangle_{\mathbb{R}^{n^2}}$. We may thus rewrite the above equality as:

$$\langle A, \mathrm{Cov}_p(\mu) \rangle_{\mathbb{R}^{n^2}} \;\; = \;\; \langle A, \kappa Id \rangle_{\mathbb{R}^{n^2}}$$

for some value $\kappa > 0$. Recall that this holds for any nonzero matrix $A \in M_n$, thus indeed $\mathrm{Cov}_p(\mu) = \kappa Id$ is scalar. □

Claim 4.6.1. *In the setting within the proof of Lemma 4.2.4,*

$$|(Id + \varepsilon A)\, x|^p \;\; = \;\; |x|^p \left(1 + \varepsilon p \frac{\langle Ax, x \rangle}{|x|^2} + \mathcal{O}(\varepsilon)^2 \right).$$

Proof.

See that $\quad |(Id + \varepsilon A)\, x|^p \;=\; \left(|(Id + \varepsilon A)\, x|^2 \right)^{\frac{p}{2}} \;=$

$$= \;\; \left(|x|^2 + 2\varepsilon \langle Ax, x \rangle + \varepsilon^2 |Ax|^2 \right)^{\frac{p}{2}} \;\; = \;\; |x|^p \left(1 + 2\varepsilon \frac{\langle Ax, x \rangle}{|x|^2} + \varepsilon^2 \frac{|Ax|^2}{|x|^2} \right)^{\frac{p}{2}}.$$

We expand the function $f(\tau) = (1 + \tau)^{\frac{p}{2}}$ defined for $\tau \in \left[-\frac{1}{2}, \frac{1}{2} \right]$ into a power series around 0 ($p \neq 0$). Calculate:

$$f'(\tau) \;\; = \;\; \frac{p}{2} (1 + \tau)^{\frac{p}{2} - 1} \qquad ; \qquad f''(\tau) \;\; = \;\; \frac{p}{2} \left(\frac{p}{2} - 1 \right) (1 + \tau)^{\frac{p}{2} - 2}.$$

We have $f(0) = 1$, $f'(0) = \frac{p}{2}$. By Lagrange's formula for the residual of the Taylor series, $f(\tau) = 1 + \frac{p}{2}\tau + \frac{f''(\tilde{\tau})}{2}\tau^2$ where $|\tilde{\tau}| < |\tau|$. As f'' is continuous over the compact interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ it is bounded, thus $f(\tau) = 1 + \frac{p}{2}\tau + \mathcal{O}(\tau^2)$ when $\tau \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Now see that

$$2\varepsilon \frac{\langle Ax, x\rangle}{|x|^2} + \varepsilon^2 \frac{|Ax|^2}{|x|^2} \ \leq \ 2\varepsilon \left\|A\right\|_{\text{op}} + \varepsilon^2 \left\|A\right\|_{\text{op}}^2 \ \leq \ \frac{1}{49}$$

hence we may use the approximation of $\tau \mapsto (1+\tau)^{\frac{p}{2}}$ obtained above to get

$$|(Id + \varepsilon A)x|^p = |x|^p \Bigg(1 + \varepsilon p \frac{\langle Ax, x\rangle}{|x|^2} + \varepsilon^2 \frac{p}{2} \frac{|Ax|^2}{|x|^2} + \\ + \mathcal{O}\left(\left(2\varepsilon \frac{\langle Ax, x\rangle}{|x|^2} + \varepsilon^2 \frac{|Ax|^2}{|x|^2} \right)^2 \right) \Bigg).$$

Having fixed p, A, all the coefficients of $\varepsilon, \varepsilon^2$ above are bounded;

$$\frac{\langle Ax, x\rangle}{|x|^2} \ \leq \ \left\|A\right\|_{\text{op}} \quad ; \quad \frac{|Ax|^2}{|x|^2} \ \leq \ \left\|A\right\|_{\text{op}}^2 .$$

It follows that

$$|(Id + \varepsilon A)\, x|^p \ = \ |x|^p \left(1 + \varepsilon p \frac{\langle Ax, x\rangle}{|x|^2} + \mathcal{O}\left(\varepsilon^2\right) \right).$$

□

Claim 4.6.2. *In the setting within the proof of Lemma 4.2.4,*

$$|\det\left(Id + \varepsilon A\right)|^{-\frac{p}{n}} \ = \ 1 - \varepsilon \frac{p}{n} TrA + \mathcal{O}\left(\varepsilon^2\right).$$

Proof. It is well known that for a fixed matrix A, $\det(Id+\varepsilon A)$ is a polynomial of degree at most n in ε, and $\det(Id + \varepsilon A) = 1 + \varepsilon TrA + \mathcal{O}\big(\varepsilon^2\big)$. For $\varepsilon > 0$ small enough, the value $\varepsilon \text{Tr} A + \mathcal{O}\left(\varepsilon\right)^2$ above is at most $\frac{1}{2}$ in absolute value. We may thus use the expansion into a power series of $\tau \mapsto (1+\tau)^{-\frac{p}{n}}$ as in Appendix 4.6.1 to get that

$$|\det\left(Id + \varepsilon A\right)|^{-\frac{p}{n}} \ = \ \left(1 + \varepsilon \text{Tr} A + \mathcal{O}\left(\varepsilon^2\right)\right)^{-\frac{p}{n}} = 1 - \frac{p}{n}\varepsilon \text{Tr} A - \frac{p}{n}\mathcal{O}\left(\varepsilon^2\right) + \\ + \mathcal{O}\left(\left(\varepsilon \text{Tr} A + \mathcal{O}\left(\varepsilon^2\right)\right)^2 \right) \ = \ 1 - \varepsilon \frac{p}{n} \text{Tr} A + \mathcal{O}\left(\varepsilon^2\right).$$

□

4.6.2

We prove the reduction in Lemma 4.3.5.

Claim 4.6.3. *We may assume that S is positive definite.*

Proof. Assume we have proven that if both $\mathrm{Cov}_p(\mu)$ and $\mathrm{Cov}_p(T_*\mu)$ are scalar and T is positive definite, then $T = Id$. Define $T = \sqrt{S^tS}$ as the positive definite root of the positive definite matrix S^tS. Then $T \in SL(n)$ and $|Tx| = |Sx|$ for all $x \in \mathbb{R}^n$. Also $\mathrm{Cov}_p(T_*\mu)$ is scalar; Indeed, writing $\mathrm{Cov}_p(S_*\mu) = \kappa Id$, we see that

$$\begin{aligned}
\mathrm{Cov}_p(T_*\mu) &= \int_{\mathbb{R}^n} xx^t\ |x|^{p-2}\ dT_*\mu \ = \int_{\mathbb{R}^n} Txx^tT\ |Tx|^{p-2}\ d\mu \ = \\
&= \int_{\mathbb{R}^n} TS^{-1}Sxx^tS^tS^{-t}T\ |Tx|^{p-2}\ d\mu \ = \\
&= TS^{-1}\left(\int_{\mathbb{R}^n} (Sx)\,(Sx)^t\ \ |Sx|^{p-2}\ \ d\mu\right)\left(TS^{-1}\right)^t \ = \\
&= TS^{-1}\mathrm{Cov}_p(S_*\mu)S^{-t}T \ = \ TS^{-1}\kappa IdS^{-t}T \ = \ \kappa TS^{-1}S^{-t}T \ = \\
&= \kappa T\left(S^tS\right)^{-1}T \ = \ \kappa TT^{-2}T \ = \ \kappa Id.
\end{aligned}$$

Hence by the reduction step $T = Id$, and $S^tS = T^2 = Id$ thus $S \in O_n$. □

Claim 4.6.4. *We may assume that $S = diag(\lambda_1, ..., \lambda_n)$, $\lambda_1 \geq ... \geq \lambda_n > 0$.*

Proof. Assume we have proven that if both $\mathrm{Cov}_p(\mu)$ and $\mathrm{Cov}_p(T_*\mu)$ are scalar and $T = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ with $\lambda_1 \geq ... \geq \lambda_n > 0$, then $T = Id$. By the first reduction step (Claim 4.6.3) we may assume that S is positive definite, and we need to prove that $S = Id$. Use the orthogonal diagonalization $S = U^tTU$ for $U \in O_n$, $T = \mathrm{diag}(\lambda_1, ...\lambda_n)$ with $\lambda_1 \geq ... \geq \lambda_n$. We have $\det T = \det S = 1$ and also both $\mathrm{Cov}_p(U_*\mu)$, $\mathrm{Cov}_p(T_*U_*\mu)$ are scalar; First, Claim 4.3.4 entails that $\mathrm{Cov}_p(U_*\mu)$ is scalar as $\mathrm{Cov}_p(\mu)$ is. Second, $\mathrm{Cov}_p(S_*\mu)$ is scalar and $T_*U_*\mu = U_*\left(U^tTU\right)_*\mu = U_*S_*\mu$, hence $\mathrm{Cov}_p(T_*U_*\mu)$ is scalar by the same claim again. □

Chapter 5

Exponential Tail Bound

5.1 Introduction

So far we have improved the moment bounds implied by Lévy's inequality in the case of $p = 2$. We have shown, for a log-concave measure μ on $\mathbb{R}^n$, that

$$\|F_\mu\|_2 \;\le\; \frac{C}{n}$$

rather than the bound implied by Lévy's inequality, which is

$$\|F_\mu\|_2 \;\le\; \frac{C}{\sqrt{n}}.$$

The improvement is thus by a factor of $1/\sqrt{n}$. In the log-concave case, it is possible to achieve a $1/\sqrt{n}$ improvement for all $p \ge 1$.

We will obtain the moment bounds via the ψ_1 norm.

Proposition 5.1.1. *Let $n \ge C'$ and let μ be an L^1-isotropic log-concave probability measure on $\mathbb{R}^n$. Then,*

$$\int_{S^{n-1}} e^{n|F_\mu - \mathbb{E}F_\mu|/C} \, d\sigma_{n-1} \;\le\; 2.$$

Here, $C, C' > 0$ are universal constants.

This is equivalent to stating that $\|F_\mu\|_{\psi_1} \le C/n$. In general, a ψ_1 bound implies the moment bounds $\|f\|_p \;\le\; p\,\|f\|_{\psi_1}$ for any $p \ge 1$ (see, e.g.

Lemma 2.1 in [3]). Also, it implies an exponential tail bound of the form $\mathbb{P}\left[|f - \mathbb{E}f| > t\right] \le 2e^{-t/\|f\|_{\psi_1}}$ (for a proof, see Appendix 5.4.1). We thus obtain the main result of this chapter.

Corollary 5.1.2. *Let $n \ge C'$ and let μ be a centered Leb-a.c. log-concave probability measure on $\mathbb{R}^n$. Then there exists a position of μ in which it is L^1-isotropic and*

$$\|F_\mu\|_p \;\le\; C\frac{p}{n} \quad \forall p \ge 1 \quad ; \quad \mathbb{P}\left[|F_\mu - \mathbb{E}F_\mu| > t\right] \;\le\; 2e^{-nt/C}.$$

Here, $C, C' > 0$ are universal constants.

Compare these to the bounds Lévy's inequality produces:

$$\|F_\mu\|_p \;\le\; C\frac{\sqrt{p}}{\sqrt{n}} \quad \forall p \ge 1 \quad ; \quad \mathbb{P}\left[|F_\mu - \mathbb{E}F_\mu| > t\right] \;\le\; 2e^{-nt^2/C}$$

See Appendix 5.4.2 for a proof that Lévy's inequality implies the above moment bounds. We thus improve the dependence on n in the moment bounds by a factor of $\frac{1}{\sqrt{n}}$.

The proof of Proposition 5.1.1 relies on the following theorem:

Theorem 5.1.3. *Let $f : S^{n-1} \to \mathbb{R}$ be a C^2-smooth function such that $\|f''_S(\theta)\|_{op} \le 1$ for every $\theta \in S^{n-1}$. Then,*

$$\int e^{\frac{n-1}{2}(f - \mathbb{E}f)}\, d\sigma \;\le\; e^{\frac{n-1}{2}\int |\nabla_S f|^2 d\sigma}. \tag{5.1}$$

The integrals above are taken over S^{n-1}, with respect to $\sigma = \sigma_{n-1}$. The matrix $f''_S(\theta)$ is the *spherical second derivative* of f at θ, defined as

$$f''_S(\theta) \;=\; P_{\theta^\perp}\left(f''(\theta) - \langle \nabla_S f(\theta), \theta\rangle\, Id\right) P_{\theta^\perp} \tag{5.2}$$

where $f''(\theta)$ is the Euclidean second derivative (or Hessian matrix) of a smooth extension of f to an open neighborhood containing S^{n-1}. Similarly, we say that $f : S^{n-1} \to \mathbb{R}$ is *twice-differentiable* (C^2*-smooth*) if some smooth extension of f to $\mathbb{R}^n$ is twice-differntiable (C^2-smooth). We refer to Appendix A in [2] for a detailed introduction of the notion of the spherical second derivative.

Theorem 5.1.3 was proven by S. G. Bobkov, G. P. Chistyakov and F. Götze (Corollary 3.2 in [2]). In order to apply it to F, need to show that F is C^2-smooth, and that $\|F''_S(\theta)\|_{\text{op}}$ is bounded by a universal constant.

In what follows we use F and its 1-homogeneous extension to $\mathbb{R}^n$ interchangeably.

5.2 Calculating the second derivative

Proposition 5.2.1. *Let μ be a centered Leb-a.c. log-concave probability measure on $\mathbb{R}^n$. Then F_μ is C^2-smooth, and for any $\theta \in S^{n-1}$*

$$F''_\mu(\theta) \ = \ \int_{\theta^\perp} x \otimes x \ \rho(x) \ dx$$

where ρ is the density of μ.

The integral here is a Lebesgue integral over the $n-1$-dimensional space $\theta^\perp$.

Lemma 5.2.2. *Let $\rho : \mathbb{R}^n \to [0, \infty)$ be log-concave with $\int_{\mathbb{R}^n} \rho(x)dx > 0$ and $\int_{\mathbb{R}^n} x \ \rho(x)dx = 0$. Denote by A the set of discontinuity points of ρ. Then $Vol_{n-1}(A \cap H) = 0$ for any hyperplane $H \subseteq \mathbb{R}^n$ through the origin.*

Proof.
Step 1. Let $f : \mathbb{R}^n \to [0, \infty)$ be log-concave. Then the support of f is convex.

Proof. Let $x, y \in$ supp f. For any $z = \lambda x + (1 - \lambda)y \in [x, y]$, we have $f(z) \geq f(x)^\lambda f(y)^{1-\lambda} > 0$ hence $z \in$ supp f. □

Step 2. Let $f : \mathbb{R}^n \to [0, \infty)$ be log-concave and centered, i.e. $\int_{\mathbb{R}^n} x \, f(x) \, dx = 0$, and assume $\int_{\mathbb{R}^n} f(x)dx > 0$. Then $0 \in$ int supp f.

Proof. Write $M =$ supp f. Assume $0 \notin \text{int}M$. By Step 1 M is convex, and there is a closed half-space $I = \{x : \ \langle x, \theta \rangle \geq 0\} \subseteq \mathbb{R}^n$ such that $M \subseteq I$. This raises a contradiction:

$$0 \ = \ \int_{\mathbb{R}^n} \langle x, \theta \rangle \ \ f(x) \ dx \ = \ \int_{M \cap \text{int} I} \langle x, \theta \rangle \ \ f(x) \ dx \ > \ 0.$$

The last integral is positive due to $\langle x, \theta \rangle f(x)$ being strictly positive on $M \cap \text{int}I$, while $\text{Vol}_n(M \cap \text{int}I) = \text{Vol}_n(M \cap I) = \text{Vol}_n(M) > 0$ since $0 < \int_{\mathbb{R}^n} f = \int_M f$. ☐

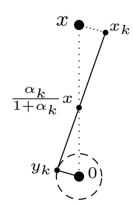
Step 3. Let $K \subseteq \mathbb{R}^n$ be convex with $0 \in \text{int}K$. Then $x \in \partial K \Rightarrow [0, x) \subseteq K$.

Proof. Take some $r > 0$ such that $B = \bar{B}(0, r) \subseteq K$, and take $(x_k)_{k=1}^{\infty} \subseteq K$ a sequence in K such that $x_k \to x$. For every k, define

$$y_k = \frac{r}{|x - x_k|}(x - x_k).$$

Then $|y_k| = r$ meaning $y_k \in B \subseteq K$. Writing $\alpha_k = r/|x - x_k|$ we have by convexity that:

$$K \ \ni \ \frac{1}{1+\alpha_k} y_k \ + \ \frac{\alpha_k}{1+\alpha_k} x_k \ = \ \frac{\alpha_k}{1+\alpha_k}(x - x_k + x_k) \ = \ \frac{\alpha_k}{1+\alpha_k} x.$$



When $k \to \infty$, $|x - x_k| \to 0$ and $\frac{\alpha_k}{1+\alpha_k} \to 1$. Convexity ensures that the entire segment $[0, x)$ is contained in K (since $0 \in K$). ☐

Write $M = \text{supp}\ \rho$. By Step 1, M is convex.

Step 4. For any hyperplane $H \subseteq \mathbb{R}^n$ through the origin, $\partial M \cap H \subseteq \partial_{n-1}(M \cap H)$. By ∂_{n-1} we mean the boundary in $H \simeq \mathbb{R}^{n-1}$.

Proof. Let $x \in \partial M \cap H$. By Step 2, $0 \in \text{int}M$ hence by Step 3, $[0, x) \subseteq M$. A hyperplane is convex thus also $[0, x] \subseteq H$ therefore $[0, x) \subseteq M \cap H$ and $x \in \partial(M \cap H)$. ☐

Step 5. $A \subseteq \partial M$.

Proof. Let $x \in \text{int}M$, take $x \in B \subseteq M$ a closed ball inside M containing x. We have $\rho = e^{-h}$ with $h|_B$ being a real-valued convex function on a compact convex set. Then h is continuous at x (see, e.g. [5]), thus so is ρ. If $x \in \text{ext}M$, then ρ vanishes around x hence continuous at x. $\square$

Step 6.

By Steps 4 and 5, for any hyperplane $H \subseteq \mathbb{R}^n$ through the origin, $A \cap H \subseteq \partial M \cap H \subseteq \partial_{n-1}(M \cap H)$. Any convex set has a boundary of Lebesgue measure zero (see, e.g. [10]), hence $\text{Vol}_{n-1}\left(\partial_{n-1}(M \cap H)\right) = 0$. $\square$

Proof of Proposition 5.2.1. Fix $\theta \in S^{n-1}$. To prove that F is twice-differentiable at θ, we are required to prove that

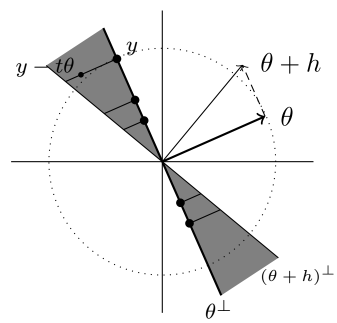
$$F(\theta + h) \;=\; F(\theta) + \langle \nabla F(\theta), h \rangle + \frac{1}{2}\int_{\theta^\perp} \langle x, h\rangle^2 \;\rho(x)\; dx + o(|h|^2) \quad \text{as } |h| \to 0.$$

As F is 1-homogeneous, it is enough to show the above under the assumption that $h \perp \theta$. A detailed proof of this reduction is given in Appendix 5.4.3. We calculate:

$$\begin{aligned}
&F(\theta + h) - F(\theta) - \langle \nabla_S F(\theta), h\rangle \;= \\
&= \int_{H_{\theta+h}} \langle x, \theta + h\rangle \;\rho(x)\; dx - \int_{H_\theta} \langle x, \theta\rangle \;\rho(x)\; dx - \int_{H_\theta} \langle x, h\rangle \;\rho(x)\; dx \\
&= \int_{H_{\theta+h}\setminus H_\theta} \langle x, \theta + h\rangle \;\rho(x)\; dx - \int_{H_\theta \setminus H_{\theta+h}} \langle x, \theta + h\rangle \;\rho(x)\; dx \\
&= \int_{H_{\theta+h}\Delta H_\theta} |\langle x, h\rangle| \;\rho(x)\; dx - \int_{H_{\theta+h}\Delta H_\theta} |\langle x, \theta\rangle| \;\rho(x)\; dx.
\end{aligned}$$

Observe that we have, up to a set of Lebesgue-measure zero,

$$H_{\theta+h}\Delta H_\theta \;=\; \left\{x:\; -1 < \frac{\langle x, \theta\rangle}{\langle x, h\rangle} < 0\right\} \;=\; \left\{y - t\theta:\; y \perp \theta,\; \substack{0<|t|<|\langle y,h\rangle| \\ \text{sign}(t)=\text{sign}(\langle y,h\rangle)}\right\}.$$



We continue by decomposing the left-hand side integral above using Fubini's theorem.

$$\int_{H_{\theta+h}\Delta H_\theta} |\langle x,h\rangle| \;\rho(x)\;dx \;=\; \int_{y\in\theta^\perp}\int_{t=0}^{\langle y,h\rangle} \langle y-t\theta,h\rangle\;\rho(y-t\theta)\;dt\;dy \;=$$

$$= \int_{\theta^\perp} \langle y,h\rangle^2 \left(\frac{1}{\langle y,h\rangle}\int_0^{\langle y,h\rangle} \rho(y-t\theta)\;dt\right)\;dy \;=\; \int_{\theta^\perp} \langle y,h\rangle^2 f_1(\langle y,h\rangle)\;dy$$

since $h \perp \theta$ and having defined

$$f_1(s) \;=\; \frac{1}{s}\int_0^s \rho(y-t\theta)\;dt.$$

Note that when ρ is continuous at y, $f_1(\langle y,h\rangle) \to \rho(y)$ as $h \to 0$. Similarly as to the right-hand side integral from before,

$$\int_{H_{\theta+h}\Delta H_\theta} |\langle x,\theta\rangle| \;\rho(x)\;dx \;=\; \int_{y\in\theta^\perp}\int_{t=0}^{\langle y,h\rangle} \left(-\langle y-t\theta,\theta\rangle\right)\;\rho(y-t\theta)\;dt\;dy \;=$$

$$= \int_{\theta^\perp} \langle y,h\rangle^2 \left(\frac{1}{\langle y,h\rangle^2}\int_0^{\langle y,h\rangle} t\rho(y-t\theta)\;dt\right)\;dy \;=\; \int_{\theta^\perp} \langle y,h\rangle^2 f_2(\langle y,h\rangle)\;dy$$

having defined

$$f_2(s) \;=\; \frac{1}{s^2}\int_0^s t\rho(y-t\theta)\;dt.$$

Defining $g(t) = t\rho(y-t\theta)$ we see that when ρ is continuous at y it holds that $g(0) = 0$, $g'(0) = \rho(y)$, hence $f_2(\langle y,h\rangle) \to g'(0)/2 = \rho(y)/2$ as $h \to 0$. We may now plug what we have calculated into the main equation:

$$\frac{1}{|h|^2}\left(F(\theta+h) - F(\theta) - \langle \nabla F(\theta),h\rangle - \frac{1}{2}\int_{\theta^\perp} \langle y,h\rangle^2\;\rho(y)\;dy\right) \;=$$

$$= \int_{\theta^\perp} \left\langle y, \frac{h}{|h|} \right\rangle^2 \left(f_1(\langle y,h\rangle) - f_2(\langle y,h\rangle) - \frac{1}{2}\rho(y) \right) \, dy$$

with $f_1(\langle y,h\rangle) - f_2(\langle y,h\rangle) - \frac{1}{2}\rho(y) \to 0$ when ρ is continuous at $y \in \theta^\perp$, which happens Leb-a.e. in $\theta^\perp$ due to Lemma 5.2.2. By the dominated convergence theorem, we are left to show that the integrand above is bounded by a convergent function. Indeed, first

$$\left| \left\langle y, \frac{h}{|h|} \right\rangle^2 f_1(\langle y,h\rangle) \right| \;\leq\; \frac{|y|^2}{|\langle y,h\rangle|} \left| \int_0^{\langle y,h\rangle} \rho(y - t\theta) \, dt \right| \;\leq\; |y|^2 \sup_{t\in\mathbb{R}} \rho(y + t\theta).$$

Second,

$$\left| \left\langle y, \frac{h}{|h|} \right\rangle^2 f_2(\langle y,h\rangle) \right| \;=\; \frac{1}{|h|^2} \left| \int_0^{\langle y,h\rangle} t\rho(y - t\theta) \, dt \right| \;\leq\; |y|^2 \sup_{t\in\mathbb{R}} \rho(y + t\theta).$$

Third,

$$\frac{1}{2} \left| \left\langle y, \frac{h}{|h|} \right\rangle^2 \rho(y) \right| \;\leq\; |y|^2 \sup_{t\in\mathbb{R}} \rho(y + t\theta).$$

Any non-negative integrable log-concave function decays exponentially (see Lemma 2.2.1 in [4]), i.e. there are $A, B > 0$ such that

$$\rho(x) \;\leq\; Ae^{-B|x|}$$

for any $x \in \mathbb{R}^n$. Hence, for $y \perp \theta$ we have

$$\sup_{t\in\mathbb{R}} \rho(y + t\theta) \;\leq\; \sup_{t\in\mathbb{R}} Ae^{-B|y+t\theta|} \;=\; Ae^{-B|y|}.$$

Thus the function $y \mapsto 3A\,|y|^2\, e^{-B|y|}$ bounds our integrand from above, and is of course integrable. This establishes that F is twice-differentiable at any $\theta \in S^{n-1}$, with

$$F''(\theta) \;=\; \int_{\theta^\perp} y \otimes y \; \rho(y) \; dy.$$

We now wish to prove continuity of F''. As F is 1-homogeneous, the above implies it is twice-differentiable at any $x \in \mathbb{R}^n \setminus \{0\}$, with

$$F''(x) \;=\; \frac{1}{|x|} F''\left(\frac{x}{|x|} \right) \;=\; \frac{1}{|x|} \int_{x^\perp} y \otimes y \; \rho(y) \; dy$$

(for a proof, see Appendix 5.4.4). To show continuity of the matrix F'' at $x \in \mathbb{R}^n \setminus \{0\}$, is suffices to show continuity of $\langle F''(\cdot)u, v\rangle$ at x for any $u, v \in \mathbb{R}^n$. Fix $u, v \in \mathbb{R}^n$ and define

$$f(y) = \langle F''(y)u, v\rangle \;=\; \frac{1}{|y|}\int_{y^\perp} \langle z, u\rangle \langle z, v\rangle \;\rho(z)\; dz.$$

To show continuity of f at x we need to show:

$$f(x+h) - f(x) \;\to\; 0 \qquad \text{as } |h| \to 0.$$

Note that f is a (-1)-homogeneous function, therefore it is enough to show the above under the assumption $h \perp x$ (for a detailed proof of this reduction, see Appendix 5.4.5). We represent $f(x+h)$ as an integral over $x^\perp$; For $z \perp (x+h)$ decompose $z = y - \frac{\langle y,h\rangle}{|x|^2}x$ with $y \perp x$. Then,

$$f(x+h) \;=\; \int_{(x+h)^\perp} \frac{\langle z, u\rangle \langle z, v\rangle}{|x+h|}\;\rho(z)\;dz \;=$$

$$=\; \frac{1}{|x+h|}\int_{x^\perp}\left\langle y - \frac{\langle y,h\rangle}{|x|^2}x, u\right\rangle \left\langle y - \frac{\langle y,h\rangle}{|x|^2}x, v\right\rangle \;\rho\left(y - \frac{\langle y,h\rangle}{|x|^2}x\right)\; dy.$$

When ρ is continuous at y, the integrand tends to $\langle y, u\rangle \langle y, v\rangle \rho(y)/|x|$ as $h \to 0$, which is the integrand in $f(x)$. By Lemma 5.2.2, this happens Leb-a.e. in $x^\perp$. By the dominated convergence theorem, it suffices to show that the integrand is bounded by an integrable function. Indeed, first

$$\left|\left\langle y - \frac{\langle y,h\rangle}{|x|^2}x, u\right\rangle\right| \;\le\; |u|\left|y - \frac{\langle y,h\rangle}{|x|^2}x\right| \;\le\; |u|\,|y|\left(1 + \frac{|h|}{|x|}\right).$$

Similarly,

$$\left|\left\langle y - \frac{\langle y,h\rangle}{|x|^2}x, v\right\rangle\right| \;\le\; |v|\,|y|\left(1 + \frac{|h|}{|x|}\right).$$

Finally,

$$\rho\left(y - \frac{\langle y,h\rangle}{|x|^2}x\right) \;\le\; \sup_{t\in\mathbb{R}} \rho(y + tx).$$

Then the integrand is bounded by

$$y \;\mapsto\; \frac{|u|\,|v|}{|x+h|}\left(1 + \frac{|h|}{|x|}\right)^2 |y|^2 \sup_{t\in\mathbb{R}^n} \rho(y + tx).$$

which is a family of functions, with the parameter h. When h is small enough, they are uniformly bounded by a function which is integrable, as before, and our proposition is proven.

□

5.3 Bounding the second derivative

Having shown that F is C^2-smooth, we now turn to proving that $\|F''_S\|_{\text{op}}$ is bounded by a universal constant. We do so by reducing to a problem in two dimensions.

Lemma 5.3.1. *Let $n \geq C'$ and let μ be an L^1-isotropic log-concave probability measure on $\mathbb{R}^n$. Assume that $\theta, \eta \in S^{n-1}$ are perpendicular to one another. Then there exists a centered Leb-a.c. log-concave probability measure ν on $\mathbb{R}^2$ such that*

$$\left\langle F''_\mu(\theta)\eta, \eta \right\rangle_{\mathbb{R}^n} \;=\; \langle F''_\nu(e_1)e_2, e_2\rangle_{\mathbb{R}^2}$$

where $e_1, e_2 \in \mathbb{R}^2$ are the standard basis vectors. Moreover, there exists $S \in GL_2$ such that $S_*\nu$ is isotropic and $\|S\|_{op}, \|S^{-1}\|_{op} \leq C$.
Here, $C, C' > 0$ are universal constants.

Proof. The following trivial claim states that $\left\langle F''_\mu(\theta)\eta, \eta \right\rangle$ takes into consideration only values of F_μ at points on the plane $span\,\{\theta, \eta\}$.

Claim 5.3.2. *Let $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ be a C^2-smooth function and let $u, v \in \mathbb{R}^n$, $u \neq 0$. Write $E = span\,\{u, v\}$ and define $\tilde{f} = f|_E$. Then $\langle f''(u)v, v\rangle_{\mathbb{R}^n} = \langle \tilde{f}''(u)v, v\rangle_E$.*

The proof of Claim 5.3.2 is given in Appendix 5.4.6.

Define an orthogonal projection $P : \mathbb{R}^n \to \mathbb{R}^2$ by $\theta \mapsto e_1$, $\eta \mapsto e_2$, and denote $\nu = P_*\mu$. Then ν is a Leb-a.c. centered log-concave probability measure on $\mathbb{R}^2$. For any $y \in span\,\{\theta, \eta\}$ we have

$$F_\mu(y) \;=\; \int_{\mathbb{R}^n} \langle x, y\rangle_+ \; d\mu(x) \;=\; \int_{\mathbb{R}^2} \langle x, Py\rangle_+ \; dP_*\mu(x) \;=\; F_\nu(Py)$$

thus $F_\mu|_{span\{\theta,\eta\}} \;=\; F_\nu \circ P$. Both F_μ and F_ν are C^2-smooth by Proposition 5.2.1, hence by Claim 5.3.2:

$$\left\langle F''_\mu(\theta)\eta, \eta \right\rangle_{\mathbb{R}^n} \;=\; \langle F''_\nu(e_1)e_2, e_2\rangle_{\mathbb{R}^2}$$

as we sought. As to the 'moreover' statement, let $T \in GL_n$ be such that $T_*\mu$ is isotropic. By Proposition 4.4.1, $\|T\|_{\text{op}}, \|T^{-1}\|_{\text{op}} \leq C$ for some universal constant C. The matrix $PT^{-1}T^{-t}P^t \in GL_2$ is positive definite, so we may take $S = (PT^{-1}T^{-t}P^t)^{-\frac{1}{2}}$. As S is positive definite,

$$\left\|S^{-1}\right\|_{\text{op}} \;=\; \left\|PT^{-1}T^{-t}P^t\right\|_{\text{op}}^{\frac{1}{2}} \;\leq\; \left\|T^{-1}\right\|_{\text{op}} \;\leq\; C$$

since $\|P\|_{\text{op}}, \|P^t\|_{\text{op}} = 1$. On the other hand, denote $E = span\{\theta, \eta\}$. For $x \in \mathbb{R}^2$ we have $P^t x \in E$ with $|P^t x| = |x|$. It holds that $\langle T^{-1}T^{-t}P^t x, P^t x\rangle = |T^{-t}P^t|^2 \geq |P^t x|^2 / \|T\|_{\text{op}}^2 \gtrsim |P^t x|^2 = |x|^2$ and as $P^t x \in E$ we have $|PT^{-1}T^{-t}P^t x| \geq \langle T^{-1}T^{-t}P^t x, P^t x/|P^t x|\rangle \gtrsim |x|$. Thus $|S^{-2}x| \gtrsim |x|$ so $\|S\|_{\text{op}} \leq C$.
Finally, observe that $S_*\nu$ is isotropic:

$$\int_{\mathbb{R}^2} xx^t \; dS_*\nu \;=\; \int_{\mathbb{R}^2} xx^t \; dS_*P_*\mu \;=\; \int_{\mathbb{R}^n} SPxx^tP^tS^t \; d\mu$$

$$= \int_{\mathbb{R}^n} SPT^{-1}Txx^tT^tT^{-t}P^tS^t \; d\mu \;=\; \int_{\mathbb{R}^n} SPT^{-1}xx^tT^{-t}P^tS^t \; dT_*\mu$$

$$= \; SPT^{-1}\left(\int_{\mathbb{R}^n} xx^t \; dT_*\mu\right)T^{-t}P^tS^t \;=\; SPT^{-1}T^{-t}P^tS^t \;=\; Id$$

with the middle transition in the last line due to $T_*\mu$ being isotropic.

□

In the end we will see that the problem in dimension $n = 2$ boils down to the following:

Lemma 5.3.3. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^2$ with density ρ, and let $\theta \in \mathbb{R}^2$ with $|\theta| = 1$. Then,*

$$\int_{-\infty}^{\infty} t^2 \; \rho(t\theta) \; dt \;\leq\; C$$

where $C > 0$ is a universal constant.

Proof. **Step 1.**

$$\int_{-\infty}^{\infty} \rho(t\theta) \; dt \;\leq\; \sqrt{2}.$$

Proof. Take $\eta \perp \theta$ with $|\eta| = 1$. The function $\omega(t) = \int_{-\infty}^{\infty} \rho(t\eta + s\theta)\ ds$ is log-concave as a linear image of a log-concave measures is always log-concave. In our case, ω density of the image of μ under $x \mapsto \langle x, \eta \rangle$. Also, $\omega(0) = \int_{-\infty}^{\infty} \rho(s\theta)\ ds > 0$ as ρ is a centered (non-trivial) log-concave function hence $0 \in \text{int supp}\ \rho$. We may thus apply Theorem 2.2.3 from [4] to ω obtaining

$$\int_0^{\infty} t^2\, \omega(t)\ dt \ \leq\ \frac{2}{\omega(0)^2} \left(\int_0^{\infty} \omega \right)^3 \quad ; \quad \int_{-\infty}^{0} t^2\, \omega(t)\ dt \ \leq\ \frac{2}{\omega(0)^2} \left(\int_{-\infty}^{0} \omega \right)^3 .$$

By isotropicity of μ,

$$1 \ =\ \int_{\mathbb{R}^2} \langle x, \eta \rangle^2\ \ \rho(x)\ dx \ =\ \int_{-\infty}^{\infty} t^2 \omega(t)\ dt \ \leq\ \frac{2}{\omega(0)^2} \left(\int_{-\infty}^{\infty} \omega \right)^3 \ =\ \frac{2}{\omega(0)^2}$$

as $\int_{-\infty}^{\infty} \omega = \int_{\mathbb{R}^2} \rho = 1$. □

Step 2. We apply Theorem 2.2.3 from [4] again, this time to the $t \mapsto \rho(t\theta)$ which is a log-concave function as a restriction of a log-concave function. We get:

$$\int_{-\infty}^{\infty} t^2\ \rho(t\theta)\ dt \ \leq\ \frac{2}{\rho(0)^2} \left(\int_{-\infty}^{\infty} \rho(t\theta)\ dt \right)^3 \ \leq\ \frac{\sqrt{32}}{\rho(0)^2}.$$

Finally, $\rho(0)$ is bigger than a universal constant for all isotropic log-concave densities on $\mathbb{R}^2$ (see, e.g. Proposition 2.3.12 in [4]).

□

Now we are ready to bound the operator norm of the second derivative.

Lemma 5.3.4. *Let $n \geq C'$ and let μ be an L^1-isotropic log-concave probability measure on $\mathbb{R}^n$. Then $\|(F_\mu)''_S(\theta)\|_{op} \leq C$ for any $\theta \in S^{n-1}$. Here, $C, C' > 0$ are universal constants.*

Proof. Recall the definition of the spherical second-derivative:

$$F''_S(\theta) \ =\ P_{\theta^\perp} \left(F''(\theta) - \langle \nabla_S F(\theta), \theta \rangle\, Id \right) P_{\theta^\perp}.$$

We see that F''_S is a symmetric matrix for any θ, hence

$$\|F''_S(\theta)\|_{\text{op}} \ =\ \sup_{\eta \in S^{n-1}} |\langle F''_S(\theta)\eta, \eta \rangle| \,.$$

Moreover, observing that $\langle F_S''(\theta)\eta, \eta\rangle = \langle F_S''(\theta) P_{\theta^\perp}\eta, P_{\theta^\perp}\eta\rangle$ we may assume $\eta \perp \theta$. As $|\langle F_S''(\theta)\eta,\eta\rangle| \leq |\langle F''(\theta)\eta,\eta\rangle| + |\langle \nabla_S F(\theta), \theta\rangle|$, it suffices to bound both of the latter values.

First, note that $\langle \nabla_S F(\theta), \theta\rangle = F(\theta)$; this is true in general for any 1-homogeneous function, and also evident from the formula for $\nabla_S F(\theta)$ given in Corollary 2.3.3.

Claim 5.3.5. *Let $n \geq \tilde{C}$ and let μ be an L^1-isotropic log-concave probability measure on $\mathbb{R}^n$. Then $F_\mu \leq C$. Here, $C, \tilde{C} > 0$ are universal constants.*

Proof. Let $T \in GL_n$ be such that $T_*\mu$ is isotropic. Fix any $\theta \in S^{n-1}$ and write $\eta = T^{-*}\theta$. Calculate:

$$F(\theta) \ = \ \int_{\mathbb{R}^n} \langle x, \theta\rangle_+ \ d\mu(x) \ = \ \int_{\mathbb{R}^n} \langle x, \eta\rangle_+ \ dT_*\mu(x) \ \leq$$

$$\leq \ \int_{\mathbb{R}^n} |\langle x, \eta\rangle| \ dT_*\mu(x) \ \leq \ \sqrt{\int_{\mathbb{R}^n} \langle x, \eta\rangle^2 \ dT_*\mu(x)} \ = \ |\eta|$$

and see that $|\eta| = |T^{-*}\theta| \leq \|T^{-1}\|_{\mathrm{op}} \leq C$ where $C > 0$ is a universal constant, by Proposition 4.4.1. $\square$

We now turn to bounding $\langle F''(\theta)\eta, \eta\rangle$. By Lemma 5.3.1, there is a centered Leb-a.c. log-concave probability measure ν on $\mathbb{R}^2$ such that

$$\left\langle F_\mu''(\theta)\eta, \eta\right\rangle_{\mathbb{R}^n} \ = \ \langle F_\nu''(e_1)e_2, e_2\rangle_{\mathbb{R}^2}.$$

Lemma 5.3.1 also gives us $S \in GL_2$ such that $S_*\nu$ is isotropic and $\|S\|_{\mathrm{op}}, \|S^{-1}\|_{\mathrm{op}}$ are bounded by a universal constant.

Denoting by $\omega : \mathbb{R}^2 \to [0, \infty)$ the density of ν, we have by the integral formula of F'' given in Proposition 5.2.1 that

$$\begin{aligned}\langle F_\nu''(e_1)e_2, e_2\rangle \ &= \ \int_{\{x\in\mathbb{R}^2 :\ x_1 = 0\}} x_2^2 \ \omega(x) \ dx \\ &= \ \int_{-\infty}^{\infty} t^2 \ \omega(te_2) \ dt \ = \ \int_{-\infty}^{\infty} t^2 \ \frac{\omega_{S_*\nu}(tSe_2)}{|\det S^{-1}|} \ dt\end{aligned}$$

as the density of $S_*\nu$ is given by $\omega_{S_*\nu}(x) = \omega(S^{-1}x)|\det S^{-1}|$. We make the change of variable $t = r/|Se_2|$, and writing $\xi = Se_2/|Se_2|$ we get

$$= \frac{1}{|Se_2|^3|\det S^{-1}|}\int_{-\infty}^{\infty} r^2\ \omega_{S_*\nu}(r\xi)\ dr\ \simeq\ \int_{-\infty}^{\infty} r^2\ \omega_{S_*\nu}(r\xi)\ dr$$

as $\|S\|_{\mathrm{op}}, \|S^{-1}\|_{\mathrm{op}} \simeq 1$. As $S_*\nu$ is isotropic, by Lemma 5.3.3 we are done. □

We are now in position to prove the main proposition of this chapter.

Proof of Proposition 5.1.1. By Proposition 5.2.1, F is C^2-smooth, and by Lemma 5.3.4, $\|F''_S(\theta)\|_{\mathrm{op}} \leq C$ for any $\theta \in S^{n-1}$ where C is a universal constant. We apply Theorem 5.1.3 to F/C to obtain

$$\int_{S^{n-1}} e^{\frac{n-1}{2C}(F-\mathbb{E}F)}\ d\sigma_{n-1}\ \leq\ \exp\left(\frac{n-1}{2C^2}\int_{S^{n-1}} |\nabla_S F|^2\ d\sigma\right)\ \leq\ e^{C'}$$

as $\int |\nabla_S F|^2\ d\sigma\ \leq\ \frac{\tilde{C}}{n}$, by Proposition 2.1.5. Repeating this process with $-F/C$ and since $e^{|x|} \leq e^x + e^{-x}$ we get

$$\int_{S^{n-1}} e^{\frac{n-1}{2C}|F-\mathbb{E}F|}\ d\sigma_{n-1}\ \leq\ 2e^{C'}\ =\ e^{\tilde{C}}$$

for some universal constant $\tilde{C} > 0$. By Jensen's inequality,

$$\int_{S^{n-1}} e^{\frac{n-1}{2C(1+2\tilde{C})}|F-\mathbb{E}F|}\ d\sigma\ \leq\ \left(\int_{S^{n-1}} e^{\frac{n-1}{2C}|F-\mathbb{E}F|} d\sigma\right)^{\frac{1}{1+2\tilde{C}}}\ \leq\ e^{\frac{\tilde{C}}{1+2\tilde{C}}}\ \leq\ \sqrt{e}\ <\ 2.$$

□

5.4 Appendix

5.4.1

The definition is $\|f\|_{\psi_1}\ =\ \inf\left\{r > 0:\ \mathbb{E}\left[e^{|f-\mathbb{E}f|/r}\right] \leq 2\right\}.$

Claim 5.4.1. *Assume that* $\mathbb{E}\left[e^{a|f-\mathbb{E}f|}\right] \leq 2$*. Then for any* $t > 0$*,* $\mathbb{P}\left[|f - \mathbb{E}f| > t\right] \leq 2e^{-at}$*.*

Proof. We use Chebyshev's inequality,

$$\mathbb{P}\left[|f-\mathbb{E}f|>t\right] \;=\; \mathbb{P}\left[e^{a|f-\mathbb{E}f|}>e^{at}\right] \;\leq\; \mathbb{E}\left[e^{a|f-\mathbb{E}f|}\right]e^{-at} \;\leq\; 2e^{-at}.$$

□

5.4.2

Claim 5.4.2. *Assume* $\mathbb{P}\left[|f-\mathbb{E}f|>t\right] \leq 2e^{-cnt^2}$. *Then for any* $p \geq 1$, $\|f\|_p = \left(\mathbb{E}|f|^p\right)^{1/p} \leq \frac{4}{\sqrt{c}}\cdot\frac{\sqrt{p}}{\sqrt{n}}$.

Proof.

$$\mathbb{E}\left[\left(\sqrt{2cn}\,|f-\mathbb{E}f|\right)^p\right] \;=\; \mathbb{E}\left[\int_0^{\sqrt{2cn}|f-\mathbb{E}f|} pt^{p-1}\,dt\right] \;=$$

$$=\; p\int_0^\infty t^{p-1}\mathbb{P}\left[\sqrt{2cn}\,|f-\mathbb{E}f|>t\right]\,dt \;\leq\; 2p\int_0^\infty t^{p-1}e^{-t^2/2}\,dt.$$

Substituting $t=\sqrt{2s}$ we get

$$=\; 2p\int_0^\infty (2s)^{\frac{p-1}{2}}e^{-s}\,ds \;=\; p2^{\frac{p+1}{2}}\Gamma\left(\frac{p+1}{2}\right).$$

Recall that we have already calculated and found $\Gamma\left(\frac{p+1}{2}\right)^{1/p} \leq 2\sqrt{p}$ for $p \geq 1$ (Claim 3.3.1). Hence,

$$\left(\mathbb{E}|f-\mathbb{E}f|^p\right)^{\frac{1}{p}} \;\leq\; \frac{1}{\sqrt{2cn}}\,p^{\frac{1}{p}}\,2^{\frac{p-1}{2p}}\,2\sqrt{p} \;\leq\; \frac{1}{\sqrt{2cn}}\,e^{1/e}\,\sqrt{2}\,2\sqrt{p} \;\leq\; \frac{4\sqrt{p}}{\sqrt{cn}}.$$

□

5.4.3

Claim 5.4.3. *Let* $f:\mathbb{R}^n\to\mathbb{R}$ *be 1-homogeneous and differentiable and let* $\theta\in S^{n-1}$. *Assume that for a symmetric matrix* $A\in M_n$ *with* $A\theta=0$ *we have*

$$f(\theta+h) \;=\; f(\theta)+\langle\nabla f(\theta),h\rangle+\frac{1}{2}\langle Ah,h\rangle+o(|h|^2)$$

as $|h|\to 0$ *and* $h\perp\theta$. *Then* f *is twice-differentiable at* θ *and* $A=f''(\theta)$.

Proof. We show that

$$f(\theta+h) \;=\; f(\theta)+\langle\nabla f(\theta),h\rangle+\frac{1}{2}\langle Ah,h\rangle+o(|h|^2) \qquad \text{as } h\to 0$$

without the constraint of h being perpendicular to θ. For $h\in\mathbb{R}^n$, $|h|<1$ write

$$a \;=\; 1+\langle\theta,h\rangle \qquad ; \qquad u \;=\; \frac{\theta+h}{a}-\theta.$$

Then $u\perp\theta$ and we may expand $f(\theta+u)$ by the assumption of the reduction:

$$\frac{1}{a}f(\theta+h) \;=\; f\left(\frac{\theta+h}{a}\right) \;=\; f(\theta+u) \;\overset{o(|u|^2)}{\approx}\; f(\theta)+\langle\nabla f(\theta),u\rangle+\frac{1}{2}\langle Au,u\rangle$$

$$=\; f(\theta)+\left\langle\nabla f(\theta),\frac{\theta+h}{a}-\theta\right\rangle+\frac{1}{2}\left\langle A\left(\frac{\theta+h}{a}-\theta\right),\frac{\theta+h}{a}-\theta\right\rangle$$

Recall that for 1-homogeneous functions and $\theta\in S^{n-1}$, $\langle\nabla f(\theta),\theta\rangle=f(\theta)$.

$$=\; \frac{1}{a}f(\theta)+\frac{1}{a}\langle\nabla f(\theta),h\rangle+\frac{1}{2}\left\langle A\left(\frac{h}{a}+\left(\frac{1}{a}-1\right)\theta\right),\frac{h}{a}+\left(\frac{1}{a}-1\right)\theta\right\rangle.$$

Hence,

$$f(\theta+h)-f(\theta)-\langle\nabla f(\theta),h\rangle \;\overset{o(|u|^2)}{\approx}\; \frac{1}{2a}\langle A(h+(1-a)\theta),h+(1-a)\theta\rangle \;=$$

$$\frac{1}{2a}\langle Ah,h\rangle+\frac{1-a}{2a}\langle A\theta,h\rangle+\frac{1-a}{2a}\langle Ah,\theta\rangle+\frac{(1-a)^2}{2a}\langle A\theta,\theta\rangle$$

Since $A\theta=0$ and A is symmetric, all the components above but the first vanish.

$$=\; \frac{1}{2a}\langle Ah,h\rangle \;=\; \frac{1}{2(1+\langle h,\theta\rangle)}\langle Ah,h\rangle \;=\; \frac{1}{2}\langle Ah,h\rangle(1+o(1)) \;=$$

$$=\; \frac{1}{2}\langle Ah,h\rangle+o(|h|^2).$$

To conclude we note that $o(|u|^2)=o(|h|^2)$. $\square$

5.4.4

Claim 5.4.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be 1-homogeneous and assume that f is twice-differentiable at $\theta \in S^{n-1}$. Then f is twice-differentiable at $r\theta$ for any $r > 0$ and $f''(r\theta) = \frac{1}{r} f''(\theta)$.*

Proof. We show differentiability by the Taylor expansion.

$$\begin{aligned} f(r\theta+h) &= rf\left(\theta + \frac{h}{r}\right) = rf(\theta)+r\left\langle \nabla f(\theta), \frac{h}{r}\right\rangle+\frac{r}{2}\left\langle f''(\theta)\frac{h}{r}, \frac{h}{r}\right\rangle+o\left(|h|^2\right) \\ &= f(r\theta) + \langle \nabla f(\theta), h\rangle + \frac{1}{2}\left\langle \frac{f''(\theta)}{r} h, h\right\rangle + o(|h|^2) \end{aligned}$$

and note that $\nabla f(\theta) = \nabla f(r\theta)$ when f is 1-homogeneous. □

5.4.5

Claim 5.4.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be (-1)-homogeneous, $x \in \mathbb{R}^n \setminus \{0\}$ such that $f(x + h) \to f(x)$ as $|h| \to 0$ under the assumption that $h \perp x$. Then f is continuous at x.*

Proof. For $h \in \mathbb{R}^n$, write $a = 1 + \frac{\langle h,x\rangle}{|x|^2}$, $u = \frac{x+h}{a} - x$. Then $u \perp x$, $a \to 1$ as $h \to 0$, $u \to 0$ as $h \to 0$ and

$$f(x+h) = \frac{1}{a} f\left(\frac{x+h}{a}\right) = \frac{1}{a} f(x+u) \to f(x) \quad \text{as } h \to 0.$$

□

5.4.6

Proof of Claim 5.3.2. First, see that $\frac{\partial f}{\partial u}(y) = \frac{\partial \tilde{f}}{\partial u}(y)$ for any $y \in E$;

$$\frac{\partial f}{\partial u}(y) = \lim_{t\to 0} \frac{f(y+tu) - f(y)}{t} = \lim_{t\to 0} \frac{\tilde{f}(y+tu) - \tilde{f}(y)}{t} = \frac{\partial \tilde{f}}{\partial u}(y).$$

Then,

$$\begin{aligned} \langle f''(x)u, v\rangle &= \frac{\partial}{\partial v}\frac{\partial f}{\partial u}(x) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial u}(x+tv) - \frac{\partial f}{\partial u}(x)}{t} \\ &= \lim_{t\to 0} \frac{\frac{\partial \tilde{f}}{\partial u}(x+tv) - \frac{\partial \tilde{f}}{\partial u}(x)}{t} = \frac{\partial}{\partial v}\frac{\partial \tilde{f}}{\partial u}(x) = \langle \tilde{f}''(x)u, v\rangle. \end{aligned}$$

□

Chapter 6

A Loose End; Even Stronger Concentration

6.1 Introduction

In this last chapter of the thesis we present an unfinished argument that leads to tighter concentration than was witnessed throughout the previous chapters, for some special case of even log-concave measures. Namely, we prove the following.

Proposition 6.1.1. *Let $n \geq C'$ and let μ be an L^1-isotropic log-concave probability measure on $\mathbb{R}^n$. Assume that μ is even and that for $\beta > 0$*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\langle x, y \rangle^4}{|x|^3 \, |y|^3} \, d\mu(x) \, d\mu(y) \ \leq \ (3 + \beta/n) \, Z_{1,\mu}^2. \tag{6.1}$$

Then $Var(F_\mu) \ \leq \ \frac{C(1+\beta)}{n^3}$. Here, $C, C' > 0$ are universal constants.

It is not immediately clear how to extend the above second-moment bound to include other moments, as in the spirit of Chapter 5. Regarding the integral condition presented in Inequality (6.1), it holds trivially for the standard Gaussian measure and indeed for all other spherically symmetric measures (in all of which $\mathrm{Var}(F) = 0$). It also holds in the discrete (not log-concave) case of the uniform measure on the discrete cube. We suspect that Inequality (6.1) in fact holds in general for unconditional log-concave measures in L^1-isotropic position (note that the L^1-covariance matrix of an unconditional measure is necessarily diagonal due to the constraints imposed by the measure's symmetries).

This stronger concentration is a direct consequence of applying the second-order Poincaré inequality on the sphere, introduced by S.G. Bobkov, G. P. Chistyakov and F. Götze [2]:

Theorem 6.1.2 (Second-order Poincaré inequality)**.** *Let $f : S^{n-1} \to \mathbb{R}$ be C^2-smooth and assume that $\int_{S^{n-1}} \theta f(\theta)\, d\sigma = 0$. Then,*

$$Var(f) \;\leq\; \frac{1}{2n(n+2)} \int_{S^{n-1}} \|f''_S\|^2_{HS}\, d\sigma_{n-1}.$$

In our case, μ being even implies F_μ being even which in turn implies $\int_{S^{n-1}} \theta F_\mu(\theta) d\sigma = 0$, as it is the mean of an odd function on the even sphere. We thus intend to show, assuming the condition (6.1) holds, that

$$\int_{S^{n-1}} \|F''_S\|^2_{\mathrm{HS}}\, d\sigma_{n-1} \;\leq\; C/n$$

where C is some universal constant.

First, we must develop some tools that will be required in the course of our discussion; The first regards some ways of randomizing points on the sphere; The second regards the preservation of the isotropic property under restriction to a hyperplane.

6.2 Randomizing points on the sphere

The next lemma shows that picking a point from the sphere uniformly at random is equivalent to picking any fixed point and rotating it randomly (i.e. via an orthogonal transformation).

Denote by λ_n (or simply λ) the unique Haar probability measure on O_n.

Lemma 6.2.1. *For any measurable $h : S^{n-1} \to \mathbb{R}$ and any $v \in S^{n-1}$,*

$$\int_{O_n} h(Uv)\; d\lambda(U) \;=\; \int_{S^{n-1}} h(\theta)\; d\sigma(\theta).$$

Proof. Fix $v \in S^{n-1}$ and define $T : O_n \to S^{n-1}$ by $T(U) = Uv$. We claim that $T_*\lambda = \sigma$. Indeed, take any measurable set $A \subseteq S^{n-1}$, $W \in O_n$. Then

$$T^{-1}(WA) \;=\; \{\, U \in O_n :\; Uv \in WA \,\} \;=\; \left\{\, U \in O_n :\; W^{-1}Uv \in A \,\right\}$$

$$=\; \{\, W\tilde{U} \in O_n :\; \tilde{U}v \in A \,\} \;=\; W\{\, \tilde{U} \in O_n :\; \tilde{U}v \in A \,\} \;=\; WT^{-1}(A).$$

This means that $T_*\lambda$ is rotation-invariant, by rotation-invariance of λ:

$$T_*\lambda(WA) \;=\; \lambda\left(T^{-1}(WA)\right) \;=\; \lambda\left(WT^{-1}(A)\right) \;=\; \lambda\left(T^{-1}(A)\right) \;=\; T_*\lambda(A).$$

Thus $T_*\lambda$ is indeed the unique rotation invariant probability measure on S^{n-1} and the lemma follows. □

In the next lemma we compare picking two points from the sphere S^{n-1} independently with picking two points in a lower-dimensional subsphere embedded in S^{n-1}, then randomly rotating both of them together (via an orthogonal transformation).

Denote by $S_0^{n-2} = \{\theta \in S^{n-1} : \theta_1 = 0\}$ the embedding of S^{n-2} into the hyperplane $\{x \in \mathbb{R}^n : x_1 = 0\}$, and by $\nu = \lambda \times \sigma_{n-2} \times \sigma_{n-2}$ the product measure on $O_n \times S_0^{n-2} \times S_0^{n-2}$.

Lemma 6.2.2. *Define $T : O_n \times S_0^{n-2} \times S_0^{n-2} \to S^{n-1} \times S^{n-1}$ by*

$$T(U, \theta_1, \theta_2) = (U\theta_1, U\theta_2).$$

Then $T_*\nu$ is absolutely continuous with respect to $\sigma_{n-1} \times \sigma_{n-1}$, and

$$\frac{dT_*\nu}{d(\sigma_{n-1} \times \sigma_{n-1})}\,(\theta_1, \theta_2) \;=\; \frac{\Omega_n}{\sqrt{1 - \langle \theta_1, \theta_2 \rangle^2}}$$

with the normalizing constant being $\Omega_n = \frac{n-2}{2}\left(\Gamma\left(\frac{n-1}{2}\right)/\Gamma\left(\frac{n}{2}\right)\right)^2$.

Proof.

Step 1. $T_*\nu$ is absolutely-continuous with respect to $\sigma_{n-1} \times \sigma_{n-1}$.

Proof. Denote $\tau = \sigma_{n-1} \times \sigma_{n-1}$. It suffices to show that $\tau(M) = 0 \Rightarrow T_*\nu(M) = 0$ for a measurable set $M \in S^{n-1} \times S^{n-1}$. First, we prove for product sets, that is $A \times B$ where $A, B \subseteq S^{n-1}$ are measurable. Assume that $\tau(A \times B) = 0$. By definition $\tau(A \times B) = \sigma_{n-1}(A)\sigma_{n-1}(B)$, so necessarily either $\sigma_{n-1}(A) = 0$ or $\sigma_{n-1}(B) = 0$. Assume that $\sigma_{n-1}(A) = 0$. Then,

$$\begin{aligned} T_*\nu(A \times B) &\leq T_*\nu(A \times S^{n-1}) \\ &= \int_{O_n} \int_{S_0^{n-2}} \int_{S_0^{n-2}} \mathbb{1}_{(U\theta_1, U\theta_2) \in A \times S^{n-1}} \, d\sigma_{n-2}(\theta_1) \, d\sigma_{n-2}(\theta_2) \, d\lambda(U) \\ &= \int_{S_0^{n-2}} \left(\int_{O_n} \mathbb{1}_{U\theta \in A} \, d\lambda(U) \right) d\sigma_{n-2}(\theta) \\ &= \int_{S_0^{n-2}} \left(\int_{S^{n-1}} \mathbb{1}_{\eta \in A} \, d\sigma_{n-1}(\eta) \right) d\sigma_{n-2}(\theta) = \sigma_{n-1}(A) = 0. \end{aligned}$$

We used the correspondence $\lambda \leftrightarrow \sigma_{n-1}$ stated in Lemma 6.2.1. Moving on, the product σ-algebra of $S^{n-1} \times S^{n-1}$ is generated by such products $A \times B$, $A, B \subseteq S^{n-1}$ measurable. The statement $\tau(M) = 0 \Rightarrow T_*\nu(M) = 0$ extends to countable unions of product sets hence their complements, and thus applies to the entire product σ-algebra. □

Step 2. $\frac{dT_*\nu}{d\tau}$ is invariant under rotation. That is, $\frac{dT_*\nu}{d\tau}(\theta_1, \theta_2) = \frac{dT_*\nu}{d\tau}(W\theta_1, W\theta_2)$ for any $W \in O_n$.

Proof. First, we show that $T_*\nu(A \times B) = T_*\nu(WA \times WB)$ for any measurable sets $A, B \in S^{n-1}$ and any $W \in O_n$. Indeed,

$$\begin{aligned} T^{-1}(WA \times WB) &= \left\{ (U, \theta_1, \theta_2) : \; U\theta_1 \in WA, \; U\theta_2 \in WB \right\} \\ &= \left\{ (U, \theta_1, \theta_2) : \; W^{-1}U\theta_1 \in A, \; W^{-1}U\theta_2 \in B \right\} \\ &= \left\{ (W\tilde{U}, \theta_1, \theta_2) : \; \tilde{U}\theta_1 \in A, \; \tilde{U}\theta_2 \in B \right\} \\ &= \left\{ (W\tilde{U}, \theta_1, \theta_2) : \; (\tilde{U}, \theta_1, \theta_2) \in T^{-1}(A \times B) \right\} \end{aligned}$$

and what we require follows by rotation-invariance of λ. Second, fixing $W \in$

O_n it holds for any measurable sets $A, B \in S^{n-1}$ that

$$\begin{aligned}
\int_{A\times B} \frac{dT_*\nu}{d\tau}(\theta_1,\theta_2)\ d\tau(\theta_1,\theta_2) &= \int_{A\times B} dT_*\nu(\theta_1,\theta_2) \\
&= \int_{WA\times WB} dT_*\nu(\theta_1,\theta_2) \\
&= \int_{WA\times WB} \frac{dT_*\nu}{d\tau}(\theta_1,\theta_2)\ d\tau(\theta_1,\theta_2) \\
&= \int_{A\times B} \frac{dT_*\nu}{d\tau}(W\theta_1,W\theta_2)\ d\tau(\theta_1,\theta_2)
\end{aligned}$$

and the claim in Step 2 follows. □

Step 3. $\frac{dT*\nu}{d\tau}(\theta_1,\theta_2) = \frac{\Omega_n}{\sqrt{1-\langle\theta_1,\theta_2\rangle^2}}$ with $\Omega_n = \frac{n-2}{2}\left(\Gamma\left(\frac{n-1}{2}\right)/\Gamma\left(\frac{n}{2}\right)\right)^2$.

Proof. By Step 2 we have that $\frac{dT_*\nu}{d\tau}(\theta_1,\theta_2) = f\left(\langle\theta_1,\theta_2\rangle\right)$ for some $f : [-1,1] \to [0,\infty]$. For $g : [-1,1] \to [0,\infty]$, on the one hand:

$$\int_{S^{n-1}\times S^{n-1}} g(\langle\theta_1,\theta_2\rangle)\ dT_*\nu(\theta_1,\theta_2) = \iint_{S^{n-1}\times S^{n-1}} g(\langle\theta_1,\theta_2\rangle)\ f(\langle\theta_1,\theta_2\rangle)\ d\tau(\theta_1,\theta_2)$$

The distribution of $\langle\theta_1,\theta_2\rangle$ on $[-1,1]$ is just the marginal distribution of unit sphere, which is well known to be proportional to $(1-t^2)^{\frac{n-3}{2}}$. The normalizing constant is thus $\omega_n := \Gamma\left(\frac{n}{2}\right)/\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)$. Writing $h_n(t) = \omega_n\left(1-t^2\right)^{\frac{n-3}{2}}$ the above integral equals $\int_{-1}^1 g(t)f(t)h_n(t)\ dt$. On the other hand,

$$\begin{aligned}
&\int_{S^{n-1}\times S^{n-1}} g(\langle\theta_1,\theta_2\rangle)\ dT_*\nu(\theta_1,\theta_2) \\
&\qquad= \int_{O_n}\int_{S_0^{n-2}}\int_{S_0^{n-2}} g(\langle U\theta_1,U\theta_2\rangle)\ d\sigma_{n-2}(\theta_1)\ d\sigma_{n-2}(\theta_2)\ d\lambda(U) \\
&\qquad= \int_{O_n}\left(\int_{S_0^{n-2}}\int_{S_0^{n-2}} g(\langle\theta_1,\theta_2\rangle)\ d\sigma_{n-2}(\theta_1)\ d\sigma_{n-2}(\theta_2)\right)\ d\lambda(U) \\
&\qquad= \int_{-1}^1 g(t)\ h_{n-1}(t)\ dt.
\end{aligned}$$

We see that necessarily $f(t)h_n(t) = h_{n-1}(t) \ \ \forall t \in [-1,1]$, that is

$$f(t) \ = \ \frac{\omega_{n-1}(1-t^2)^{\frac{n-4}{2}}}{\omega_n(1-t^2)^{\frac{n-3}{2}}} \ = \ \frac{\Omega_n}{\sqrt{1-t^2}} \quad ; \quad \Omega_n \ = \ \frac{\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}\right)^2}{\Gamma\left(\frac{n}{2}\right)^2}.$$

□

This concludes the proof of Lemma 6.2.2. □

6.3 Sections of a log-concave density

The next lemma shows that a section of an even log-concave density retains the isotropic property, up to a universal constant.

Lemma 6.3.1. *Let $\rho : \mathbb{R}^n \to [0,\infty)$ be an isotropic even log-concave density and let $\eta \in S^{n-1}$. Then for any $\xi \perp \eta$,*

$$c\,|\xi|^2 \ \le \ \int_{\eta^\perp} \langle x,\xi\rangle^2 \rho(x)\ dx \ \le \ C\,|\xi|^2$$

where $c, C > 0$ are universal constants.

This lemma was proven for symmetric convex bodies by V.D. Milman and A. Pajor in their foundational paper [11]. We generalize the result to log-concave measures by translating the measure to a convex body using the approach developed by K. Ball, then applying Milman & Pajor's result.

Proof. Define the following convex body:

$$K \ = \ K_{n+2}(\rho) \ = \ \left\{ x \in \mathbb{R}^n : \ \int_0^\infty \rho(tx)t^{n+1}\ dt \ \ge \ \frac{\rho(0)}{n+2} \right\}.$$

We refer to [4] for a detailed description of the convex bodies associated with log-concave functions and their properties. Since ρ is even K is symmetric hence also centered. Normalizing $\tilde{K} = \mathrm{Vol}_n(K)^{-1/n}K$ we have that $\tilde{K}$ is an isotropic convex body; Indeed, writing $a = \mathrm{Vol}_n(K)^{-1/n}$ we have for $\xi \in S^{n-1}$

that

$$
\begin{aligned}
a^{-n-2}\int_{\tilde{K}} \langle x,\xi\rangle^2\,dx &= \int_K \langle x,\xi\rangle^2\,dx \\
&= n\kappa_n \int_{S^{n-1}}\int_{r=0}^{\infty} \langle r\theta,\xi\rangle^2\, \mathbb{1}_K(r\theta) r^{n-1} dr\ d\sigma(\theta) \\
&= n\kappa_n \int_{S^{n-1}} \langle \theta,\xi\rangle^2 \left(\int_{r=0}^{\max\{\tilde{r}:\ \tilde{r}\theta\in K\}} r^{n+1} dr\right) d\sigma(\theta) \\
&= \frac{n\kappa_n}{\rho(0)}\int_{S^{n-1}}\int_{r=0}^{\infty} \langle r\theta,\xi\rangle^2\,\rho(r\theta) r^{n-1}\ dr\ d\sigma(\theta) \\
&= \frac{1}{\rho(0)}\int_{\mathbb{R}^n} \langle x,\xi\rangle^2\,\rho(x)\ dx \;=\; \frac{1}{\rho(0)}
\end{aligned}
$$

with the last transition due to ρ being isotropic and the rest by definition of K and polar integration back and forth. Hence $\tilde{K}$ is isotropic with $L_K^2 = a^{n+2}/\rho(0)$. By Milman & Pajor [11] it holds for any symmetric convex body T and any $\xi \in S^{n-1}$ that

$$
\int_T |\langle x,\xi\rangle| \frac{dx}{\mathrm{Vol}_n(T)} \;\simeq\; \frac{\mathrm{Vol}_n(T)}{\mathrm{Vol}_{n-1}(T\cap\xi^\perp)}.
$$

Fixing $\eta \in S^{n-1}$ we apply the above to $K\cap\eta^\perp$ and a unit vector $\xi \perp \eta$ to obtain

$$
I \;:=\; \int_{K\cap\eta^\perp} |\langle x,\xi\rangle| \frac{dx}{\mathrm{Vol}_{n-1}(K\cap\eta^\perp)} \;\simeq\; \frac{\mathrm{Vol}_{n-1}(K\cap\eta^\perp)}{\mathrm{Vol}_{n-2}(K\cap\eta^\perp\cap\xi^\perp)}.
$$

Again by Milman & Pajor [11], this time utilizing the fact that $\tilde{K}$ is isotropic we have that $\mathrm{Vol}_{n-1}(\tilde{K}\cap\eta^\perp) \simeq L_K^{-1}$ and $\mathrm{Vol}_{n-2}(\tilde{K}\cap\eta^\perp\cap\xi^\perp) \simeq L_K^{-2}$, hence

$$
\mathrm{Vol}_{n-1}(K\cap\eta^\perp) \;=\; a^{-n+1}\mathrm{Vol}_{n-1}(\tilde{K}\cap\eta^\perp) \;\simeq\; a^{-n+1}L_K^{-1}
$$

$$
\mathrm{Vol}_{n-2}(K\cap\eta^\perp\cap\xi^\perp) \;=\; a^{-n+2}\mathrm{Vol}_{n-2}(\tilde{K}\cap\eta^\perp\cap\xi^\perp) \;\simeq\; a^{-n+2}L_K^{-2},
$$

thus $I \simeq L_K/a$. We now relate the value of the integral I to a moment on a section of ρ via the association between ρ and K. Due to the choice of the parameter $n+2$ in the definition of $K = K_{n+2}(\rho)$, this association is manifested through the third moment on a section. To utilize the third moment, we mainly use the equivalence of moments of a log-concave function (Theorem 4.4.2). On the one hand,

$$\int_{K\cap\eta^\perp} |\langle x,\xi\rangle|^3 \frac{dx}{\mathrm{Vol}_{n-1}(K\cap\eta^\perp)} \;\simeq\; \left(\int_{K\cap\eta^\perp} |\langle x,\xi\rangle| \frac{dx}{\mathrm{Vol}_{n-1}(K\cap\eta^\perp)}\right)^3 \;\simeq\; \frac{L_K^3}{a^3}.$$

On the other hand,

$$\begin{aligned}
\int_{K\cap\eta^\perp} |\langle x,\xi\rangle|^3\, dx \;&=\; (n-1)\kappa_{n-1}\int_{S^{n-1}\cap\eta^\perp}\int_{r=0}^{\infty} |\langle r\theta,\xi\rangle|^3\, \mathbb{1}_K(r\theta) r^{n-2} dr\; d\sigma(\theta)\\
&=\; (n-1)\kappa_{n-1}\int_{S^{n-1}\cap\eta^\perp} |\langle\theta,\xi\rangle|^3 \left(\int_{r=0}^{\max\{\tilde r:\ \tilde r\theta\in K\}} r^{n+1} dr\right) d\sigma(\theta)\\
&=\; \frac{(n-1)\kappa_{n-1}}{\rho(0)}\int_{S^{n-1}\cap\eta^\perp}\int_{r=0}^{\infty} |\langle r\theta,\xi\rangle|^3\, \rho(r\theta) r^{n-2} dr\; d\sigma(\theta)\\
&=\; \frac{1}{\rho(0)}\int_{\eta^\perp} |\langle x,\xi\rangle|^3 \rho(x)\; dx \;\simeq\; \frac{1}{\rho(0)}\left(\int_{\eta^\perp} \langle x,\xi\rangle^2 \rho(x)\; dx\right)^{3/2}.
\end{aligned}$$

Recalling that $\mathrm{Vol}_{n-1}(K\cap\eta^\perp) \simeq a^{-n+1}L_K^{-1}$ and $L_K^2 = a^{n+2}/\rho(0)$ we may combine both the above to finally obtain

$$\left(\int_{\eta^\perp} \langle x,\xi\rangle^2 \rho(x)\; dx\right)^{3/2} \;\simeq\; \frac{\rho(0)L_K^2}{a^{-n-2}} \;=\; 1.$$

□

We also require a result about the volume of a section.

Lemma 6.3.2. *Let $\rho:\mathbb{R}^n\to[0,\infty)$ be an isotropic even log-concave density and let $\eta\in S^{n-1}$. Then,*

$$c \;\le\; \int_{\eta^\perp} \rho(x)\; dx \;\le\; C$$

where $c, C > 0$ are universal constants.

Proof. Define $\omega:\mathbb{R}\to[0,\infty)$ to be the marginal density of ρ in direction η, that is

$$\omega(t) \;=\; \int_{\langle x,\eta\rangle = t} \rho(x)\; dx.$$

By the Prékopa-Leindler inequality, ω is log-concave. For the lower bound, we may apply Lemma 2.2.4 from [4] to $\omega|_{[0,\infty)}$ obtaining

$$\frac{1}{\|\omega\|_\infty}\int_0^\infty \omega(t)\,dt \;\le\; \left(\frac{3}{\|\omega\|_\infty}\int_0^\infty t^2\omega(t)\,dt\right)^{1/3}$$

hence

$$\int_0^\infty t^2\omega(t)\,dt \;\ge\; \frac{1}{3\,\|\omega\|_\infty^2}\left(\int_0^\infty \omega(t)\,dt\right)^3.$$

Applying the same to $\omega|_{(-\infty,0]}$ and noting that $b^3/4 \le a^3 + (b-a)^3 \le b^3$ for $0 \le a \le b$ we may combine both results to get

$$\begin{aligned}
1 \;&=\; \int_{\mathbb{R}^n} \langle x,\eta\rangle^2 \rho(x)\,dx \;=\; \int_{-\infty}^\infty t^2\omega(t)\,dt \;\ge \\
&\ge\; \frac{1}{3\,\|\omega\|_\infty^2}\left(\left(\int_0^\infty \omega(t)\,dt\right)^3 + \left(\int_{-\infty}^0 \omega(t)\,dt\right)^3\right) \;\ge \\
&\ge\; \frac{1}{12\,\|\omega\|_\infty^2}\left(\int_{-\infty}^\infty \omega(t)\,dt\right)^3 \;\ge\; \frac{1}{12\,\|\omega\|_\infty^2}\left(\int_{\mathbb{R}^n} \rho(x)\,dx\right)^3 \;=\; \frac{1}{12\,\|\omega\|_\infty^2}.
\end{aligned}$$

Note that ρ being even implies that ω is even as well. Having that ω is even and log-concave, it holds that $\omega(0) = \|\omega\|_\infty$. We thus established that

$$\int_{\eta^\perp} \rho(x)\,dx \;=\; \omega(0) \;=\; \|\omega\|_\infty \;\ge\; 1/\sqrt{12}.$$

To obtain an upper bound, we apply the inverse Theorem 2.2.3 from [4] in a similar fashion:

$$\int_{-\infty}^\infty t^2\omega(t)\,dt \;\le\; \frac{2}{\omega(0)^2}\left(\int_{-\infty}^\infty \omega(t)\,dt\right)^3$$

and hence $\omega(0) \le \sqrt{2}$. In establishing the upper bound we did not require ω to be even. □

We now return to our ultimate goal; bounding $\int \|F_S''\|_{\mathrm{HS}}^2\,d\sigma$. We first deal in general with 1-homogeneous functions.

6.4 Derivatives of a homogeneous function

Claim 6.4.1. *Let $f : S^{n-1} \to \mathbb{R}$ be C^2-smooth and 1-homogeneous. Then* $\|f''_S\|^2_{HS} = \|f''\|^2_{HS} + (n-1)f^2 - 2f\Delta f$.

Here, $\Delta f = \mathrm{Tr} f''$ is the ordinary Laplacian of f.

Proof. Let $\theta \in S^{n-1}$. The Hilbert-Schmidt norm and the Laplacian are both indifferent to the selection of the orthonormal basis. Choose a basis $e_1, ..., e_n$ such that $e_1 = \theta$. All this to say we may assume $\theta = e_1$.
As f is 1-homogeneous it holds that $f''(\theta)\theta = 0$ (see, e.g. [2]), and since $\theta = e_1$ this means that the first column of $f''(\theta)$ vanishes. By symmetry of $f''(\theta)$ so does the first row. Now, as $f''_S(\theta) = P_{\theta^\perp}\left(f''(\theta) - \langle \nabla_S f(\theta), \theta\rangle Id\right) P_{\theta^\perp}$, the same applies to $f''_S(\theta)$ as well. We may thus easily calculate:

$$\|f''_S(\theta)\|^2_{\mathrm{HS}} = \sum_{i,j=2}^{n} f''_S(\theta)^2_{ij} = \sum_{i\neq j,\ i,j\geq 2} f''(\theta)^2_{ij} + \sum_{i=2}^{n}\Big(f''(\theta)_{ii} - f(\theta)\Big)^2$$

$$= \sum_{i\neq j,\ i,j\geq 2} f''(\theta)^2_{ij} + \sum_{i=2}^{n} f''(\theta)^2_{ii} + \sum_{i=2}^{n} f(\theta)^2 - 2f(\theta)\sum_{i=2}^{n} f''(\theta)_{ii}$$

$$= \sum_{i,j=2}^{n} f''(\theta)^2_{ij} + (n-1)f(\theta)^2 - 2f(\theta)\sum_{i=2}^{n} f''(\theta)_{ii}$$

$$= \|f''(\theta)\|^2_{\mathrm{HS}} + (n-1)f(\theta)^2 - 2f(\theta)\Delta f(\theta).$$

□

This leads to the following integral calculation:

Claim 6.4.2. *Let $f : S^{n-1} \to \mathbb{R}$ be C^2-smooth and 1-homogeneous. Then,* $\int \|f''_S\|^2_{HS}\, d\sigma = \int \|f''\|^2_{HS}\, d\sigma - (n-1)\left(\int f\ d\sigma\right)^2 - (n-1)\,Var(f) + 2\int |\nabla_S f|^2 d\sigma$.

Proof. Starting from Claim 6.4.1 we have

$$\int \|f''_S\|^2_{\mathrm{HS}}\, d\sigma = \int \|f''\|^2_{\mathrm{HS}}\, d\sigma + (n-1)\int f^2\ d\sigma - 2\int f\Delta f\ d\sigma.$$

We tend to the integral component involving the Laplacian. An explicit calculation (see [2]) gives off:

$$\Delta f(\theta) = \Delta_S f(\theta) + (n-1)\langle \nabla f(\theta), \theta\rangle + \langle f''(\theta)\theta, \theta\rangle.$$

Here, $\Delta_S f = \mathrm{Tr} f''_S$ is the spherical Laplacian of f. Recall that in our case of f being 1-homogeneous, $\langle \nabla f(\theta), \theta\rangle = f(\theta)$, and $\langle f''(\theta)\theta, \theta\rangle = 0$. We thus get:

$$\int f\Delta f \; d\sigma \;=\; \int f\Delta_S f \; d\sigma + \;(n-1)\int f^2 \; d\sigma.$$

It is well known for smooth spherical functions that $\int f\Delta_S f \; d\sigma \;=\; -\int |\nabla_S f|^2 d\sigma$ (see, e.g. [2]), hence combining all the above we obtain

$$\int \|f''_S\|^2_{\mathrm{HS}} \, d\sigma \;=\; \int \|f''\|^2_{\mathrm{HS}} \, d\sigma \;-\; (n-1)\int f^2 \; d\sigma \;+\; 2\int |\nabla_S f|^2 d\sigma.$$

By decomposing $\int f^2 \;=\; \mathrm{Var}(f) \;+\; \left(\int f\right)^2$ we finally arrive at the required:

$$\int \|f''_S\|^2_{\mathrm{HS}} \;=\; \int \|f''\|^2_{\mathrm{HS}} \;-\; (n-1)\left(\int f\right)^2 \;-\; (n-1)\mathrm{Var}(f) \;+\; 2\int |\nabla_S f|^2.$$

$\square$

At this point we note that the values $\int f$, $\mathrm{Var}(f)$ and $\int |\nabla_S f|^2$ we have already calculated in Chapter 2, with respect to F_μ. We will recall the specific values in due course. Consequently the main obstacle that remains is calculating the integral $\int \|F''\|^2_{\mathrm{HS}} \, d\sigma$.

6.5 Calculating the final integral

Writing $\rho : \mathbb{R}^n \to [0, \infty)$ for the density of μ we have

$$F''(\theta) \;=\; \int_{\theta^\perp} x \otimes x \; \rho(x) \; dx$$

(recall Proposition 5.2.1), from which a simple calculation yields

$$\int_{S^{n-1}} \|F''(\theta)\|^2_{\mathrm{HS}} \; d\sigma(\theta) \;=\; \int_{S^{n-1}} \int_{x \perp \theta} \int_{y \perp \theta} \langle x, y\rangle^2 \;\; \rho(x)\rho(y) \; dx \; dy \; d\sigma(\theta).$$

For reasons that will be clarified as we progress, it is crucial for us to drop from the integral those regions in which x and y are close to being linearly dependent. The following claim assures that we are indeed able to do so.

Claim 6.5.1.
$$\int_{S^{n-1}} \iint_{x,y\perp\theta} \langle x,y\rangle^2 \mathbb{1}_{\left|\left\langle \frac{x}{|x|},\frac{y}{|y|}\right\rangle\right|>\frac{1}{2}} \rho(x)\rho(y)\ dx\ dy\ d\sigma(\theta) \ =\ \mathcal{O}(1/n)\,.$$

From here on, the symbol $\mathcal{O}(\cdot)$ stands for a value that is at most a universal constant times the expression in the parentheses.

Proof. Denoting $\eta = y/|y|$, we first consider the inner integral:

$$\int_{\theta^\perp} \langle x,\eta\rangle^2 \mathbb{1}_{|\langle x,\eta\rangle|>|x|/2}\, \rho(x)\ dx \ \le\ \sqrt{\int_{\theta^\perp} \langle x,\eta\rangle^4 \rho(x) dx}\sqrt{\int_{\theta^\perp} \mathbb{1}_{|\langle x,\eta\rangle|>|x|/2}\ \rho(x)dx}.$$

Take $T \in GL_n$ such that $T_*\mu$ is isotropic. Then $\|T\|_{\mathrm{op}}, \|T^{-1}\|_{\mathrm{op}} \simeq 1$. The left hand side square root above is

$$\begin{aligned}\sqrt{\int_{\theta^\perp} \langle x,\eta\rangle^4 \rho(x)\ dx} \ &\simeq\ \sqrt{\int_{(T^{-*}\theta)^\perp} \langle x,T^{-*}\eta\rangle^4 \rho(T^{-1}x)\,|\det T^{-1}|\,dx} \\ &=\ \sqrt{\int_{\tilde\theta^\perp} \langle x,T^{-*}\eta\rangle^4 \rho_{T_*\mu(x)}\ dx} \\ &\simeq\ \int_{\tilde\theta^\perp} \left\langle x,T^{-*}\eta\right\rangle^2 \rho_{T_*\mu}(x)\ dx \ \lesssim\ \left|T^{-*}\eta\right|^2 \ \simeq\ 1\end{aligned}$$

by a change of variable, then by the fact that the density of $T_*\mu$ is $\rho_{T_*\mu} = |\det T^{-1}|\,\rho\circ T^{-1}$, then by the reverse Hölder property, then by Lemma 6.3.1 (note that the density $\rho_{T_*\mu}$ is even), then because $\|T^{-1}\|_{\mathrm{op}} \simeq 1$. Though $\rho_{T_*\mu}|_{\tilde\theta^\perp}$ is not normalized to be a probability density, we may omit the normalization constant $\int_{\tilde\theta^\perp}\rho_{T_*\mu}$ as it is of the order of magnitude of a constant, by Lemma 6.3.2. As to the right hand side square root from before,

$$\begin{aligned}\int_{\theta^\perp} \mathbb{1}_{|\langle x,\eta\rangle|>|x|/2}\ \rho(x)\ dx \ &\le\ 2^8 \int_{\theta^\perp} \frac{\langle x,\eta\rangle^8}{|x|^8}\ \rho(x)\ dx \\ &\lesssim\ \sqrt{\int_{\theta^\perp} \langle x,\eta\rangle^{16} \rho(x)\ dx}\sqrt{\int_{\theta^\perp} \frac{1}{|x|^{16}}\ \rho(x)\ dx}.\end{aligned}$$

The left hand side square root is treated similarly to before, and is of the order of magnitude of a constant. Tending to the right hand side root,

$$\int_{\theta^\perp} \frac{1}{|x|^{16}}\ \rho(x)\ dx \ \simeq\ \int_{\tilde\theta^\perp} \frac{1}{|x|^{16}}\ \rho_{T_*\mu}(x)\ dx \ \simeq\ \left(\int_{\tilde\theta^\perp} |x|^2 \rho_{T_*\mu}(x)\ dx\right)^{-8} \ \simeq\ n^{-8}.$$

We have used Paouris' theorem (Theorem 4.4.3) then Lemma 6.3.1 $n-1$ times. One may notice that the version of Paouris' theorem we have included in Chapter 4 demands the density to be isotropic, while $\rho_{T_*\mu}|_{\tilde{\theta}^\perp}$ is not so. The original, more comprehensive version of the theorem (see Section 5.3 in [4]) allows for the qualitatively the same result under the more relaxed assumption

$$\int_{\tilde{\theta}^\perp} \langle x, \xi \rangle^2 \frac{\rho_{T_*\mu}(x)}{\int_{\tilde{\theta}^\perp} \rho_{T_*\mu}} \, dx \;\simeq\; |\xi|^2$$

which follows from Lemmas 6.3.1 and 6.3.2. Thus we have established that

$$\int_{\theta^\perp} \langle x, \eta \rangle^2 \, \mathbb{1}_{|\langle x,\eta\rangle| > |x|/2} \, \rho(x) \, dx \;=\; \mathcal{O}(1/n^2)$$

for $\eta \perp \theta$, $|\eta| = 1$. Plugging this into the triple integral we seek to bound,

$$\int_{S^{n-1}} \iint_{x,y\perp\theta} \langle x, y \rangle^2 \mathbb{1}_{|\langle \frac{x}{|x|}, \frac{y}{|y|} \rangle| > \frac{1}{2}} \, \rho(x)\rho(y) \, dx \, dy \, d\sigma(\theta)$$
$$\lesssim \; \frac{1}{n^2} \int_{S^{n-1}} \left(\int_{\theta^\perp} |y|^2 \, \rho(y) \, dy \right) d\sigma(\theta) \;\simeq\; \frac{1}{n^2} \cdot n \;=\; \frac{1}{n}.$$

by applying Lemma 6.3.1 $n-1$ times. □

By now we have

$$\int_{S^{n-1}} \|F''(\theta)\|^2_{\mathrm{HS}} d\sigma(\theta) \;=\; \mathcal{O}(1/n)$$
$$+ \int_{S^{n-1}} \iint_{x,y\perp\theta} \langle x, y \rangle^2 \, \mathbb{1}_{|\langle \frac{x}{|x|}, \frac{y}{|y|} \rangle| < \frac{1}{2}} \, \rho(x)\rho(y) \, dx \, dy \, d\sigma(\theta)$$

Forgetting the $\mathcal{O}(1/n)$ component, we interchange integration over S^{n-1} with integration over O_n using Claim 6.2.1,

$$= \int_{O_n} \int_{x,y\perp e_1} \langle Ux, Uy \rangle^2 \, \mathbb{1}_{|\langle \frac{Ux}{|Ux|}, \frac{Uy}{|Uy|} \rangle| < \frac{1}{2}} \, \rho(Ux)\rho(Uy) \, dx \, dy \, d\lambda(U)$$

Using polar integration on the hyperplanes $\{x \perp e_1\}, \{y \perp e_1\}$,

$$= \; (n-1)^2 \kappa_{n-1}^2 \int_{O_n} \iint_{r_1,r_2=0}^{\infty} \iint_{\theta_1,\theta_2 \in S_0^{n-2}} r_1^n r_2^n \; \langle U\theta_1, U\theta_2 \rangle^2 \, \mathbb{1}_{|\langle U\theta_1, U\theta_2\rangle| < \frac{1}{2}}$$
$$\rho(r_1 U\theta_1)\rho(r_2 U\theta_2) \, dr_1 \, dr_2 \, d\sigma_{n-2}(\theta_1) \, d\sigma_{n-2}(\theta_2) \, d\lambda(U),$$

where we've embedded $S^{n-2} \hookrightarrow \mathbb{R}^n$ by $S_0^{n-2} = \{x : \ |x| = 1, x_1 = 0\}$.

$$= \ (n-1)^2 \kappa_{n-1}^2 \iint_{r_1, r_2 = 0}^{\infty} r_1^n r_2^n \Bigg(\int_{O_n} \iint_{\theta_1, \theta_2 \in S_0^{n-2}} \langle U\theta_1, U\theta_2 \rangle^2 \, \mathbb{1}_{|\langle U\theta_1, U\theta_2\rangle| < \frac{1}{2}} \, \rho(r_1 U\theta_1) \rho(r_2 U\theta_2) \ d\sigma_{n-2}(\theta_1) \ d\sigma_{n-2}(\theta_2) \ d\lambda(U) \Bigg) \ dr_1 \ dr_2$$

The distribution of $U\theta_1, U\theta_2$ over $S^{n-1} \times S^{n-1}$ is similar to $\sigma_{n-1} \times \sigma_{n-1}$ but with greater emphasis on closer points, as was established in Lemma 6.2.2.

$$= \ (n-1)^2 \kappa_{n-1}^2 \iint_{r_1, r_2 = 0}^{\infty} r_1^n r_2^n \Bigg(\iint_{S^{n-1} \times S^{n-1}} \langle \theta_1, \theta_2 \rangle^2 \, \mathbb{1}_{|\langle \theta_1, \theta_2 \rangle| < \frac{1}{2}} \, \frac{\Omega_n}{\sqrt{1 - \langle \theta_1, \theta_2 \rangle^2}} \ \rho(r_1 \theta_1) \rho(r_2 \theta_2) \ d\sigma_{n-1}(\theta_1) \ d\sigma_{n-1}(\theta_2) \Bigg) dr_1 dr_2$$

Using reverse polar integration over $\mathbb{R}^n$, twice, we reach:

$$= \ \frac{\Omega_n (n-1)^2 \kappa_{n-1}^2}{n^2 \kappa_n^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle x, y \rangle^2 \ \frac{1}{\sqrt{1 - \left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle^2}} \ \mathbb{1}_{\left|\left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle\right| < \frac{1}{2}} \ \frac{d\mu(x)}{|x|} \ \frac{d\mu(y)}{|y|}$$

with

$$\Omega_n \ = \ \frac{\left(\frac{n-2}{2}\right) \Gamma\left(\frac{n-1}{2}\right)^2}{\Gamma\left(\frac{n}{2}\right)^2} \qquad ; \qquad \kappa_n \ = \ \frac{\pi^{\frac{n}{2}}}{\frac{n}{2} \Gamma\left(\frac{n}{2}\right)}.$$

One may calculate and see that the coefficient is:

$$\frac{\Omega_n (n-1)^2 \kappa_{n-1}^2}{n^2 \kappa_n^2} \ = \ \frac{n}{2\pi} \ - \ \frac{1}{\pi}.$$

As for the integral, we expand $t \mapsto 1/\sqrt{1-t^2}$ into a power series around 0. Here we rely heavily on the fact that $|t| < 1/2$. Then,

$$\iint \langle x, y \rangle^2 \ \frac{1}{\sqrt{1 - \left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle^2}} \ \mathbb{1}_{\left|\left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle\right| < \frac{1}{2}} \ \frac{d\mu(x)}{|x|} \ \frac{d\mu(y)}{|y|} \ =$$

$$= \ \iint \frac{\langle x, y \rangle^2}{|x||y|} \left(1 \ + \ \frac{\langle x, y \rangle^2}{2|x|^2|y|^2} \ + \ \mathcal{O}\left(\frac{\langle x, y \rangle^4}{|x|^4 |y|^4} \right) \right) \ \mathbb{1}_{\left|\left\langle \frac{x}{|x|}, \frac{y}{|y|} \right\rangle\right| < \frac{1}{2}} \ d\mu(x) \ d\mu(y)$$

$$\leq \ \iint \frac{\langle x, y \rangle^2}{|x||y|} \ d\mu(x) \ d\mu(y) \ + \ \iint \frac{\langle x, y \rangle^4}{2|x|^3|y|^3} \ d\mu(x) \ d\mu(x) \ + \ \mathcal{O}\Bigg(\iint \frac{\langle x, y \rangle^6}{|x|^5 |y|^5} \ d\mu(x) \ d\mu(y) \Bigg).$$

Based on calculations done in the previous chapters the first integral equals $nZ_{1,\mu}^2$ (see proof of Proposition 2.1.5) and the third is $\mathcal{O}\left(n^{-2}\right)$ (similar to the integral bounded in the proof of Proposition 4.1.3). We arrive at:

$$\begin{aligned}\int_{S^{n-1}} \|F''(\theta)\|_{\mathrm{HS}}^2 \ d\sigma(\theta) &= \left(\frac{n}{2\pi} - \frac{1}{\pi}\right)\left(nZ_{1,\mu}^2 + \frac{1}{2}\iint \frac{\langle x,y\rangle^4}{|x|^3|y|^3} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \\ &= \frac{n^2 Z_{1,\mu}^2}{2\pi} - \frac{nZ_{1,\mu}^2}{\pi} + \frac{n}{4\pi}\iint \frac{\langle x,y\rangle^4}{|x|^3|y|^3} + \mathcal{O}\Big(\frac{1}{n}\Big)\end{aligned}$$

as $\iint \frac{\langle x,y\rangle^4}{|x|^3|y|^3} = \mathcal{O}\left(\frac{1}{n}\right)$. Recall Claim 6.4.2:

$$\int \|F_S''\|_{\mathrm{HS}}^2 = \int \|F''\|_{\mathrm{HS}}^2 - (n-1)\left(\int F\right)^2 - (n-1)\mathrm{Var}(F) + 2\int |\nabla_S F|^2.$$

We remember that $\mathrm{Var}(F) = \mathcal{O}\left(\frac{1}{n^2}\right)$ and $\int |\nabla_S F|^2 = \mathcal{O}\left(\frac{1}{n}\right)$ (see Corollary 4.1.4 and the proof of Proposition 2.1.2 via Proposition 2.1.5). Moreover, as we have calculated before (see the proof of Proposition 2.1.2 in the general setting, appearing in Section 2.4):

$$\begin{aligned}\left(\int F \ d\sigma\right)^2 &= \frac{C_{n,1}^{-2}}{2\pi}\left(\int_{\mathbb{R}^n} |x| \ d\mu\right)^2 = \frac{n^2 Z_{1,\mu}^2}{2\pi C_{n,1}^2} \\ &= \frac{n^2 Z_{1,\mu}^2}{2\pi}\left(\frac{1}{n} + \frac{1}{2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right) = \frac{nZ_{1,\mu}^2}{2\pi} + \frac{Z_{1,\mu}^2}{4\pi} + \mathcal{O}\Big(\frac{1}{n^2}\Big).\end{aligned}$$

We finally arrive at:

Corollary 6.5.2. $\int \|F_S''\|_{HS}^2 = \frac{n}{4\pi}\left(\iint \frac{\langle x,y\rangle^4}{|x|^3|y|^3} - 3Z_{1,\mu}^2\right) + \mathcal{O}\left(\frac{1}{n}\right).$

Proof.

$$
\begin{aligned}
\int \|F_S''\|_{\mathrm{HS}}^2 &= \int \|F''\|_{\mathrm{HS}}^2 - (n-1)\left(\int F\right)^2 - (n-1)\mathrm{Var}(F) + 2\int |\nabla_S F|^2 \\
&= \left(\frac{n^2 Z_{1,\mu}^2}{2\pi} - \frac{n Z_{1,\mu}^2}{\pi} + \frac{n}{4\pi}\iint \frac{\langle x,y\rangle^4}{|x|^3|y|^3} + \mathcal{O}\left(\frac{1}{n}\right)\right) \\
&\qquad - (n-1)\left(\frac{n Z_{1,\mu}^2}{2\pi} + \frac{Z_{1,\mu}^2}{4\pi} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) + \mathcal{O}\left(\frac{1}{n}\right) \\
&= \frac{n^2 Z_{1,\mu}^2}{2\pi} - \frac{n Z_{1,\mu}^2}{\pi} + \frac{n}{4\pi}\iint \frac{\langle x,y\rangle^4}{|x|^3|y|^3} - \frac{n^2 Z_{1,\mu}^2}{2\pi} \\
&\qquad + \frac{n Z_{1,\mu}^2}{2\pi} - \frac{n Z_{1,\mu}^2}{4\pi} + \mathcal{O}\left(\frac{1}{n}\right) \\
&= \frac{n}{4\pi}\left(\iint \frac{\langle x,y\rangle^4}{|x|^3|y|^3} - 3Z_{1,\mu}^2\right) + \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}
$$

□

Combining with the second-order Poincaré inequality (Theorem 6.1.2) and recalling also that $Z_{1,\mu} \simeq \frac{1}{\sqrt{n}}$ (see proof of 4.1.3), Proposition 6.1.1 is proven.

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