Lovász Local Lemma [Erdös & Lovász 1975]

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Lecture 7

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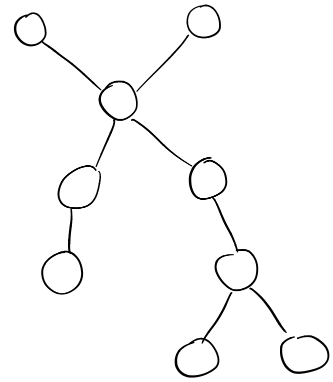
$\mathcal{A} = \{A_1, \dots, A_n\}$



① $Pr[A_i] \leq P \quad , \forall A_i$

② Each event depends on at most $\underline{d}$ other events.

LLL: $e \cdot P \cdot (d+1) < 1$, then $Pr[\wedge \bar{A}_i] > 0$

LLL can be seen as generalizing a local property to a global one:

"If there is positive probability that no bad event occurs in each neighborhood, then there is positive prob. no bad event happens globally."

Example 1: (2-Coloring hypergraphs)

$e_1, \dots, e_m$

k-uniform hypergraph

Every edge intersects $\leq d$ other edges.

LLL: If $e(d+1) < 2^{k-1}$, $\exists$ 2 coloring of the vertices of the hypergraph s.t. no edge is monochromatic.

Example 2: k-CNF

n vars, m clauses.

- clause has k vars.
- Every var in $\leq \frac{2^k}{ke}$ clauses.

$$\underbrace{(x_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(x_2 \vee x_3 \vee x_4)}_{C_2} \dots$$

LLL: $\exists$ satisfying assignment.

Constructive LLL (Moser & Tardos 2010)

$\mathcal{X} = \{X_1, \ldots, X_m\}$: indep. discrete r.v.'s

$\mathcal{A} = \{A_1, \ldots, A_n\}$: events

$vrb(A_i) \subseteq \mathcal{X}$

A_i and A_j are dep. if $vrb(A_i) \cap vrb(A_j) \neq \emptyset$.

Goal: Find a good assignment to X_i's s.t. no bad event happens.

Dep graph $G_{\mathcal{A}}: (\mathcal{A}, E)$

$$E = \{(A_i, A_j) \mid vrb(A_i) \cap vrb(A_j) \neq \emptyset\}$$

max-deg $\leq d$

$\Pr[A_i] \leq p$

$e \cdot p(d+1) < 1 - \varepsilon$

$\Longrightarrow$ By LLL, we know a construction exists, but how do we find it?

MT Algorithm

centralized algorithm

1) Initialize X_i's with random assignment.
2) While $\exists$ bad event:

   Pick one bad event (arbitrarily)

   and resample all its variables.

Witness tree

Execution: $A_{i_1}, A_{i_2}, \ldots \overset{j}{|} \ldots, A_{i_\ell}$

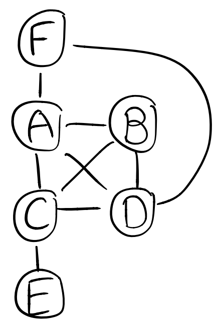
Witness tree T_j: (j'th resampling step)

Root A_{i_j}

For every $A \in \{A_{i_{j-1}}, \ldots, A_{i_1}\}$:

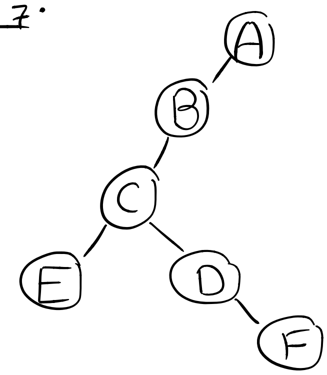
Add A as the child of its deepest neighbor in T_j.

A, B, C, D, E, F



F, D, E, C, E, B, A, D

T_7:



Lemma: $\forall i<j \quad T_i \neq T_j$.

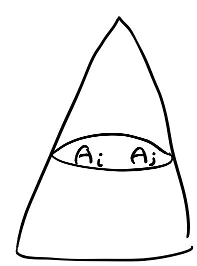
Pf: If the two trees have the same root then T_i must be a (strict) sub-tree of T_j. $\square$

Fix tree T. $|T| = s$.

What is the probability that T occurs in MT execution?

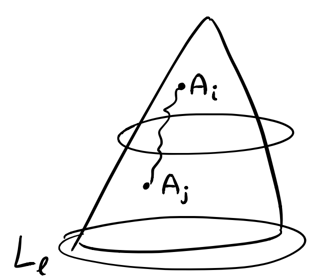
Lemma: $\Pr[T \text{ occurs}] \le p^s$.

Pf:



Obs: If A_i and A_j are in the same level, then they are independent.

$\Rightarrow \Pr[L_\ell] \le p^{|L_\ell|}$



The fact that A_j is a descendent of A_i does not increase the probability of A_i, because we use fresh randomness.

$\Rightarrow \Pr[L_i \mid L_\ell, \dots, L_{i+1}] = \Pr[L_i] \le p^{|L_i|}$.

$\square$

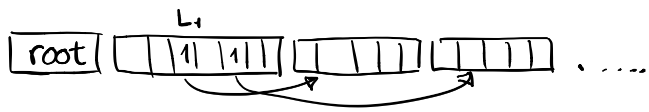
Lemma: $\mathbb{E}[\#\text{trees occuring in MT execution}) = O(n)$

Pf: $\mathcal{T}_s$: collection of all trees of size s.

$$\#\text{ trees of size-}s \le \overset{\text{root}}{n} \cdot \binom{s(d+1)}{s-1}$$

Can represent each tree by a binary string of length $s(d+1)$

with exactly $s-1$ 1 bits.



$$n \cdot \binom{s(d+1)}{s-1} \le n\,((d+1)e)^s$$

$$\binom{a}{b} \le \left(\frac{ea}{b}\right)^b$$

$$\mathbb{E}[\#\text{ witness trees}] = \sum_{s=1}^{\infty} \sum_{T \in \mathcal{T}_s} \Pr[T \text{ occurs}]$$

$$\le \sum_{s=1}^{\infty} n \cdot ((d+1)e)^s \cdot p^s$$

$$= n \sum_{s=1}^{\infty} (\underbrace{(d+1)e\,p}_{eP(d+1) < 1-\varepsilon})^s = O(n) \qquad \square$$

Thm: W.h.p. there is no tree of size $s = c \cdot \log_{\frac{1}{eP(d+1)}} n$

Pf: $\mathbb{E}[\#\text{ trees of size } s] = |\mathcal{T}_s| \cdot p^s$

$$\le (eP(d+1))^s = \frac{1}{n^c}. \qquad \square$$

Distributed LLL

Communication graph : dep. graph

$\mathcal{A} = \{A_1, \dots, A_n\}$, $\mathcal{X} = \{X_1, \dots, X_m\}$

Node v_i : event A_i

$\Pr[A_i] \le p$

LLL: $p \cdot e \cdot (d+1) \le 1 - \varepsilon$

Goal: Find good assignment.

Example: Defective Coloring

* Graph $G = (V, E)$, max-deg $= \Delta$, $f \geq 60 \ln \Delta$
* Vertex has $\lceil \frac{2\Delta}{f} \rceil$ colors.

Goal: each vertex has at most f neighbors with the same color.

→ bad event : $> f$ neighbors with same color.

→ Dep. graph : vertices that share a neighbor (i.e., distance $= 2$).

Thm 1: there is a distributed LLL algorithm that runs in $O(T_{MIS} \cdot \log_{\frac{1}{p \cdot e(d+1)}} n)$ rounds, w.h.p.

Thm 2: [Chung, Pettie, Su]: LLL in $O(\log_{\frac{1}{ped^2}} n)$.

Distributed LLL

1) Initialize with random assignment.

2) While bad event:

   $B :=$ bad event

   $M = MIS(G[B])$

   Resample all vars in M.

Want to show: Step 2 repeats $O(\log_{\frac{1}{ep(d+1)}} n)$ iterations.

<u>Lemma</u>: W.h.p. $O(\log_{\frac{1}{eP(d+1)}})$ resampling phases.

$\underbrace{M_1}_{v_1,\dots,v_k}, M_2, \dots, M_\ell$

<u>Obs</u>: $u \in M_i$ it holds that T_u has depth $\geq i-1$.

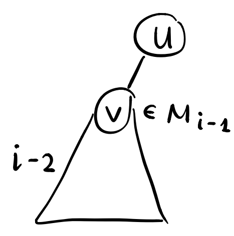
<u>Pf</u>: $i=1$: easy.

<u>step</u>: $N^+(u) \cap M_{i-1} \neq \emptyset$, why? →

If u is bad in $i-1$ ✓

If u is good in $i-1$, we must have resampled a neighbor.

⇒ we are done by induction.



$\square$

The obs completes the proof of the lemma (using arguments from centralized alg).

Improved Distributed LLL

LLL: $ped^2 \leq 1-\varepsilon$.

1) Initialize with random assignment.
2) While bad event:

   $B :=$ bad event

   $I = \{u \mid u \in B,\ ID(u) < ID(v),\ \forall v \in N(u) \cap B\}$

   Resample all vars in I.

<u>Remark:</u> I can be much smaller than MIS. Consider the path graph.

$$1 - 2 - 3 - \dots\dots - n$$

<u>2-Witness tree</u> $\qquad I_1, \dots\dots, I_\ell$

<u>step j</u>:

Root A_{ij}

For every $A \in \{A_{ij-1}, \dots, A_{i1}\}$:

Connect A to its deepest node in $N^{+2}(u)$ in the tree.

- 2-Witness tree of size S: p^S
- # trees of size S: $n \cdot \binom{S(d^2+1)}{S-1} \approx n\,(e\,(d^2+1))^S$
- w.h.p. no 2-witness tree of size $S = O(\log_{\frac{1}{ped^2}} n)$

Lemma: $u \in I_j$ has 2-witness tree of depth $\geq j-1$

Pf: $N^{+2}(u) \cap I_{j-1}$

* u is good in $j-1$
* u is bad in $j-1$

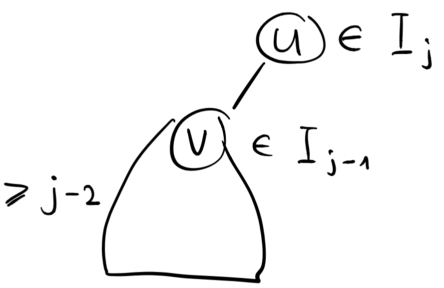
$u \in I_{j-1}$

$v < u$, v is bad $v \in N(u)$

$v \in I_{j-1}$

v is good in j but bad in $j-1$

$\Rightarrow N^{+2}(u) \cap I_{j-1} \neq \emptyset$



$v \in N^{+2}(u) \cap I_{j-1}$

$\square$