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By
Yam Eitan

מאת
ים איתן

הגוף הקמור הממורכז עם ההטלות
החד-ממדיות בעלות הזנב הגדול
ביותר
The Centered Convex Body
Whose Marginals Have The
Heaviest Tails

Advisor: Prof. Bo'az Klartag

מנחה: פרופ' בועז קלרטג

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חשוון תשפ"ב

THE CENTERED CONVEX BODY WHOSE MARGINALS HAVE THE HEAVIEST TAILS

YAM EITAN

Chapter 1. Main Results

ABSTRACT

Given any real numbers $1 < p < q$, we study the norm ratio (i.e. the ratio between the q-norm and the p-norm) of marginals of *centered* convex bodies. We first show that some marginal of the simplex maximizes said ratio in the class of n-dimensional centered convex bodies. We then pass to the dimension independent (i.e. log-concave) case where we find a 1-parameter family of random variables in which the maximum ratio must be attained, and find the exact maximizer of the ratio when $p = 2$ and q is even. In addition, we find another interesting maximization property of marginals of the simplex involving functions with positive third derivatives.

1. INTRODUCTION

Let $1 < p < q$, $K \subset \mathbb{R}^n$ be a convex body and $X \sim Uni(K)$ be a random vector uniformly distributed over K. From here forth we shall assume K is centered (i.e. $\mathbb{E}[X] = 0$). Define:

$$\alpha(K) := \max_{\theta \in S^{n-1}} \frac{\|X \cdot \theta\|_q}{\|X \cdot \theta\|_p}.$$

Here, S^{n-1} is the $(n-1)-$dimensional unit sphere, centered at the origin of $\mathbb{R}^n$. This quantity can be thought of as a measurement of the "heaviest tail" of any 1-dimensional projection of X. Our main result is the following:

Theorem 1. *Let $K \subset \mathbb{R}^n$ be a centered convex body and let Δ_n be an n-dimensional centered simplex. Then:*

$$(1.1) \qquad \qquad \alpha(K) \leq \alpha(\Delta_n).$$

While one might not be surprised that the simplex is a maximizer of $\alpha(\cdot)$, as it is a maximizer of many other functionals over the family of convex bodies (e.g. [17]), the unit vector for which the maximum ratio is achieved is not always one of the normals of the simplex. In other words, for certain values of p and q there exist non-conic maximizers of $\alpha(\cdot)$. The relation between p, q, n, and the unit vector maximizing the norm ratio for the simplex remains elusive thus far. Recall now the definition of a random vector in isotropic position:

Definition 2. A random vector $X = (X_1, \ldots, X_n)$ is said to be in isotropic position if :

$$\mathbb{E}[X] = 0, \qquad \mathbb{E}[X_i X_j] = \delta_{i,j}, \qquad (\forall 0 \leq i, j \leq n),$$

where $\delta_{i,j}$ is the Dirac delta. A convex body is said to be in isotropic position if $X \sim Uni(K)$ is in isotropic position.

Any convex body can be transformed into an isotropic one through an affine map (see [3] for proof). Our second main result is the following:

Theorem 3. *Let $K \subset \mathbb{R}^n$ be a convex body in isotropic position and $X \sim Uni(K)$ be a random variable uniformly distributed over K. Then for any $k \in \{3, 4\}$, $\phi \in C^k(\mathbb{R})$ such that $\phi^{(k)}(x) > 0 \ \forall x \in \mathbb{R}$, we have:*

$$(1.2) \qquad \max_{\theta \in S^{n-1}} \mathbb{E}[\phi(X \cdot \theta)] \leq \max_{\theta \in S^{n-1}} \mathbb{E}[\phi(\Gamma_n \cdot \theta)],$$

where $\Gamma_n \sim Uni(\Delta_n)$ is also in isotropic position.

Here if $k = 3$, then the inequality is strict unless K is a cone, in which case the maximizing unit vector is normal to (one of) the cone's base(s). However, if $k = 4$ this isn't the case, as for some ϕ there exist non-conic maximizers. A version of Theorems 1 and 3 for the symmetric log-concave case can be found in Eskenazis, Nayar, and Tkocs [5]. In fact, we shall use techniques similar to those presented in the said paper here. Recall that a random variable X is called log-concave if it has a density f_X with respect to the Lebesgue measure such that $log\,(f_x(\cdot))$ is concave on the support of f_X, where log is the natural logarithm. The set of log-concave random variables can be defined equivalently as the closure of the set of random variables of the form $X \cdot \theta$ for all convex bodies (regardless of the dimension). Thus our main result concerning the dimension independent case is the following:

Theorem 4. *Let X be a centered log concave random variable, and let n be an even integer. We then have:*

$$(1.3) \qquad \frac{\|X\|_n}{\|X\|_2} \leq \left(n! \sum_{i=0}^{n} \frac{(-1)^k}{k!} \right)^{\frac{1}{n}} = (!n)^{\frac{1}{n}},$$

with equality if and only if X coincides with $\pm\Gamma$ in law. Here, !n is the subfactorial of n and Γ is a random variable with density $g(x) = e^{-(x+1)}$ supported on the line $[-1, \infty)$.

In addition, we prove the following result regarding odd moments of log-concave random variables:

Theorem 5. *Let X be a centered log-concave random variable. If q is an odd integer, and p is a real number such that $1 < p < q$, then we have:*

$$(1.4) \qquad \frac{|\mathbb{E}[X^q]|^{\frac{1}{q}}}{\|X\|_p} \leq \frac{|\mathbb{E}[\Gamma^q]|^{\frac{1}{q}}}{\|\Gamma\|_p},$$

with equality if and only if X coincides with Γ in law (The difference here is that the absolute value in the numerator is taken outside of the expectation).

A different proof of Theorem 5 for the case $p = 2$, $q = 3$ can be found in Bubeck and Eldan [4]. The problem of finding the maximum norm ratio over different families of random variables has been discussed in several other instances in the mathematical literature. The fact that the norm ratios of marginals of convex bodies are bounded by some universal constant was proven first by Berwald in [1] and then by Borell in [1] (one can also find this result in a survey [14] by Milman and Pajor). Borel and Berwald also found the maximum norm ratio of

log-concave random variables supported on the real half-line (see [15] for proof). Another example for a problem of a similar type is the Khintchine inequality, which deals with the maximum norm ratio of linear combinations of independent Bernoulli random variables. The sharp Khintchine inequality for $p = 2$ was established by Haagerup in [6]. A simpler proof by Nazarov and Podkorytov can be found in [16]. The case where p and q are even numbers was also solved. For a more in-depth history of the Khintchine inequality as well as a proof of the even case see [15]. Eskenazis, Nayar, and Tkocs also found the maximum norm ratio for the s-norm ball when $p = 2$ in the aforementioned [5]. Throughout this paper, by a convex body we mean a convex compact set with non-empty interior, and by a cone we mean the convex hull of a subset of a an affine space of co-dimension one and a point outside of the said affine space.

2. THE DIMENSION DEPENDENT CASE

Recall $X \sim Uni(K)$ for some n-dimensional, centered, convex body K. We start by stating the well-known Brunn principle (see e.g. [18] for proof):

Lemma 6. *For any $\theta \in S^{n-1}$, the random variable $X \cdot \theta$ has a density $f_\theta(x)$ w.r.t. the Lebesgue measure. In addition f_θ is supported on a compact interval and $f_\theta^{1/n-1}$ is concave in its support. Finally, if $f_\theta^{1/n-1}$ is affine in its support, then K is a (pherhaps truncated) cone and θ is normal to K's base.*

Bruun's principle gives us valuable information about the way the graph of f_θ can intersect the graphs of other certain functions of interest. We shall make use of this information in conjunction with the theory of Chebyshev systems (for an in-depth discussion on Chebyshev systems see [10]):

Definition 7. A system of real functions $\{u_i(t)\}_0^k$ is called a Chebyshev system of order k if any linear combination:

$$(2.1) \qquad p(t) = \sum_{i=0}^{k} a_i u_i(t) \qquad \qquad (\sum_{i=0}^{k} a_i^2 > 0)$$

has at most k roots.

One readily sees that $\{u_i(t)\}_0^k$ is a Chebyshev system if and only if the determinant of the matrix $\{u_i(t_j)\}_{i \times j}$ doesn't vanish for any $t_0 < \cdots < t_k \in \mathbb{R}$. This in turn is equivalent to the existence of a linear combination of the form (2.1) whose roots are exactly $t_1, \ldots, t_k$. For a fixed set of numbers $p_0 < \cdots < p_k$ where $p_0 = 0$, $p_1 = 1$ we define:

$$(2.2) \qquad u_i(t) = \begin{cases} |t|^{p_i} sgn(t) & i \text{ is odd} \\ |t|^{p_i} & i \text{ is even} \end{cases}.$$

Lemma 8. *For $k < 5$, $\{u_i(t)\}_0^k$ is a Chebyshev system.*

Proof. We shall prove the lemma for $k = 4$ (the other cases can be proved in a similar yet easier way). Let $p(t)$ be of the form (2.1), we may assume $a_4 = 1$ since if $a_4 = 0$ we get the case $k = 3$. By Rolle's theorem it is enough to show $p'(t)$ has at most 3 roots. Define:

$$q(t) = a_2 u_2'(t) + a_3 u_3'(t) + u_4'(t).$$

FIGURE 2.1. Illustrations of number of possible solutions (2.4) may have in cases 1, 2, and 3 respectively

It is enough to show that there exists a line segment $[b,c] \subset \mathbb{R}$ such that q is monotonically decreasing in $[b,c]$ and monotonically increasing both in $[c,\infty)$ and in $(-\infty, b]$. For $t > 0$ we have:

$$(2.3) \qquad q'(t) = \tilde{a}_2 t^{p_2-2} + \tilde{a}_3 t^{p_3-2} + \tilde{a}_4 t^{p_4-2}$$

$$\tilde{a}_i = a_i p_i (p_i - 1).$$

Notice that a_i and $\tilde{a}_i$ have the same sign. Again by normalizing we may assume $\tilde{a}_4 = 1$. We now have $q'(t) = 0$ if and only if:

$$\tilde{a}_2 + \tilde{a}_3 t^{p_3 - p_2} + t^{p_4 - p_2} = 0.$$

Changing variables to $x = t^{p_3 - p_2}$ and defining $\beta = \frac{p_4 - p_2}{p_3 - p_2} > 1$ we get:

$$(2.4) \qquad \tilde{a}_2 + \tilde{a}_3 x + x^\beta = 0.$$

Since x^β is strictly convex, it intersects any affine function at most twice. Further inspection shows that the number of positive roots of (2.4) depends on a_2, a_3 in the following way (depicted in figure 2.1):

(1) If $a_2 \leq 0$, then (2.4) has exactly one positive solution, since the function $|x|^\beta$ is strictly convex in all of $\mathbb{R}$, but the line $-\tilde{a}_2 - \tilde{a}_3 x$ must intersect it once in $(0, \infty)$ and one in $(-\infty, 0]$.

(2) If $a_2 > 0$, $a_3 \leq 0$, then (2.4) has at most two positive solutions.

(3) If $a_2 > 0, a_3 > 0$, then (2.4) has no positive solutions, since in this case, the left-hand side of (2.4) is positive.

We now examine $t < 0$. Setting $s = -t > 0$ we get:

$$(2.5) \qquad q'(s) = \tilde{a}_2 s^{p_2-2} - \tilde{a}_3 s^{p_3-2} + \tilde{a}_4 s^{p_4-2}.$$

But this is just (2.3) with the sign of a_3 changed. Thus, if case 1 holds, for $t > 0$, $q(t)$ can only decrease up to some point $c \geq 0$ and then must increase for $t > c$ (the limit of q at infinity is infinity). For $t < 0$, $q(t)$ must increase up to some point $b \leq 0$ and may only decrease when $t > b$. Thus the claim of the lemma holds for case 1. If case 2 holds, for $t > 0$, $q(t)$ may change its monotonicity at most twice, but for $t < 0$, $q(t)$ must be monotonically increasing (since for $t < 0$ when checking our list of possible roots of the derivative, we change the sign of a_3). Thus the claim of the lemma still holds. Case 3 is just a reflection of case 2 and so the lemma holds for all cases. □

Now let θ_1, θ_2 be two outer normal unit vectors to the centered simplex Δ_n and recall that $\Gamma_n \sim Uni(\Delta_n)$. Define:

$$(2.6) \qquad \Gamma_n^s = \frac{s\theta_1 - (1-s)\theta_2}{|s\theta_1 - (1-s)\theta_2|} \cdot \Gamma_n \qquad 0 \leq s \leq 1$$

and let $g_n^s(t)$ be the density of Γ_n^s. We now state a few important properties of g_n^s:

Lemma 9. *The following properties hold for g_n^s:*

(1) $g_n^0(x) = g_n^1(-x)$.

(2) $\left(g_n^0(x)\right)^{\frac{1}{n-1}}$ *is affine on its support.*

(3) *For $0 < s < 1$, let $\operatorname{Supp} g_n^s = [a_s, b_s]$, then $\exists c_s \in [a_s, b_s]$ such that $(g_n^s(x))^{\frac{1}{n-1}}$ is affine in $[a_s, c_s]$ and in $[c_s, b_s]$. In addition, g_n^s is continuous.*

Proof. (1) follows from the symmetry of the simplex. (2) follows from the equality case of Brunn's principle. To see (3) holds notice that we can split Δ_n into two simplices whose intersection is a face orthogonal to the unit vector $\frac{s\theta_1 - (1-s)\theta_2}{|s\theta_1 - (1-s)\theta_2|}$. Thus, using the equality case of Brunn's principle once again, we see that the claim holds (see e.g. [12] for more details). $\square$

Before proving Theorem 1, we give a similar result with regard to the following auxiliary operator:

$$(2.7) \qquad \alpha^*(K) := \max_{\theta \in S^{n-1}} \frac{|\mathbb{E}[|X \cdot \theta|^q sgn(x)]|^{\frac{1}{q}}}{\|X \cdot \theta\|_p}.$$

This operator can be thought of as measuring the largest asymmetry between tails of the marginals of K.

Theorem 10. *For any centered convex body $K \subset \mathbb{R}^n$:*

$$(2.8) \qquad \alpha^*(K) \le \alpha^*(\Delta_n),$$

where equality holds if and only if K is a cone. In addition, the unit vectors in which the maximum is achieved are normal to (one of the) the base(s) of the cone.

Proof. As before, for some $\theta \in S^{n-1}$, let $f_\theta(x)$ be the density of $X \cdot \theta$. We shall also abbreviate g_n^0 by g_n for the duration of this proof. By the homogeneity of (2.8) we may assume:

$$(2.9) \qquad \int |x|^p f_\theta(x) dx = \int |x|^p g_n(x) dx = 1.$$

Assume by contradiction that:

$$(2.10) \qquad \int |x|^q sgn(x) f_\theta(x) dx > \int |x|^q sgn(x) g_n(x) dx.$$

Our general idea is to use Lemma 8 to interlace the difference function $h := g_n - f_\theta$ with appropriate linear combinations of $\{1, x, |x|^p, |x|^q sgn(x)\}$. We then use four linear constraints (i.e. (2.9) (2.10) and the fact that f_θ and g_n are centered densities), to force h to change sign at least four times. This will contradict Lemmas 6 and 9, which state that $(g_n)^{\frac{1}{n-1}}$ is non-negative, affine on $[a_0, b_0]$ and continuous at every point but a_0, and $(f_\theta)^{\frac{1}{n-1}}$ is non-negative and concave on its support. Thus h may change sign at most twice at (a_1, ∞) and not at all at $(-\infty, a_1)$, making for a maximum of three possible sign changes. We now show (2.10) implies h must change sign at least four times. First, since f_θ and g_n are densities, we get:

$$\int h(x) dx = 0.$$

Thus h must change sign at least once. Now if h changes sign only once, say at a point $x_1 \in \mathbb{R}$, then $(x-x_1)h(x)$ never changes sign, but since f_θ and g_n are centered densities, we get:

$$\int (x - x_1)h(x)dx = 0,$$

which contradicts (2.10). Assume now h changes sign exactly twice at points x_1, x_2. Then by Lemma 8 there exists $p(x) = d_1 + d_2 x + |x|^p$ whose roots are exactly x_1, x_2 and so $p(x)h(x)$ never changes sign. Using (2.9) we now get:

$$\int p(x)h(x)dx = 0,$$

which contradicts (2.10). Finally, assume h changes sign exactly three times at points x_1, x_2, x_3, again by Lemma 8 there exists $p(x) = d_1 + d_2 x + d_3 |x|^p + |x|^q sgn(x)$, whose roots are exactly x_1, x_2, x_3. Also note that in this case f_θ and g_n must intersect twice on (a_1, ∞) and so $h(x)$ must be non-negative for large values of x, thus $h(x)p(x) \geq 0$ for all $x \in \mathbb{R}$. However (2.9) and (2.10) give us:

$$\int p(x)h(x)dx < 0,$$

which is a contradiction. We thus showed:

$$\frac{\mathbb{E}[|X \cdot \theta|^q sgn(X \cdot \theta)]}{\|X \cdot \theta\|_p} \leq \frac{\mathbb{E}[|\Gamma_n^1|^q sgn(\Gamma_n^1)]}{\|\Gamma_n^1\|_p}$$

and that equality implies that $X \cdot \theta$ coincides with Γ_n^1 in law, which in turn implies K is a cone and θ is normal to K's base by the equality case of Brunn's principle. $\square$

Proof of Theorem 1. The proof is very similar to that of Theorem 10. Using the same notations as before we can again assume by homogeneity :

$$(2.11) \qquad \int |x|^p f_\theta(x)dx = \int |x|^p g_n^s(x)dx = 1 \qquad \forall 0 \leq s \leq 1$$

and assume by contradiction:

$$(2.12) \qquad \int |x|^q f_\theta(x)dx > \int |x|^q g_n^s(x)dx \qquad \forall 0 \leq s \leq 1.$$

Now choose some $p < r < q$. Theorem 10 tells us:

$$\int |x|^r sgn(x) g_n^1(x)dx \leq \int |x|^r sgn(x) f_\theta(x)dx \leq \int |x|^r sgn(x) g_n^o(x)dx,$$

where the first inequality just follows from property (1) in Lemma 9. Thus from continuity we may choose some $0 \leq s_0 \leq 1$ such that:

$$(2.13) \qquad \int |x|^r sgn(x) f_\theta(x)dx = \int |x|^r sgn(x) g_n^{s_0}(x)dx.$$

We now proceed exactly as we did before; by Lemmas 6 and 9 the difference function $h = g_n^{s_0} - f_\theta$ can change sign at most 4 times. The same arguments as before show that h must change sign at least 4 times, but if h changes sign exactly 4 times at points $x_1, \ldots, x_4$, again by Lemma 8 there exists $p(x) = d_1 + d_2 x + d_3 |x|^p + d_4 |x|^r sgn(x) + |x|^q$, whose roots are exactly $x_1, \ldots, x_4$. Since $g_n^{s_0}$ and f_θ must intersect twice in $[c_{s_0}, b_{s_0}]$ we have $h(x) \geq 0$ for large values of x and so

$p(x)h(x) \geq 0$. But (2.11) (2.12) and the fact $g_n^{s_0}$ and f_θ are centered densities give us:

$$\int p(x)h(x)dx < 0,$$

which is a contradiction. We thus showed:

$$\frac{\|X \cdot \theta\|_q}{\|X \cdot \theta\|_p} \leq \frac{\|\Gamma_n \cdot \theta_{s_0}\|_q}{\|\Gamma_n \cdot \theta_{s_0}\|_p}$$

and so:

$$\alpha(K) \leq \alpha(\Delta_n).$$

∎

As mentioned in the Introduction, the proof of Theorem 1 shows that the marginal for which the maximum norm ratio in the simplex (and in any other convex body) is achieved belongs to the 1-parameter family Γ_n^s and thus any convex body which maximizes α can be split into two cones by some hyperplane. Quite unexpectedly, calculations show that for different values of p and q, the maximum norm ratio is achieved in different values of s.

Notice also that all we needed for the proof of Theorem 1 was the Chebyshev system property of $\{1, x, |x|^p, |x|^r sgn(x), |x|^q\}$. Thus, using other Chebyshev systems might bring forth new inequalities. Let us give one more example of a Chebyshev system:

Lemma 11. *For any $k \in \mathbb{N}$, let $\phi \in C^k(\mathbb{R})$, if $\phi^{(k)} > 0$ the set $\{1, x, ..., x^{k-1}, \phi\}$ is a Chebyshev system.*

Proof. We use Rolle's Theorem iteratively. If for some $p(x) = \sum_{i=0}^{k-1} d_i x^i + d_k \phi(x)$ there are more then k roots, then $p^{(k)}(x) = \phi^{(k)}(x)$ has at least one root, which is, of course, a contradiction (this proof is taken from [10, Chapter 2, Example E]). □

Proof of Theorem 3. The proofs when $k = 3, 4$ are identical to those of Theorems 10 and 1 respectively.

∎

3. THE DIMENSION INDEPENDENT CASE

Looking at the log-concave case, we notice the role of our one-parameter family Γ_n^s is taken by following simpler family of random variables:

$$(3.1) \qquad \Gamma^s = s\Gamma - (1-s)\Gamma'.$$

Here, Γ, Γ' are i.i.d with density $g(x) = e^{-(x+1)}1_{[-1,\infty)}$. Indeed, replacing the power of $\frac{1}{n-1}$ by log, Lemma 6 becomes the definition of a log-concave random variable, and Lemma 9 is easy to verify for $g^s(x)$, the densities of Γ^s. Thus, one can reformulate theorems 1,3 and 10 for the log-concave case and prove them in the exact same way:

Theorem 12. *Let X be a centered log concave random variable, then:*

$$(3.2) \qquad \frac{\|X\|_q}{\|X\|_p} \leq \max_{0 \leq s \leq 1} \frac{\|\Gamma^s\|_q}{\|\Gamma^s\|_p}.$$

Theorem 13. *Let X be a log concave random variable in isotropic position and let $\phi \in C^k(\mathbb{R})$ and $\phi^{(k)}(x) > 0$, $\forall x \in \mathbb{R}$ where $k \in \{3,4\}$. We then have:*

$$(3.3) \qquad \mathbb{E}[\phi(X)] \leq \max_{0 \leq s \leq 1} \mathbb{E}[\phi(\Gamma^s)].$$

If $k = 3$ the maximum is achieved at $s = 0$ and equality implies X coincides with Γ in law.

Theorem 14. *Let X be a centered log concave random variable, then:*

$$(3.4) \qquad \frac{\mathbb{E}[|X|^q sgn(X)]}{\|X\|_p} \leq \frac{\mathbb{E}[|\Gamma|^q sgn(\Gamma)]}{\|\Gamma\|_p},$$

with equality if and only if X coincides with Γ in law.

Note that Theorem 5 follows from Theorem 14. The following question naturally arises: which is the maximizer of the norm ratio among Γ^s for $0 \leq s \leq 1$? We shall use the new convenient formula (3.1) to answer this question when $p = 2$ and q is an even integer, but first, let us recall that the cumulants $\{k_n(X)\}_{n=0}^{\infty}$ of a random variable X are defined by:

$$(3.5) \qquad log\left(\mathbb{E}[e^{tX}]\right) = \sum_{n=0}^{\infty} \frac{k_n(X)}{n!} t^n.$$

See e.g. [9] for proof that when X is log-concave $\mathbb{E}[e^{tX}]$ is analytic in a small neighborhood of zero. Let us also define $\mu_n(X)$ to be the n-th moment of X. Two useful properties of the cumulants of a random variable are the following:

(1) For any X, Y independent random variables, $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, we have:

$$(3.6) \qquad k_n(aX + bY) = a^n k_n(X) + b^n k_n(Y).$$

(2) Knowing the value of the cumulants of X (and assuming again that X is centered) we can restore the value of its moments using the following formula:

$$(3.7) \qquad \mu_n(X) = \sum_{i=1}^{n} \binom{n-1}{i-1} k_i(X)\mu_{n-i}(X).$$

We now calculate the cumulants of Γ:

$$log\left(\mathbb{E}[e^{t\Gamma}]\right) = log\left(\int_{-1}^{\infty} e^{-x-1} \cdot e^{tx} dx\right) = log\left(\frac{e^{-t}}{1-t}\right) = -t - log(1-t) = \sum_{n=2}^{\infty} \frac{t^n}{n},$$

and so:

$$(3.8) \qquad k_0(\Gamma) = k_1(\Gamma) = 0, \text{ and } k_n(\Gamma) = (n-1)! \qquad n \geq 2.$$

Proof of Theorem 4. Reparametrizing (3.2) in Theorem 12, we get:

$$(3.9) \qquad \left(\frac{\|X\|_n}{\|X\|_2}\right)^n \leq \max_{0 \leq t \leq \frac{\pi}{2}} \mu_n\left(\cos(t)\Gamma - \sin(t)\Gamma'\right),$$

with equality if and only if $\exists 0 \leq t_0 \leq \frac{\pi}{2}$ such that:

$$(3.10) \qquad \frac{X}{\|X\|_2} = \cos(t_0)\Gamma - \sin(t_0)\Gamma'.$$

Thus, the only remaining thing to prove is:

$$\mu_n\left(\cos(t)\Gamma - \sin(t)\Gamma'\right) < \mu_n(\Gamma) \qquad \forall 0 < t < \frac{\pi}{2}.$$

From the discussion above we see that:

$$|k_n \left(\cos(t)\Gamma - \sin(t)\Gamma'\right)| \quad = \quad |\cos^n(t) + (-\sin(t))^n|(n-1)!$$

$$(3.11) \qquad\qquad < \quad (n-1)! = |k_n(\Gamma)| \qquad\qquad \forall n \geq 2,\ 0 < t < \frac{\pi}{2}.$$

Here, the strict inequality comes from the classic norm inequality $\|x\|_n < \|x\|_2$ for any vector x with at least two non zero coordinates. We now assume by induction that:

$$|\mu_i \left(\cos(t)\Gamma - \sin(t)\Gamma'\right)| < |\mu_i(\Gamma)| \qquad \forall 2 < i < n,\ 0 < t < \frac{\pi}{2}.$$

We thus have:

$$|\mu_n \left(\cos(t)\Gamma - \sin(t)\Gamma'\right)| \leq \sum_{i=1}^{n} \binom{n-1}{i-1} |k_i(\cos(t)\Gamma - \sin(t)\Gamma')||\mu_{n-i}(\cos(t)\Gamma - \sin(t)\Gamma')|$$

$$< \sum_{i=1}^{n} \binom{n-1}{i-1} |k_i(\Gamma)||\mu_{n-i}(\Gamma)| = |\mu_n(\Gamma)|.$$

This completes our proof.

∎

Theorem 12 reduces the problem of finding the maximum norm ratio over all log-concave random variables into finding the maximum norm ratio over Γ^s. If p and q are even integers, this problem is equivalent to finding:

$$(3.12) \qquad\qquad \max_{0 \leq s \leq 1} \frac{(r_q(s))^p}{(r_p(s))^q},$$

where $r_p(s) = \mathbb{E}[s\Gamma + (s-1)\Gamma']^p$ is a family of polynomials. We now present a technique, which worked for all even integers up to 100, to prove the maximum is achieved exactly at $s = 0$ and $s = 1$.

Theorem 15. *Let X be a centered log concave random variable and let p, q be even numbers such that $p < q < 100$ then:*

$$(3.13) \qquad\qquad \frac{\|X\|_q}{\|X\|_p} \leq \frac{\|\Gamma\|_q}{\|\Gamma\|_p} = \frac{(!q)^{\frac{1}{q}}}{(!p)^{\frac{1}{p}}},$$

where equality holds if and only if X coincides with $\pm\Gamma$ in law.

Proof. Unfortunately, this proof is computer-assisted. First notice that the rational function in (3.12) is symmetric with respect to $\frac{1}{2}$, and that it is enough to prove our hypothesis for $p = q - 2$. Thus our goal will be to show:

$$\frac{d}{ds} log\left(\frac{(r_q(s))^{q-2}}{(r_{q-2}(s))^q} \right) \leq 0 \qquad\qquad \forall 0 \leq s \leq \frac{1}{2},$$

or equivalently:

$$(3.14) \quad h_q(s) := (q-2) \cdot r_q'(s) \cdot r_{q-2}(s) - q \cdot r_{q-2}'(s) \cdot r_q(s) \leq 0 \qquad \forall 0 \leq s \leq \frac{1}{2}.$$

Define now:

$$\tilde{h}_q(s) = \frac{h_q\left(s - \frac{1}{2}\right)}{s\left(s - \frac{1}{2}\right)\left(s + \frac{1}{2}\right)} = \sum a_{i,q} s^{2i}.$$

Calculating the coefficients of the first few polynomials we get:

$$\tilde{h}_4(s) = -96$$

$$\tilde{h}_6(s) = -720(15 + 8s^2 + 144s^4)$$

$$\tilde{h}_8(s) = -1680(1485 - 2880s^2 + 105696s^4 + 104448s^6 + 268544s^8)$$

$$\tilde{h}_{10}(s) = -5040(269325 - 1323000s^2 + 72560880s^4 + 280339200s^6$$

$$+ 1409629440s^8 + 1162622976s^{10} + 1050406912s^{12}).$$

One can easily check that when $q \leq 10$, $a_{i.q} \leq 0$ for all $i \neq 1$ and $a_1^2 - 4a_0a_2 < 0$. Thus $\tilde{h}_q$ never changes sign and $h_q(s)$ has the same sign as $s(s-1)(s+1)$, which proves the theorem. Using computer assistance one can see the same proof still holds for $q < 100$. □

We finish this paper with a conjecture which generalizes these empiric results:

Conjecture 16. *Let X be a centered log-concave random variable. Then for even integers $p < q$, we have:*

$$(3.15) \qquad \frac{\|X\|_q}{\|X\|_p} \leq \frac{\|\Gamma\|_q}{\|\Gamma\|_p} = \frac{(!q)^{\frac{1}{q}}}{(!p)^{\frac{1}{p}}},$$

with equality if and only if X coinsides with $\pm\Gamma$ in law, where Γ is a random variable with density $g(x) = e^{-(x+1)}1_{[-1,\infty)}$.

Chapter 2. **Empirical data and other results**

4. INTRODUCTION

In the first part of this chapter, we present other results which use intersection arguments similar to those we presented before, as well as a recursion formula for the polynomials r_n defined in the last chapter. The results will be stated for the log-concave case but can be easily stated for the dimensional-dependent case as well. In the second part of this chapter, we present different calculations, techniques, and empirical data which we could not turn into rigorous proofs, but which we still feel give insight into the unanswered problems presented in chapter 1.

5. OTHER RESULTS

We start with a few quick propositions which can be easily proven by the same methods used thus far.

Proposition 17. *Let X be a centered log-concave random variable, whose density function $f(x)$ is supported on the interval $[-\alpha, \infty)$ for some $\alpha > 0$, and let $\phi(x)$ be a strictly convex function. Then:*

$$(5.1) \qquad \mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(\alpha \cdot \Gamma)],$$

where equality holds if and only if X coincides with $\alpha \cdot \Gamma$ in law.

Proof. Letting $g_\alpha(x)$ be the distribution function of $\alpha \cdot \Gamma$, one easily verifies that g_α and f may intersect at most twice. Using Lemma 11 to see that $\{1,\ x,\ \phi\}$ is a Chebyshev system, we reach a contradiction using the same arguments we saw in the last chapter. □

Proposition 18. *Let $f(x)$ be the density of a log-concave centered random variable X. Then:*

$$(5.2) \qquad\qquad f(x) \leq \frac{1}{|x|} \qquad\qquad \forall x \neq 0,$$

where equality at a point α implies that X coincides with $\alpha \cdot \Gamma$ in law.

Proof. Again, for any $\alpha > 0$, since X is centered and log-concave, f and g_α must intersect at least twice. Assume now that $\frac{1}{\alpha} = g_\alpha(\alpha) < f(\alpha)$. This implies that g_α and f cannot intersect at all in the ray $(-\infty, -\alpha]$ and at most once at $[-\alpha, \infty)$, a contradiction. The case of $\alpha < 0$ follows by symmetry. $\qquad\square$

We can also provide an elementary proof of the one-dimensional case of a result by Harge [7] and Hu [8].

Proposition 19. *Let X be a centered random variable with density $f(x)$ such that $\log (f(x)) \in C^2 (\mathbb{R})$ and for some $\alpha > 0$ we have:*

$$(5.3) \qquad\qquad \frac{d^2}{dx^2} \log (f(x)) \leq -\alpha \qquad\qquad \forall x \in \mathbb{R}.$$

Then for every convex function ϕ we have:

$$(5.4) \qquad\qquad \mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(\mathcal{N}(0, \frac{1}{\alpha}))],$$

with equality if and only if X coincides with $\mathcal{N}(0, \frac{1}{\alpha})$ in law (here $\mathcal{N}(0, \frac{1}{\alpha})$ is the normal distribution with variance α).

Proof. Let $z_\alpha(x)$ be the density of $\mathcal{N}(0, \frac{1}{\alpha})$ and define $h(x) = z_\alpha(x) - f(x)$. Inequality (2.3) implies h can change sign at most twice, and so we can repeat the argument used in Proposition 18. $\qquad\square$

Proposition 20. *Let X be a log-concave (not necessarily centered) random variable. If $\exists a, b, t_0, t_1 \in \mathbb{R}$ are such that $b > 0$, $t_0 < t_1$ and:*

$$(5.5) \qquad\qquad \mathbb{E}[e^{t_i X}] = \mathbb{E}[e^{t_i (a+b\Gamma)}] \qquad\qquad i \in \{0, 1\}.$$

Then for every $t > max\{0, t_1\}$, we have:

$$(5.6) \qquad\qquad \mathbb{E}[e^{tX}] \leq \mathbb{E}[e^{t(a+b\Gamma)}],$$

with equality if and only if X coincides with $a + b\Gamma$ in law.

Proof. Using Rolle's Theorem inductively, one readily verifies that $\{e^{a_0 x}, e^{a_1 x}, \ldots, e^{a_n x}\}$, $(a_0 < a_1 < \ldots < a_n)$ is a Chebyshev system. Following the usual technique one proves the proposition. $\qquad\square$

Let us now focus our efforts towards a better understanding of the polynomials r_n and h_n in hopes of proving Conjecture 16 presented in the first chapter. First, we provide a recursive formula for the polynomials r_n. Abusing the notation, we redefine r_n in the following way:

$$(5.7) \qquad\qquad r_n(z) = \begin{cases} 1 & n = 0 \\ \frac{1}{(n-1)!} \mathbb{E}[z\Gamma + (z-1)\Gamma']^n & n > 0 \end{cases}.$$

Here we just multiplied the original polynomials by $\frac{1}{(n-1)!}$.

Proposition 21. *The following recursion formula holds:*

$$(5.8) \qquad r_n(z) = z^n + (z-1)^n + \sum_{i=0}^{n-2} \frac{r_i(z)}{i} \left((z-1)^{n-i} + z^{n-i} \right).$$

Proof. Recalling the moment-cumulant formula we have:

$$r_n(z) = \frac{1}{(n-1)!} \mu_n \left(z\Gamma + (z-1)\Gamma' \right)$$

$$= \frac{1}{(n-1)!} \sum_{i=0}^{n-2} \binom{n-1}{i} \mu_i \left(z\Gamma + (z-1)\Gamma' \right) k_{n-i} \left(z\Gamma + (z-1)\Gamma' \right)$$

$$= \sum_{i=0}^{n-2} \frac{\mu_i \left(z\Gamma + (z-1)\Gamma' \right)}{i!} \cdot \frac{(n-1-i)! \left(z^{n-i} + (z-1)^{n-i} \right)}{(n-1-i)!}$$

$$= z^n + (z-1)^n + \sum_{i=0}^{n-2} \frac{r_i(z)}{i} \left(z^{n-i} + (z-1)^{n-i} \right).$$

$\square$

We now present useful empirical data regarding properties of the polynomials r_n as well as other related polynomials.

6. Empirical data

Calculating the roots of r_n for all values of n smaller than 100 with computer assistance, we found, to our surprise, that they all lie on the line $Re(z) = \frac{1}{2}$. This leads us to the following conjecture:

Conjecture 22. *All roots of the polynomials $\{r_n(z)\}_{n=0}^{\infty}$ lie on the line $Re(z) = \frac{1}{2}$.*

We now present a few of our best attempts to prove this conjecture. By (5.7) and the fact that Γ and Γ' are independent i.i.d with moments

$$(6.1) \qquad \mu_n(\Gamma) = !n = \left(n! \sum_{i=0}^{n} \frac{(-1)^k}{k!} \right)^{\frac{1}{n}} = \left[\frac{n!}{e} \right]$$

where $[\cdot]$ is the nearest integer function, we get

$$(6.2) \qquad r_n(z) = n \sum_{i=0}^{n} \frac{!i \, !(n-i)}{i! \, (n-i)!} z^i (z-1)^{n-i}.$$

As the Mobius map $z \to \frac{z}{z-1}$ maps the unit circle to the line $Re(z) = \frac{1}{2}$, it is enough to prove the roots of the polynomials

$$q_n(z) = \frac{1}{n}(z-1)^n r_n(\frac{z}{z-1}) = \sum_{i=0}^{n} \frac{!i \, !(n-i)}{i! \, (n-i)!} z^i$$

all lie in the unit circle. Equation (6.1) suggests that the coefficients of q_n are close to being equal and so q_n can be thought of as close to the polynomial $\sum_{i=0}^{n} z^i$ whose roots all lie in the unit circle. Notice also that the coefficients of q_n are symmetric around $\frac{n}{2}$, and so by [11] it is enough to show that the roots of the derivative q_n' all lie in the unit disc. Unfortunately, we were not yet able to use the facts above to prove the conjecture. Figure 6.1 depicts the coefficients
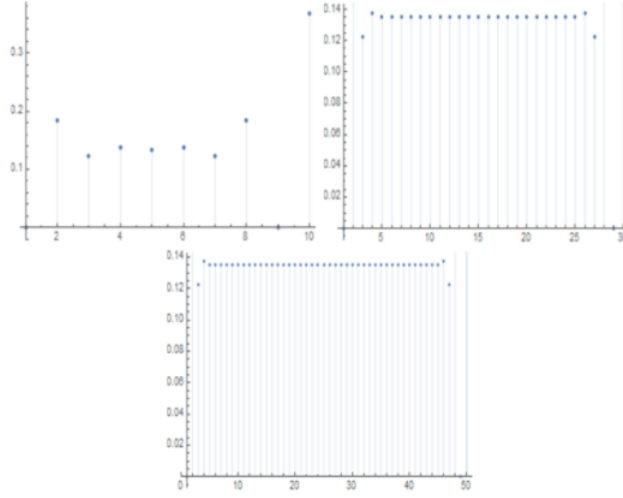
FIGURE 6.1. The coefficients of $q_n(z)$ for $n = 10$, 30, 50 respectively.

of q_n for different values of n. As can be seen, these coefficients are indeed almost constant, but are not monotone or even unimodular.

Our second approach was the following: noticing that in the proof of Theorem 4 the only property of Γ we used was that it has positive cumulants, we conjectured that this property might be enough to prove Conjecture 6. Combining this with the moment-cumulant formula, we tried first to prove the following conjecture which would have implied Conjecture 6:

Conjecture 23 (False). *Let $\{\lambda_{n,i}\}_{n,m\in\mathbb{N}}$ be a doubly indexed sequence of positive numbers. Define:*

$$p_n(z) = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ \sum_{m=0}^{n-2} \lambda_{n,m} p_m(x)\left((z-1)^{n-m} + z^{n-m}\right) & n \geq 2 \end{cases}$$

The roots of $\{p_n(z)\}_{n=0}^{\infty}$ all lie on the line $\mathrm{Re}(z) = \frac{1}{2}$.

Unfortunately, (using arguments presented in [13, Section 2] for example) one can easily verify that Conjecture 23 is equivalent to the following false claim: for every $n \in \mathbb{N}$ the polynomials:

$$\left\{ \left((z-1)^{i_1} + z^{i_1}\right) \cdot \ldots \cdot \left((z-1)^{i_k} + z^{i_k}\right) \right\}_{i_1 + \ldots + i_k = n}$$

are commonly interlacing. However, empirical data shows that for many different choices of $\lambda_{n,i}$ Conjecture 23 is true, and we hope that some other (hopefully weak) constraint on $\lambda_{n,i}$ will make Conjecture 23 true. Specifically, we still hope to prove the following:

Conjecture 24. *Let X be a centered random variable with positive cumulants. Then Conjecture 7 holds for $\lambda_{n,i} = \binom{n-1}{i} k_{n-i}(X)$ or equivalently, the roots of the polynomials*

$$p_n(z) = \mathbb{E}[zX + (z-1)X']^n$$

all lie on the line $\mathrm{Re}(z) = \frac{1}{2}$, where X, X' are i.i.d.

We now turn our discussion towards the polynomials $h_n(z)$ defined at the end of the last chapter. The roots of h_n are of greater importance to us since, as we saw at the end of the last chapter, Conjecture 16 is equivalent to proving that the only real roots of h_n are $0, \frac{1}{2}$, and 1. Theorem 15 documents some of the empirical data we have gathered on this family of polynomials. For completeness, let us now give a brief description of the computer-assisted part of the proof of Theorem 15.

Computer Assistance in Proof of Theorem 15: To verify the empirical data used to prove theorem 15, we have used the following simple code in mathematica:

```
r[t_, n_] := Sum[Binomial[n, k] Subfactorial[k] Subfactorial[n - k] Power[t
+ 1/2, k] Power[t - 1/2, n - k], {k, 0, n}];

h[t_, n_] := (n - 2)*D[r[t, n], t]*r[t, n - 2] - n*D[r[t, n - 2], t]*r[t,
n];

a[i_, k_] := CoefficientList[dkhin[t, i]/(t*(t - 1/2)*(t + 1/2)), t][[k]];

len[i_] := Length[CoefficientList[dkhin[t, i]/(t*(t - 1/2)*(t + 1/2)),
t]];

test = 0;

For[i = 2, i < 50, i++, For[k = 4, len[2*i] >= k, k++, If[a[2*i, k] >
0, test = 1]]];

For[i = 2, i < 50, i++, If[(a[2*i, 3])^2 - 4*a[2*i, 1]*a[2*i, 5] >= 0,
test = 1]];

test.
```

 In addition to this, it appears that all other roots of h_n lie on a hyperbola centered at $\frac{1}{2}$ with foci $0, 1$ as can be seen in figure 6.2. The geometric nature of the positions of the roots of both r_n and h_n made us wonder what will happen if we replace the role of the random variable Γ in the definition of r_n and h_n with other random variables. For a centered random variable X, we define:

$$r_{X,n}(z) = \mathbb{E}[zX + (z - 1)X']^n$$
$$h_{X,n}(z) = (n - 2) \cdot r'_{X,n}(z) \cdot r_{X,n-2}(z) - n \cdot r'_{X,n-2}(z) \cdot r_{X,n}(z).$$

We end this chapter by presenting the positions of the roots of these families of polynomials for a variety of random variables. The geometric nature of all of these is quite surprising.
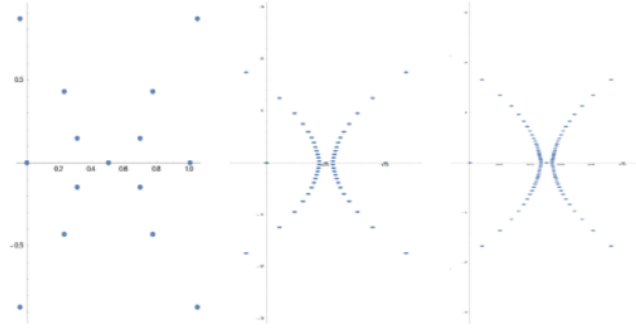
FIGURE 6.2. The roots of the polynomials $h_n(z)$ for $n = 10,\ 30,\ 50$ respectively.
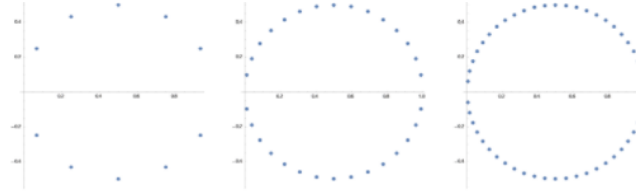

FIGURE 6.3. The roots of the polynomials $r_{U,n}(z)$ for $n = 10,\ 30,\ 50$ respectively, where U is the uniform distribution on $[-1, 1]$ (it appears that they lie on a circle).
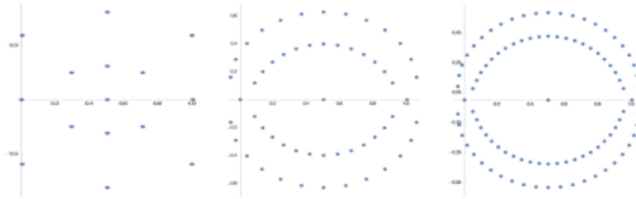

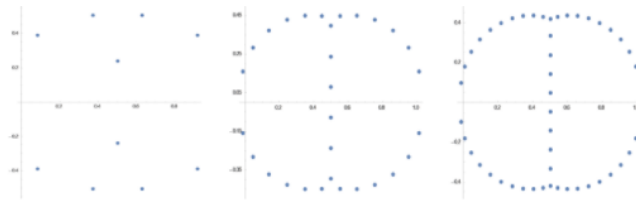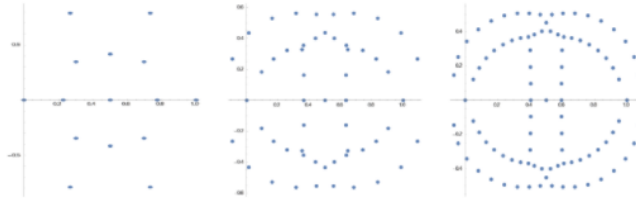FIGURE 6.4. The roots of the polynomials $h_{U,n}(z)$ for $n = 10,\ 30,\ 50$ respectively.

FIGURE 6.5. The roots of the polynomials $r_{\Gamma_{\frac{1}{2}}^1, n}(z)$ for $n = 10,\ 30,\ 50$ respectively (Γ_2^1 was defined last chapter).

FIGURE 6.6. The roots of the polynomials $h_{\Gamma_2^1,n}(z)$ for $n = 10, 30, 50$ respectively.
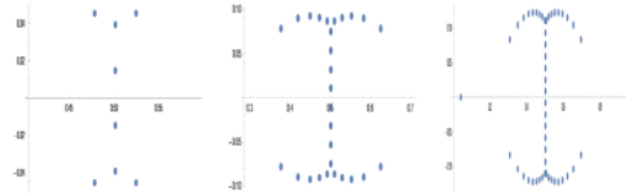

FIGURE 6.7. The roots of the polynomial $r_{B,n}(z)$ for $n = 10, 30, 50$, where B has the beta distribution with parameters 2 and 20 (i.e. the PDF of B is proportional to $x(1-x)^{19}$ and supported on the unit interval).
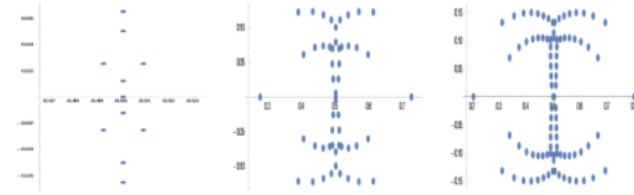

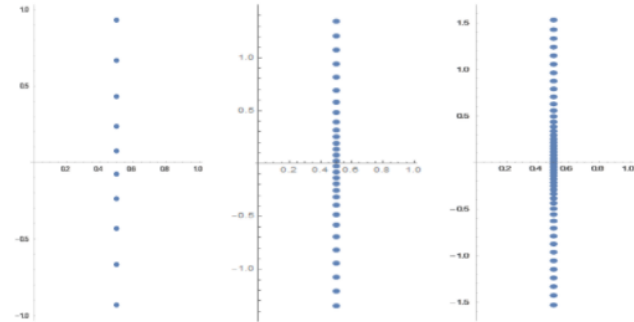FIGURE 6.8. The roots of the polynomial $h_{B,n}(z)$ for $n = 10, 30, 50$ respectively.

FIGURE 6.9. The roots of the polynomials $r_{P,n}(z)$ for $n = 10, 30, 50$ respectively, where P is a Poisson random variable (Notice P also has positive cumulants and so the images are in line with Conjecture 24).
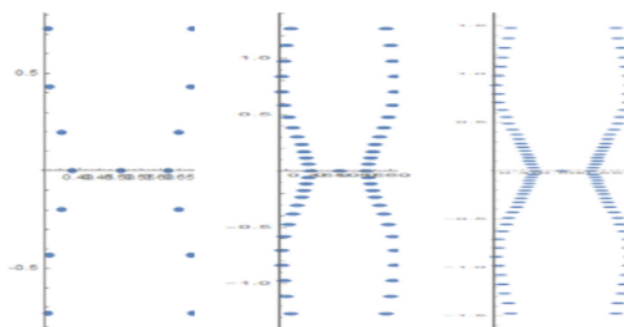
FIGURE 6.10. The roots of the polynomials $h_{P,n}(z)$ for $n = 10, 30, 50$ respectively.

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