Complex Legendre Duality

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Several points of view

We try to combine points of view from several mathematical fields:

1. Transportation of Measure
2. Convex Geometry
3. Interpolation of Banach Spaces
4. Complex geometry

Questions and comments are welcome!

Convex duality

Given a convex set $K \subseteq \mathbb{R}^n$ with $0 \in K$, its polar or dual body is

$$K^\circ = \{ x \in \mathbb{R}^n \, ; \, \forall y \in K \ \langle x, y \rangle \leq 1 \}.$$

For example:

- The unit ball of ℓ_p^n transforms to that of ℓ_q^n with $q = \frac{p}{p-1}$.
- For polytopes, faces transform to vertices and vice versa.

- *Involution:* When $K \subseteq \mathbb{R}^n$ is also closed,

$$(K^\circ)^\circ = K.$$

- *Unique fixed point:* $K = K^\circ$ if and only if $K = B^n$.
- *Order reversal:*

$$K \subseteq T \qquad \Longrightarrow \qquad K^\circ \supseteq T^\circ$$

Characterizing convex duality

Theorem (Böröczky-Schneider '08, V. Milman's conjecture)

Convex duality is the only involution on the space of compact, convex bodies in $\mathbb{R}^n$ with zero in their interior that reverses order.

(up to linear transformations)

- A functional version of convex duality is the **Legendre transform**. For a function $f : \mathbb{R}^n \to \mathbb{R}$ we define

$$f^*(x) = \sup_{y \in \mathbb{R}^n} \left[\langle x, y \rangle - f(y) \right]$$

- Typically, the supremum is attained when $x = \nabla f(y)$.
- If $f(x) = \frac{\|x\|^p}{p}$ then $f^*(x) = \frac{\|x\|_*^q}{q}$ with $q = \frac{p}{p-1}$.

The Legendre transform

Usage in optimal transport

In the smooth and strongly-convex case, if $g = f^*$ then

$$y \mapsto \nabla g(y) \text{ is the inverse map to } x \mapsto \nabla f(x).$$

Hence, if ∇f is the Brenier map from μ to ν, then ∇f^* is the Brenier map from ν to μ.

Some properties of the Legendre transform are analogous to those of convex duality:

1. The function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is always convex.
2. *Involution:* If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and lsc then $(f^*)^* = f$.
3. *Order reversal:* $$f \leq g \qquad \Longrightarrow \qquad f^* \geq g^*.$$
4. *Unique fixed point:* $f = f^*$ if and only if $f(x) = |x|^2/2$.

Characterization theorems

(Artstein-Avidan, V. Milman '09)

The Legendre transform is the only order-reversing involution defined on the space of proper, lsc, convex functions.

(up to linear transformations).

... and now let us jump to the very end of this lecture:

Theorem (Lempert '18)

Roughly speaking: The complex Legendre **transforms** *are the only smooth, order-reversing local involutions around a fixed point that are symmetries of a certain complex Monge-Ampère equation.*

- My goal in this talk is to tell you the story, how to progress from one theorem to the other, and explain what are these complex Legendre transforms that we found.

Complex Interpolation

We consider complex interpolation of two Banach spaces, or of two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on the space $\mathbb{C}^n$. The unit balls are:

$$K_j = \left\{ z \in \mathbb{C}^n \, ; \, \|z\|_j \leq 1 \right\} \qquad\qquad (j = 0, 1).$$

- Each $K = K_j$ is a **circled** convex body, i.e.,

$$e^{i\theta} K = K \qquad\qquad (\theta \in \mathbb{R}).$$

Definition (Calderón, Lions 1960s)

Write $S = \{z \in \mathbb{C} \, ; \, 0 \leq Re(z) \leq 1\}$ for the strip. For $0 < \lambda < 1$,

$$[K_0 : K_1]_\lambda = \left\{ \varphi(\lambda) \, ; \, \begin{array}{l} \varphi : S \to \mathbb{C}^n \text{ is holomorphic and bounded,} \\ \forall t \in \mathbb{R}, \ \ \varphi(it) \in K_0 \text{ and } \varphi(1+it) \in K_1 \end{array} \right\}.$$

Properties of complex interpolation

1. **Complex interpolation is contained in the Minkowski sum:**
$$[K_0 : K_1]_\lambda \subseteq (1-\lambda)K_0 + \lambda K_1$$
(The Minkowski sum corresponds to a harmonic function φ, a weaker requirement than being holomorphic).
2. **Geometric average:** $[aK : bK]_\lambda = a^{1-\lambda} b^\lambda K$.
3. **Duality theorem:** $[K^\circ, T^\circ]_\lambda = [K, T]_\lambda^\circ$.
It follows that $[K, K^\circ]_{1/2} = B^n$ (because it is self-dual).
4. **A volume inequality stronger than Brunn-Minkowski** (Berndtsson '98, Cordero-Erausquin '02):
$$Vol_n([K_0, K_1]_\lambda) \geq Vol_n(K_0)^{1-\lambda} \cdot Vol_n(K_1)^\lambda.$$
This implies both the Brunn-Minkowski inequality and the Santaló inequality for circled convex bodies.

The evolution equation of complex interpolation

Let $\|\cdot\|_t$ be the norm with unit ball $[K_0 : K_1]_t$ for $t \in [0,1]$.

- How does the norm $z \mapsto \|z\|_t$ evolve with the time parameter t?

Rochberg '84, Semmes '88

Fix $p > 0$. For $t \in [0,1]$ and $z \in \mathbb{C}^n$ write

$$\phi_t(z) = \|z\|_t^p.$$

Then, independently of p, we have

$$\ddot{\phi} = (\nabla^2_{\mathbb{C}} \phi)^{-1} \partial_z \dot{\phi} \cdot \partial_{\bar{z}} \dot{\phi}$$

where $\partial_z f = (\partial f / \partial z_1, \ldots, \partial f / \partial z_n)$ and $\nabla^2_{\mathbb{C}} f = (\partial^2 f / \partial z_j \partial \bar{z}_k)_{j,k}$ is the complex Hessian. Here, $z \cdot w = \sum_j z_j w_j$ for $z, w \in \mathbb{C}^n$.

- Looks a bit like a geodesic equation, isn't it?

Homogeneous Complex Monge-Ampère (HCMA)

Once again: We have $\phi_t(z)$, with $t \in [0,1], z \in \mathbb{C}^n$, and

$$\ddot{\phi} = (\nabla^2_{\mathbb{C}} \phi)^{-1} \partial_z \dot{\phi} \cdot \partial_{\bar{z}} \dot{\phi}$$

- The evolution equation looks nicer if we assume $t \in S$, where $S \subseteq \mathbb{C}$ is the strip, and ϕ depends only on $Re(t)$.

The evolution equation with a complexified $t \in S$ is given by:

$$\frac{\partial^2 \phi}{\partial t \partial \bar{t}} = (\nabla^2_{\mathbb{C}} \phi)^{-1} \partial_{tz} \phi \cdot \partial_{\bar{t}\bar{z}} \phi.$$

Equivalently, the determinant of the complex Hessian in (t, z) vanishes:

$$\det \nabla^2_{\mathbb{C},(t,z)} \phi = \det \left(\begin{array}{c|c} \frac{\partial^2 \phi}{\partial t \partial \bar{t}} & \frac{\partial^2 \phi}{\partial t \partial z} \\ \hline \frac{\partial^2 \phi}{\partial \bar{t} \partial \bar{z}} & \nabla^2_{\mathbb{C}} \phi = \frac{\partial^2 \phi}{\partial z \partial \bar{z}} \end{array} \right) = 0.$$

- If the t-dependence in $\phi(t, z)$ is not through $Re(t)$, this resembles a **complex geodesic** or a **harmonic map**.

Complex interpolation as a boundary value problem

Coifman, Cwikel, Rochberg, Sagher, Weiss '82: Replace the strip $S \subseteq \mathbb{C}$ by the unit disc $D \subseteq \mathbb{C}$. For every $s \in \partial D$ we are given **boundary values**:

$$\phi_s(\cdot) = \phi(s, \cdot) : \mathbb{C}^n \to \mathbb{R}.$$

HCMA in $n+1$ complex variables

We would like to consider the (maximal) solution of *HCMA*, the function

$$\phi(t, z) \qquad\qquad t \in D, z \in \mathbb{C}^n$$

that solves the HCMA equation in all variables: $\det \nabla^2_{\mathbb{C},(t,z)} = 0$.

Example: The case of complex interpolation of two norms

We have $\phi(0, z) = \|z\|^p_{1/2}$ where $\phi(s, z) = \begin{cases} \|z\|^p_0 & s \in A_0 \\ \|z\|^p_1 & s \in A_1 \end{cases}$

Here, $A_0, A_1 \subseteq \partial D$ may be any two disjoint sets of measure π.

The duality theorem in complex interpolation

We need to discuss existence & uniqueness. Why "maximal" solution? because otherwise there can be many solutions.

- Recall the duality theorem: **Complex interpolation of norms commutes with convex duality**.

In fact, this amounts to the observation (related to Lempert '85) that the fiberwise Legendre transform is a symmetry of HCMA.

The fiberwise Legendre transform

For $\phi(t, z)$ with $t \in D, z \in \mathbb{C}^n$ we set

$$\phi^*(t, z) = \sup_{w \in \mathbb{C}^n} \left[2Re(z \cdot w) - \phi(t, w) \right],$$

where $z \cdot w = \sum_j z_j w_j$. Typically, sup is attained at $z = \partial\phi/\partial w$.

- Under the fiberwise Legendre transform, we expect a solution of HCMA to transform to a solution of HCMA.

Complex interpolation of convex functions

Consider **fiberwise-convex** boundary data: A measurable

$$\phi(s,z) \qquad\qquad (s \in \partial D, z \in \mathbb{C}^n)$$

which is **convex in** $z \in \mathbb{C}^n$, for any $s \in \partial D$.

Technical assumption: p-uniform growth condition

For some $p \in (1, +\infty)$, there exist $c, A > 0$ with

$$c|z|^p - A \le \phi_s(z) \le \frac{|z|^p}{c} + A = \mu_p(z) \qquad (s \in \partial D, z \in \mathbb{C}^n)$$

Definition of the "maximal HCMA solution"

The supremum over all psh (pluri-sub-harmonic) functions $\psi(t,z)$ with $\psi(t,z) \le \mu_p(z)$ and with radial boundary values:

$$\forall s \in \partial D, z \in \mathbb{C}^n, \qquad \limsup_{r \to 1, w \to z} \psi(rs, w) \le \phi(s,z).$$

(i.e., the Perron solution for the Dirichlet problem).

Structure of the maximal HCMA solution

We assume fiberwise-convex boundary values, and *p*-uniform growth conditions.

- Our results on the maximal HCMA solution are related to Alexander-Wermer '85 and to Slodkowski '88-'90.

Theorem ("The solution is psh and fiberwise-convex")

1. *The solution $\phi(t, z)$ is fiberwise-convex in z also for $t \in D$.*
2. *For any holomorphic function $f : D \to \mathbb{C}^n$ (need Hardy space H^p) and for any $t_0 \in D$,*

$$\phi(t_0, f(t_0)) \leq P\left[\phi(s, f(s))\right](t_0)$$

where $P[\cdot]$ is the Poisson integral, the harmonic extension of the function $s \mapsto \phi(s, f(s))$ from ∂D to the disc.

- The psh condition means: sub-harmonic along any holomorphic disc.

Holomorphic discs along which the solution is harmonic

Recall: For any holomorphic $f : D \to \mathbb{C}^n$ (in H^p) and $t_0 \in D$,

$$\phi(t_0, f(t_0)) \leq P\left[\phi(s, f(s))\right](t_0) \tag{1}$$

- Now assume that the boundary values are strictly-convex (plus p-uniform growth conditions).

Theorem ("Foliation by holomorphic discs")

1. *For any $t_0 \in D$ and $z_0 \in \mathbb{C}^n$ there exists a unique holomorphic $f = f_{t_0,z_0} : D \to \mathbb{C}^n$ (in H^p) with*

$$f(t_0) = z_0$$

   such that equality is attained in (1). The function $t \mapsto \phi(t, f(t))$ is harmonic in D.

2. *If f_{t_0,z_0} and f_{t_1,z_1} agree at a single point in D – they coincide in D. These holomorphic discs foliate $D \times \mathbb{C}^n$.*

An analogy with optimal transportation

- Thus, the foliation associates between points in one boundary fiber and points in another boundary fiber.
- Moreover, this association is given as the boundary values of a holomorphic $f : D \to \mathbb{C}^n$.

What is an analogous picture in the theory of optimal transport?

For $i = 0, 1$ look at the optimal transport map S_i from a fixed reference probability measure in $\mathbb{R}^n$ to the uniform measure on a convex domain $\Omega_i \subseteq \mathbb{R}^n$ of volume one.

The map $S_1 \circ S_0^{-1}$ associates between points in Ω_0 and Ω_1. Set

$$\Omega = \{(t, (1-t)S_0(x) + tS_1(x)) \in [0,1] \times \mathbb{R}^n ; \, x \in \mathbb{R}^n\}.$$

Then Ω is foliated by line segments. A certain solution to real homogeneous Monge-Ampére is **linear** along these segments.

- Moreover $t \mapsto Vol_n(\Omega_t)$ is log-concave (McCann, Barthe).

Revisiting The duality theorem in complex interpolation

- Now assume that the boundary values are smooth and strictly-convex (plus p-uniform growth conditions).
- Write $\mathcal{F}(\phi) = \{f_{t_0,z_0}\}_{t_0 \in D, z_0 \in \mathbb{C}^n}$ for the collection of all holomorphic discs appearing in the foliation. I.e., any $f \in \mathcal{F}(\phi)$ is a holomorphic $f : D \to \mathbb{C}^n$ with

$$\phi(t, f(t)) \text{ is harmonic.}$$

Theorem ("A dual foliation through fiberwise Legendre")

1. *The fiberwise Legendre transform ϕ^* is also a maximal HCMA solution.*
2. *If $f \in \mathcal{F}(\phi)$ then $g(t) := \frac{\partial \phi}{\partial z}(t, f(t)) \in \mathcal{F}(\phi^*)$ and*

$$\phi(t, f(t)) + \phi^*(t, g(t)) = 2Re(f(t) \cdot g(t))$$

(the sup-attaining point in the fiberwise Legendre transform).

Example: quadratic boundary values

- Boundary values: Suppose for $s \in \partial D$,

$$\phi(s, z) = M(s) z \cdot \bar{z}$$

for a positive-definite Hermitian matrix $M(s) \in \mathbb{C}^{n \times n}$, say smoothly depending on $s \in \partial D$.

Theorem (Wiener-Masani, 1950s)

There exists a unique holomorphic map $B : D \to \mathbb{C}^{n \times n}$ which is invertible everywhere and with boundary values

$$B^*(s) B(s) = M(s) \qquad \forall s \in \partial D.$$

In this case, the holomorphic discs and the maximal HCMA solution are

$$\begin{cases} f_{t_0, z_0}(t) = B^{-1}(t) B(t_0) z_0 \\ \phi(t, z) = B^*(t) B(t) z \cdot \bar{z}. \end{cases}$$

- Thus $t \mapsto \phi(t, f(t))$ is not only harmonic but in fact constant.

Role of homogeneity and S^1-invariance

- How come the parameter $p > 0$ is not important in complex interpolation of complex norms? We have boundary values

$$\phi_s(z) = \|z\|_s^p$$

(It follows that ϕ is in fact p-homogeneous throughout D)

Lemma

If ϕ is a non-negative psh function on $D \times \mathbb{C}^n$ with $\phi(t, \lambda z) = |\lambda|^p \phi(t, z)$, then $\log \phi$ is psh.

- For any $f \in \mathcal{F}(\phi)$, the function $t \mapsto \phi(t, f(t))$ is harmonic, with a sub-harmonic logarithm. Hence it is constant!
- Our function ϕ here is not only harmonic but even **constant** along the leaves of the foliation, and ϕ^q is again a maximal HCMA solution for any $q > 0$.

What properties of the Legendre transform are used?

- For the duality theorem, we used a fiberwise transform of the form

$$\mathcal{L}f(w) = \sup_{z \in \mathbb{C}^n} \left[2Re(z \cdot \bar{w}) - f(z) \right].$$

The unique fixed point is $f(z) = |z|^2$. (we put a "bar" on w, no big deal).

The scalar product $z \cdot \bar{w}$ above indicates a linear theory, but in the complex world we expect something more holomorphic.

- What other fiberwise transforms are symmetries of HCMA? What properties of the map $(z, w) \mapsto z \cdot \bar{w}$ were used?

Answer

Mostly, the fact that $z \cdot \bar{w}$ is holomorphic in z, anti-holomorphic in w, and real on the diagonal $\Delta = \{(z, w)\, ;\, z = w \in \mathbb{C}^n\}$.

The Ψ-Legendre transform

Definition

Let $\Psi : \mathbb{C}^n \to \mathbb{R}$ be a real-analytic, strongly psh function.

Extend Ψ to a function $\Psi_{\mathbb{C}}(z, w)$, defined near the diagonal in $\mathbb{C}^n \times \mathbb{C}^n$, that is holomorphic in z and anti-holomorphic in w.

Define the **Ψ-Legendre transform** of a function $f : \mathbb{C}^n \to \mathbb{R}$ via

$$\mathcal{L}_{\Psi} f(w) = \sup_{z} \left[2 Re\, \Psi_{\mathbb{C}}(z, w) - f(z) \right].$$

- Locally, near $z_0 \in \mathbb{C}^n$, we have the Taylor expansion

$$\Psi(z + z_0) = \sum_{\alpha,\beta} a_{\alpha\beta} z^{\alpha} \bar{z}^{\beta}$$

and we look at the polarization

$$\Psi_{\mathbb{C}}(z + z_0, w + z_0) = \sum_{\alpha,\beta} a_{\alpha\beta} z^{\alpha} \bar{w}^{\beta}.$$

The Ψ-Legendre transform as an HCMA symmetry

Recall: A strongly-convex function is in particular strongly-psh.

Examples

1. If $\Psi(z) = |z|^2$ then $\Psi_{\mathbb{C}}(z, w) = z \cdot \bar{w}$ and $\mathcal{L}_\Psi = \mathcal{L}$, we recover the usual Legendre transform.
2. In $\mathbb{C}$, if $\Psi(z) = |z|^4 + |z|^2$ then $\phi_{\mathbb{C}}(z, w) = z^2\bar{w}^2 + z\bar{w}$, and $\mathcal{L}_\Psi$ is a new transform.

Theorem ("A locally-defined HCMA symmetry")

There exists a C^2-neighborhood U of the function $\Psi : \mathbb{C}^n \to \mathbb{R}$ such that $\mathcal{L}_\Psi$ is a well-defined, order-reversing involution in U with Ψ **as its unique fixed point**.

Moreover, $\mathcal{L}_\Psi f$ is psh for all $f \in U$, and the fiberwise Ψ-Legendre transform is a symmetry of the HCMA equation.

(i.e., it transforms smooth, maximal HCMA solutions near Ψ to smooth maximal HCMA solutions).

Remarks on the Ψ-Legendre transform

1. It was conjectured by Semmes and Donaldson that these local symmetries of HCMA should exist (their evidence: parallel curvature tensor in infinite-dimension).
2. **What is the domain $U = U_\Psi$ of the Ψ-Legendre transform?** When $\Psi(z) = |z|^2$, we know that essentially

$$U_\Psi = \{\text{convex functions}\}.$$

3. The transform

$$u \mapsto \mathcal{L}_\Psi(\Psi + u) - \Psi$$

depends on Ψ only through $\nabla^2_{\mathbb{C}} \Psi$, or equivalently, only through $\omega_\Psi = \partial\bar{\partial}\Psi$. It follows that $\mathcal{L}_{\omega_\Psi} u = \mathcal{L}_\Psi(\Psi + u) - \Psi$ may be defined on **Kähler manifolds**.
4. For circled boundary data in $\mathbb{C}^n$ we have the Berndtsson inequality. This yields a **Santaló-type inequality** for the Ψ-Legendre transform.

Why is Ψ-Legendre transform an HCMA symmetry?

Recall

$$\hat{F}(z) = \mathcal{L}_\Psi F(z) = \sup_w \left[2Re\,\Psi_{\mathbb{C}}(z,w) - F(w)\right].$$

- The fact that $\mathcal{L}_\Psi \Psi = \Psi$ follows from Ψ being strongly psh (Use Taylor expansion of second order).
- If $\|F - \Psi\|_{C^2} < \varepsilon$, then for any $z \in \mathbb{C}^n$ the supremum is uniquely attained at some point $w = g(z)$.

Lemma (A variant of Lempert's theorem for classical Legendre)

$$\partial\bar{\partial}\hat{F} = g^*(\partial\bar{\partial}F).$$

Proof: Set $\Lambda = graph(g) \subseteq \mathbb{C}^n \times \mathbb{C}^n$. Then,

$$\hat{F}(z) + F(w) - \Psi_{\mathbb{C}}(z,w) - \overline{\Psi_{\mathbb{C}}(z,w)} \geq 0,$$

in a neighbourhood of Λ, with equality if and only if $(z,w) \in \Lambda$.

Proof of key lemma

- Minimum in z is attained at Λ, and similarly for w. Hence,

$$\frac{\partial \hat{F}(z)}{\partial z_j} = \frac{\partial \Psi_{\mathbb{C}}(z,w)}{\partial z_j} \qquad \text{on } \Lambda.$$

$$\frac{\partial F(w)}{\partial \bar{w}_j} = \frac{\partial \Psi_{\mathbb{C}}(z,w)}{\partial \bar{w}_j} \qquad \text{on } \Lambda.$$

- To summarize, on Λ,

$$d\Psi_{\mathbb{C}}(z,w) = \partial \hat{F}(z) + \bar{\partial} F(w).$$

- Apply exterior derivative on Λ, to obtain

$$0 = d^2\Psi_{\mathbb{C}}(z,w) = \bar{\partial}\partial \hat{F}(z) + \partial\bar{\partial} F(w).$$

- But Λ is the graph of $w = g(z)$. Hence,

$$\partial\bar{\partial}\hat{F}(z) = \partial\bar{\partial}F(w) \text{ on } \Lambda \qquad \Longrightarrow \qquad \partial\bar{\partial}\hat{F} = g^*(\partial\bar{\partial}F).$$

The Mabuchi metric

- The HCMA equation is the complex geodesic (harmonic map) equation for the **Mabuchi metric**. Write

  $\mathcal{M} = \{$smooth, strongly psh functions on $\mathbb{C}^n$, growth condition$\}$.

The metric tensor of the Mabuchi metric at $F \in \mathcal{M}$ is, for $u, v \in C_0^\infty(\mathbb{C}^n)$,

$$g_F(u,v) = \int_{\mathbb{C}^n} uv \det \nabla^2_{\mathbb{C}} F = (i/2)^n \int_{\mathbb{C}^n} uv(\partial\bar{\partial}F)^n.$$

- Complex geodesics (or harmonic maps) are exactly the HCMA solutions.
- The Ψ-Legendre transform $\mathcal{L}_\Psi$ is a Mabuchi isometry, since by the lemma

$$\int_{\mathbb{C}^n} \chi^2(\partial\bar{\partial}F)^n = \int_{\mathbb{C}^n} (\chi \circ g)^2(\partial\bar{\partial}\hat{F})^n$$

  and $d\mathcal{L}_\Psi F.\chi = -\chi \circ g$ by the Alexandrov lemma.

The end

Thank you!