Succinct Graph Structures and Applications **Spring 2020**

Lecture 1: May 03

Lecturer: Merav Parter

Girth and Short Cycles

Recall that the girth $g(G)$ of a graph G is the length of the shortest cycle in G. Erdős girth conjecture states that for every $k \geq 1$ and sufficiently large n, there exist n-vertex graphs with $\Omega(n^{1+1/k})$ edges and girth at least $2k+1$. A weaker lower bound can be shown via the probabilistic approach. Specifically, we will prove that there exists an n-vertex graph G^* with $\Omega(n^{1+1/(2k-1)})$ edges and girth $g(G)$ at least $2k+1$.

Exercise 1. The existence of G^* can be shown in two steps. (I) Consider a $G(n,p)$ graph[1] with $p = \Theta(1/n^{1-1/(2k-1)})$ and bound the expected (total) number of cycles of length $t \leq 2k$ in this graph. (II) Prove the existence of n-vertex graph G' with $\Theta(n^{1+1/(2k-1)})$ edges and a small number of cycles and turned it into the desired graph G^* while keeping the same order of the number of edges as in G'.

Exercise 2. The number of $2k$-cycles in a graph grows with number of edges. In this exercise, we will understand this function for the case $k = 2$.
(2a) Show that every graph with no 4-cycles has $O(n^{3/2})$ edges. Hint: A cherry in a graph is an ordered set $\langle u, \{v, w\}\rangle$ where v, w are neighbors of u. Bound the number of distinct cherries in the graph from below and above and use it to bound the number of edges in 4-cycle free graph.
(2b* (bonus)) Prove that any n-vertex graph G with average degree $\Delta = \Omega(\sqrt{n})$ has $\Omega(\Delta^4)$ 4-cycles. Hint: Consider a randomly chosen pair u, v in G and show that there are Δ^2/n 2-paths between u and v, use it to bound the number of 4-cycles with u, v on the opposite corners of these cycles.
Remark: The two claims above imply that the constant factor hidden in the $O(n^{3/2})$ edges is important. A graph with less than $c_1 \cdot n^{3/2}$ edges has *no* 4-cycle and every graph with at least $c_2 \cdot n^{3/2}$) edges has $\Omega(n^2)$ 4-cycles for $c_2 > c_1$.

Multiplicative Spanners

We saw in class the Baswana-Sen algorithm for $(2k-1)$-spanners for any $k \geq 1$. This algorithm has k phases where in each phase $i \geq 1$ it computes a clustering $\mathcal{C}_i$ that consists of $O(n^{1-i/k})$ clusters. Each cluster has a center vertex, and all the vertices in a cluster are connected in the spanner via a depth-i tree rooted at the center. One of the benefits of the Baswana-Sen algorithm is that for vertices that stopped being clustered in level $i \geq 1$ it provides a stretch of at most $2i-1$ for each of their incident edges. See Fig. 8 of [1] for a short description of the algorithm.

Exercise 3. Our goal in this exercise is to compute a subgraph $H \subseteq G$ for an n-vertex graph $G = (V, E)$ with $\widetilde{O}(n^{1+1/k})$ edges such that for every vertex pair u, v such that $\text{dist}_G(u, v) \geq \sqrt{k}$, it would hold that $\text{dist}_H(u, v) = O(\sqrt{k}) \cdot \text{dist}_G(u, v)$. Note that the spanners we saw in class with $\widetilde{O}(k \cdot n^{1+1/k})$ edges provide a stretch of $2k-1$ for *every* pair of nodes. Thus our goal is to improve this stretch bound for sufficiently far

[1] In $G(n,p)$ graph, each of the $\binom{n}{2}$ edges exists with probability p.

vertex pairs in G. Consider the following algorithm for this purpose. Throughout assume that $k' = \sqrt{k}$ is an integer.

- Run the first $k' = \sqrt{k}$ phases of the Baswana-Sen algorithm. Let $\mathcal{C}_{k'}$ be the level-k' clustering, $Z_{k'}$ be the cluster centers of the clusters of $\mathcal{C}_{k'}$. Also, let H_1 be the output subgraph containing all edges added by the algorithm throughout the first k' phases (this includes the internal trees of each cluster in $\mathcal{C}_i$ for $i \in [1, k']$, as well as the edges added due to unclustered vertices).
- Consider the graph $G^* = (Z_{k'}, E^*)$ where $E^* = \{(u, v) \in Z_{k'} \mid \text{dist}_G(u, v) \leq 5\sqrt{k}\}$. That is, each vertex in G^* corresponds to a *center* of a cluster in $\mathcal{C}_{k'}$, and every two centers are connected by an edge in G^* if their distance in G is at most $5\sqrt{k}$.
- Let $H^* \subseteq G^*$ be a $(2\sqrt{k} - 1)$-spanner of G^*.
- Let H_2 be the subgraph of G obtained by adding the u-v shortest path $\pi(u, v)$ in G for any $(u, v) \in H^*$. That is, $H_2 = \bigcup_{(u,v)\in H^*} \pi(u, v)$.
- Output $H = H_1 \cup H_2$.

(3a) Show that $|E(H)| = \widetilde{O}(\sqrt{k} \cdot n^{1+1/k})$ w.h.p.

(3b) Call a vertex u *clustered* if it belongs to one of the clusters of $\mathcal{C}_{k'}$, otherwise u is *unclustered*. Show that every edge incident to an unclustered vertex u has a stretch of at most $2\sqrt{k} - 1$ in H_1. That is, $\text{dist}_H(u, v) \leq 2\sqrt{k} - 1$ for every $v \in N(u)$ where $N(u)$ are the neighbors of u in G.

(3c) Consider a pair of clustered vertices u and v at distance $\sqrt{k}$ in G. Show that z_u and z_v are neighbors in G^* where z_u, z_v are the centers of u, v (respectively) in $\mathcal{C}_{k'}$. Use it to deduce that $\text{dist}_H(u, v) = O(k)$.

(3d) Show that $\text{dist}_H(u, v) = O(k)$ for *every* pair of vertices u and v at distance $\sqrt{k}$ in G.

(3e) Use (5d) to deduce that $\text{dist}_H(u, v) = O(\sqrt{k}) \cdot \text{dist}_G(u, v)$ for every u, v satisfying that $\text{dist}_G(u, v) \geq \sqrt{k}$.

References

[1] Baswana, Surender and Kavitha, Telikepalli and Mehlhorn, Kurt and Pettie, Seth Additive spanners and (α, β)-spanners In *ACM Transactions on Algorithms (TALG)*, 1–26, 2010.