Distance Sensitivity Oracles (DSO)

$d(x, y, G \setminus \{u\})$ (x, y, u)

Complexity measures:

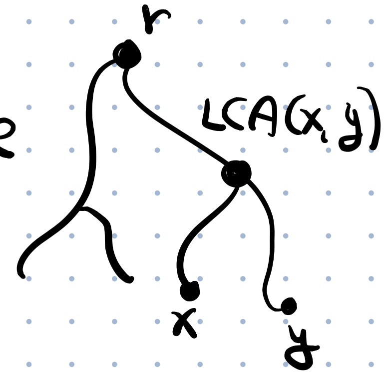
1. space
2. query time

Tradeoff

space	query time
n^3	$O(1)$
$O(m)$	$O(m)$
$\boxed{\tilde{O}(n^2)}$	$\boxed{O(1)}$

Theorem [DTR03]:
There exists 1-DSO with space $\tilde{O}(n^2)$ and query time $O(1)$.

Tool: LCA data structure



Fact: $O(n)$-space, $O(1)$ query time for LCA DS.

Description of DSO

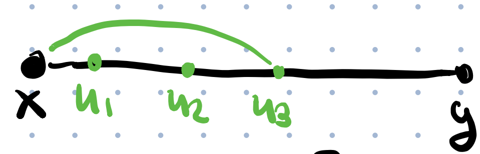
① $\forall x \in V$, LCA for T_x (BFS of x)

② B_0: $V \times V$ distances in G

③ $\forall x, y \in V$, $B_1(x,y)$ list $O(\log^2 n)$ distances:

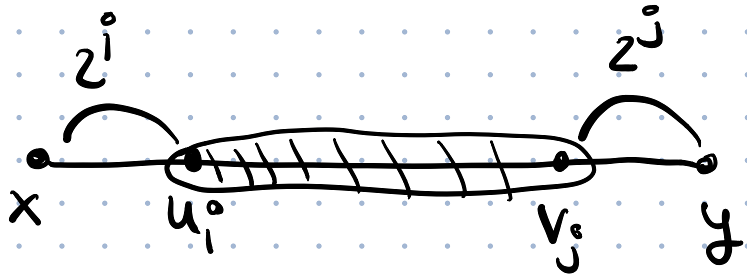
$u_i \in \pi(x,y)$ at dist. 2^i from x

$v_i \in \pi(x,y)$ " " y, $\forall_{i \in [0, \log(d(x,y))]}$



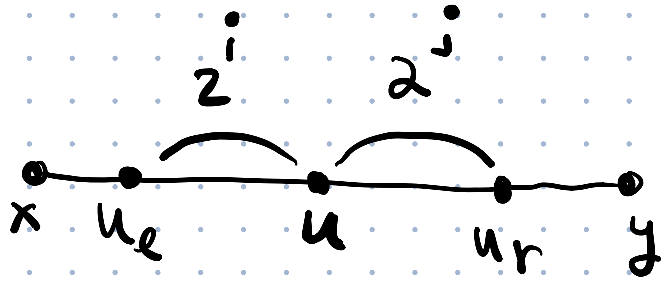
① keep $d(x,y,G \setminus u_i)$
$d(x,y,G \setminus v_i)$, $\forall i \in [\lg(d(x,y))]$

② keep $d(x,y,G \setminus \pi(u_i, v_j))$, $\forall i,j \in [\lg(d(x,y))]$



Intuition: (x, y, u)

u_ℓ: closest node to x s.t. $d(u, u_\ell) = 2^i$

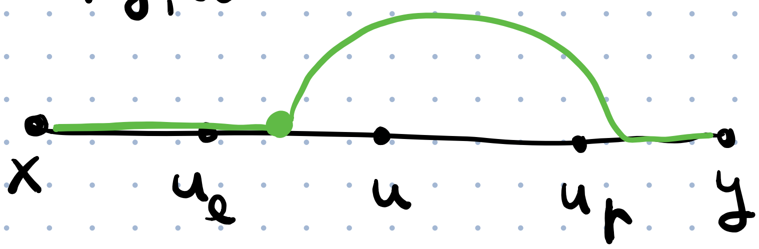


u_r: closest to y,

$d(u, u_r) = 2^j$

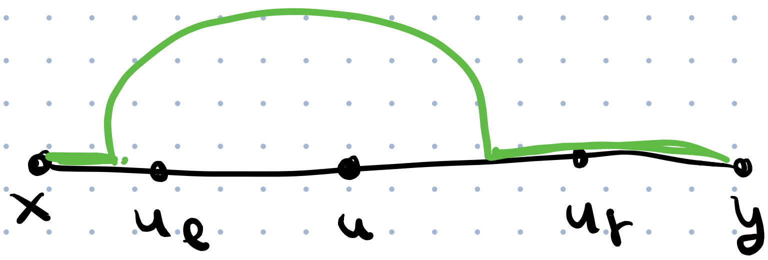
3 option $P_{\cancel{x}, y, u}$:

(1)

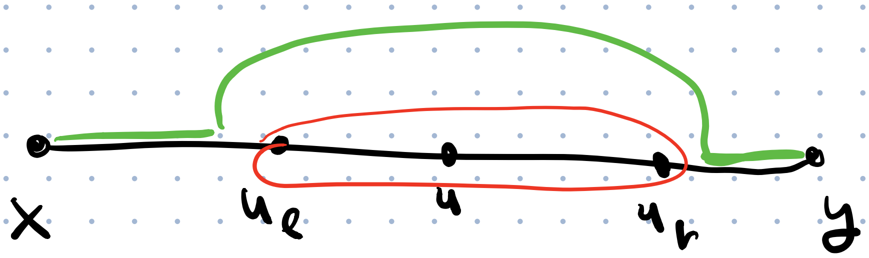


$u_\ell \in P_{x,y,u}$

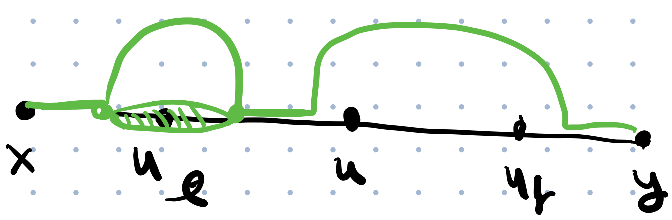
(2) $u_r \in P_{x,y,u}$



(3)

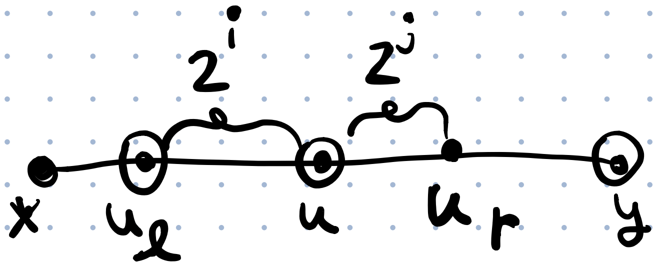


detour avoids $\pi(u_\ell, u_r)$



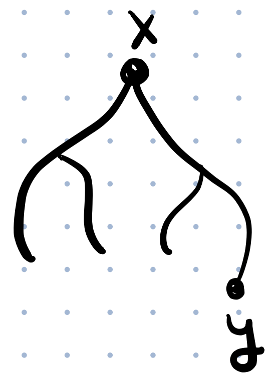
$B_1(u_e, y)$

$$d(x, y, G\setminus u) = \min\{ d(x, u_e) + d(u_e, y, G\setminus u),$$
$$d(x, u_r, G\setminus u) + d(u_r, y), \quad B_1(x, u_r)$$
$$d(x, y, G\setminus \pi(u_e, u_r)) \}$$



Query algorithm: (x, y, u)

① Query LCA (u, y) in T_x

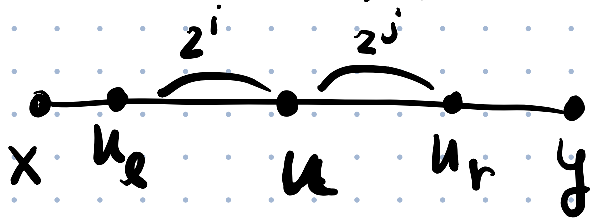


If $u \neq LCA(u, y, T_x)$: return $d(x, y)$

Assume $u \in \pi(x, y)$.

② ⊛ If $u \in \{ u_i, v_i,\ i \in [1 \ldots \lg(d(x, y))] \}$:

return $d(x, y, G\setminus u)$



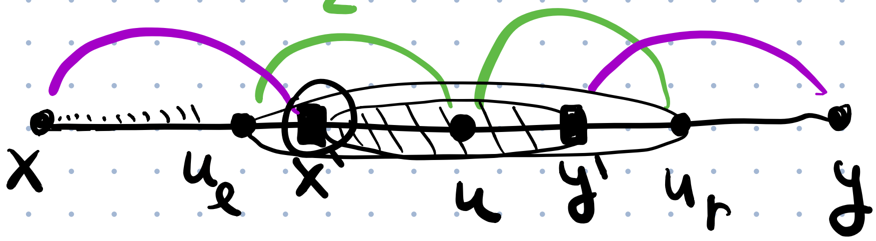
⊛ Compute u_e, u_r in $O(1)$ time

⊛ $x' \in \pi(x, y)$ s.t $d(x, x') = 2^i$

$y' \in \pi(x, y)$ s.t $(y, y') = 2^j$

2^i 2^i 2^j 2^j

$u \in \pi(x', y')$



Return:

$$\min \{ d(x, u_\ell) + d(u_\ell, y, G \setminus u), \quad d(x, u_r, G \setminus u) + d(u_r, y, G), \quad d(x, y, G \setminus \pi(x', y')) \}$$

$$d(x, y, G \setminus u) = d(x, y, G \setminus \pi(u_\ell, u_r)) = d(x, y, G \setminus \pi(x', y'))$$

$$\pi(x', y') \subseteq \pi(u_\ell, u_r)$$

FT- spanner

$G=(V,E)$, subgraph $H \subseteq G$ is an

f-TF $(2k-1)$ spanner if

$$d(s,t,H\setminus F) \leq (2k-1)\, d(s,t,G\setminus F),$$

$$\forall s,t,\ F \subseteq E,\ |F| \leq f$$

Diff. f-FT-BFS:

① FT-BFS exact
Spanner approx

② FT-BFS $s \times V$
Spanner $V \times V$

FT- spanner for f edge faults

Theorem [CLPR'10]: $\forall f, k$ one can compute f-FT $(2k-1)$ spanner with $O(f \cdot n^{1+1/k})$ edges.

Alg.

Set $G' \leftarrow G$, $H \leftarrow \emptyset$

For $i = 1, \ldots, f+1$:

$\quad H_i = \text{Spanner}(G', 2k-1)$ — standard spanner alg.

$\quad H \leftarrow H \cup H_i$

$\quad G' \leftarrow G' \setminus H_i$

$$H = \underbrace{H_1 \cup H_2 \cup \ldots H_{f+1}}$$

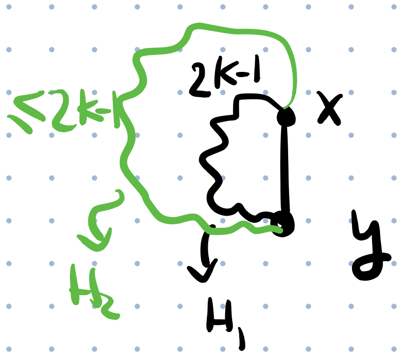
union of $f+1$ "disjoint spanners".

Correctness: s, t, F

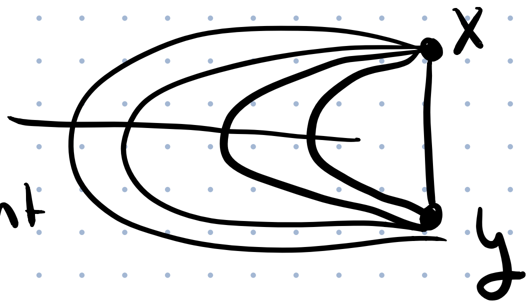
Obs: Suff. to show

$$d(x, y, H \setminus F) \leq (2k-1)\, w(x,y), \quad \forall (x,y) \in P_{s,t,F}.$$

If $(x,y) \in H$ ✓



In H, there are $\geq f+1$ edge-disjoint paths btw. x, y of length $(2k-1)\,w(x,y)$



$\Rightarrow$ One survives in $H \setminus F$

FT-spanners for vertex faults.

[CLPR 10]: $O(k^{f} \cdot n^{1+1/k})$

[DK 11]: $O(f^{2-\frac{1}{k}} \cdot n^{1+\frac{1}{k}})$

[BDPW18]: $O(f^{1-\frac{1}{k}} \cdot n^{1+\frac{1}{k}})$ → tight under girth conj.

[BP]: simplified

Algorithm of [DK 11]:

For $i = 1 \ldots O(f^3 \log n)$:

* define $V_i \subseteq V$ by sampling each v into V_i w.p $\min\left(\frac{1}{f}, \frac{1}{2}\right)$

* $H_i = \text{Spanner}(G[V_i], 2k-1)$

* $H = \bigcup H_i$

<u>Size ana:</u> $|V_i| = \frac{n}{f}$

$|H_i| = \left(\frac{n}{f}\right)^{1+\frac{1}{k}}$

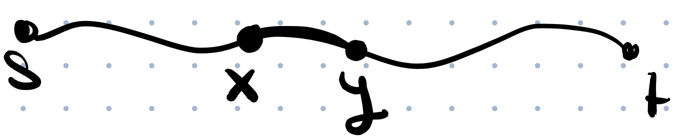
$|H| = f^3 \cdot \log n \cdot \left(\frac{n}{f}\right)^{1+\frac{1}{k}}$

$= \tilde{O}\left(f^{2-\frac{1}{k}} \cdot n^{1+\frac{1}{k}}\right)$

<u>Correctness</u>:

s, t, F

$P_{s,t,F}$: s-t SP in $G \setminus F$



<u>Show</u>: $d(x, y, H \setminus F) \leq (2k-1)\, w(x,y)$

<u>Def</u>: Iteration i is <u>good</u> for (x, y, F) if:

① $x, y \in V_i$

② $F \cap V_i = \emptyset$

Claim 1: For a fixed x, y, F, $\exists$ good iteration w.p $1 - \frac{1}{n^{5f}}$

Iteration i is good (x, y, F) w.p

$$\frac{1}{f^2} \cdot \underbrace{\left(1 - \frac{1}{f}\right)^f} = \Omega\left(\frac{1}{f^2}\right)$$

Prob. no iteration is good for (x, y, F) is

$$\left(1 - \frac{1}{c f^2}\right)^{c' f^3 \lg n} = \frac{1}{n^{c' f}}$$

$\Rightarrow$ by the union bound, every (x, y, F) has at least one good iteration.

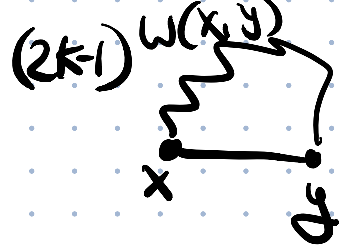
Claim 2: $d(x, y, H \setminus F) \leq (2k-1)\, \omega(x, y)$

Iteration i good for (x, y, F) we have:

$$x, y \in G[V_i], \quad F \cap V_i = \emptyset$$

$\Rightarrow$ Either $(x, y) \in H_i$ or

$$d(x, y, H_i \setminus F) \leq \underbrace{(2k-1)\, \omega(x, y)}$$


(2k-1) w(x,y)
x
y