Advanced Distributed Algorithms	**Spring 2019**
Exercise 1 (April 4)	
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Ruling-Sets. Recall that an (s, r) ruling-set R for a subset of vertices W satisfies that (i) $\texttt{dist}_G(u, v) \geq s$ for every $u, v \in R$, and (ii) $\texttt{dist}_G(w, R) \leq r$ for every $w \in W$. We saw in class an algorithm that computes an $(\alpha, \alpha \cdot \log n)$ ruling set in $O(\alpha \cdot \log n)$ rounds in the LOCAL model.

Exercise 1. Recall that in the $\mathsf{CONGEST}$ model, neighbors can exchange only $O(\log n)$ bits in each round. In this exercise we will try to compute an $(\alpha, \alpha \cdot \log n)$ using small messages of $O(\log n)$ bits. (i) Can the *same* algorithm be implemented *as is* in the $\mathsf{CONGEST}$ model within the same number of rounds? If so, explain why. If not, show how to modify the algorithm so that it can be implemented within the same number of rounds in the $\mathsf{CONGEST}$ model.

(ii) Assume that the given graph is legally colored with k colors, and that each vertex knows its color in $[1, k]$. Show that in such a case the algorithm we saw in class can be modified to compute an $(2, 2 \cdot \log k)$ ruling-set within $O(\log k)$ rounds.

Network Decomposition. A (c, d) network decomposition decomposes the vertex-set V into vertex-disjoint clusters such that the diameter of each cluster is at most d, and the cluster graph can be colored with at most c colors. We also saw that given a (c, d) network decomposition we can solve various local problems in $O(c \cdot d)$ rounds.

(iii) Show that the randomized $(O(\log n), O(\log n))$ decomposition that we saw in class is a *weak* rather than *strong* decomposition. I.e., for every cluster C, it holds that $\max_{u,v \in C} \texttt{dist}_G(u, v) = O(\log n)$, but the strong diameter $\max_{u,v \in C} \texttt{dist}_{G[C]}(u, v)$ might be large. Hint: Consider the shortest path from a clustered vertex v to its cluster center r, and let u be the neighbor of v on this shortest path, does it always hold that u and v are in the same cluster?

Low Diameter Ordering via Network Decomposition. This exercise illustrates another type of (combinatorial) application of network decomposition.

Definition .1 (Low Diameter Ordering) *Given a n-node graph $G = (V, E)$, a $d(n)$-diameter ordering of G is an assignment of unique labels*[1] *to all nodes V such that any path P on which the labels are increasing along P, any two nodes of P are within a distance $d(n)$ in the graph G.*

(iv) Show that any graph has a $d(n)$-diameter ordering with $d(n) = O(\log^2 n)$. Hint: Use the fact that every graph has an $(O(\log n), O(\log n))$ network decomposition, and use the color of the cluster to which v belongs to define its unique label in the ordering. Double Hint: The label of the vertex will consist of two numbers $L(u) = \langle a, b \rangle$ to be treated lexicographically, that is for $L(u') = \langle a', b' \rangle$, it holds that $L(u) < L(u')$ if either $a < a'$, or if $a = a'$ but $b < b'$.

[1]The label is simply a unique identifier to the vertices of $O(\log n)$ bits, i.e., a number in $[1, poly(n)]$.