Fault-Tolerant (FT) network design

Goal: Graph $G=(V,E)$

$S(G)$ logical structure

Fault-tolerant $S(G)$ that maintains functionality under faults.

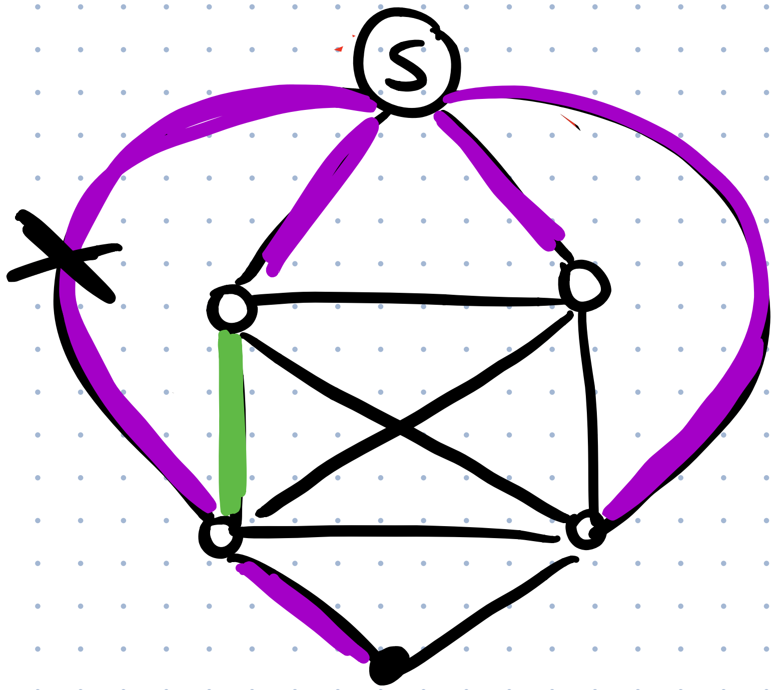
Examples:

FT-BFS: G, $\underline{s}$

$H \subseteq G$

$$\mathrm{dist}(s,t,H\setminus e) = \mathrm{dist}(s,t,G\setminus e)$$

$$\forall\, t,\ e\in E$$



— BFS

BFS in every $G\setminus\{e\}$

$$e \in BFS(s)$$

$$H = \bigcup_{e} BFS(G\setminus e)$$

② FT-Spanners

FT t-spanner $H \subseteq G$

$$\mathrm{dist}(s,t,H\setminus F) \leq t\,\boxed{\mathrm{dist}(s,t,G\setminus F)}$$

$\forall s,t,\ F \subseteq E,\ |F| \leq f$

③ FT-distance oracles

$\mathcal{D}$ data structure

$$\underbrace{(s,t,F)}_{\text{query}} \Rightarrow \underbrace{\mathrm{dist}(s,t,G\setminus F)}_{\text{answer}}$$

④ FT-labels

$L(u), L(v), L(e_1) \ldots L(e_f)$

$\Rightarrow$ dist$(u, v, G \setminus F)$

(5) FT-Routing scheme

FT-BFS (P, Peleg '13)

G unweighted

$s \in V$

$H \subseteq G$ is FT-BFS

$$\text{dist}(s, t, H \setminus e) = \text{dist}(s, t, G \setminus e) \quad \forall t, e$$

$\Rightarrow$ H contains $n-1$ BFS trees. (one per edge in BFS(s))

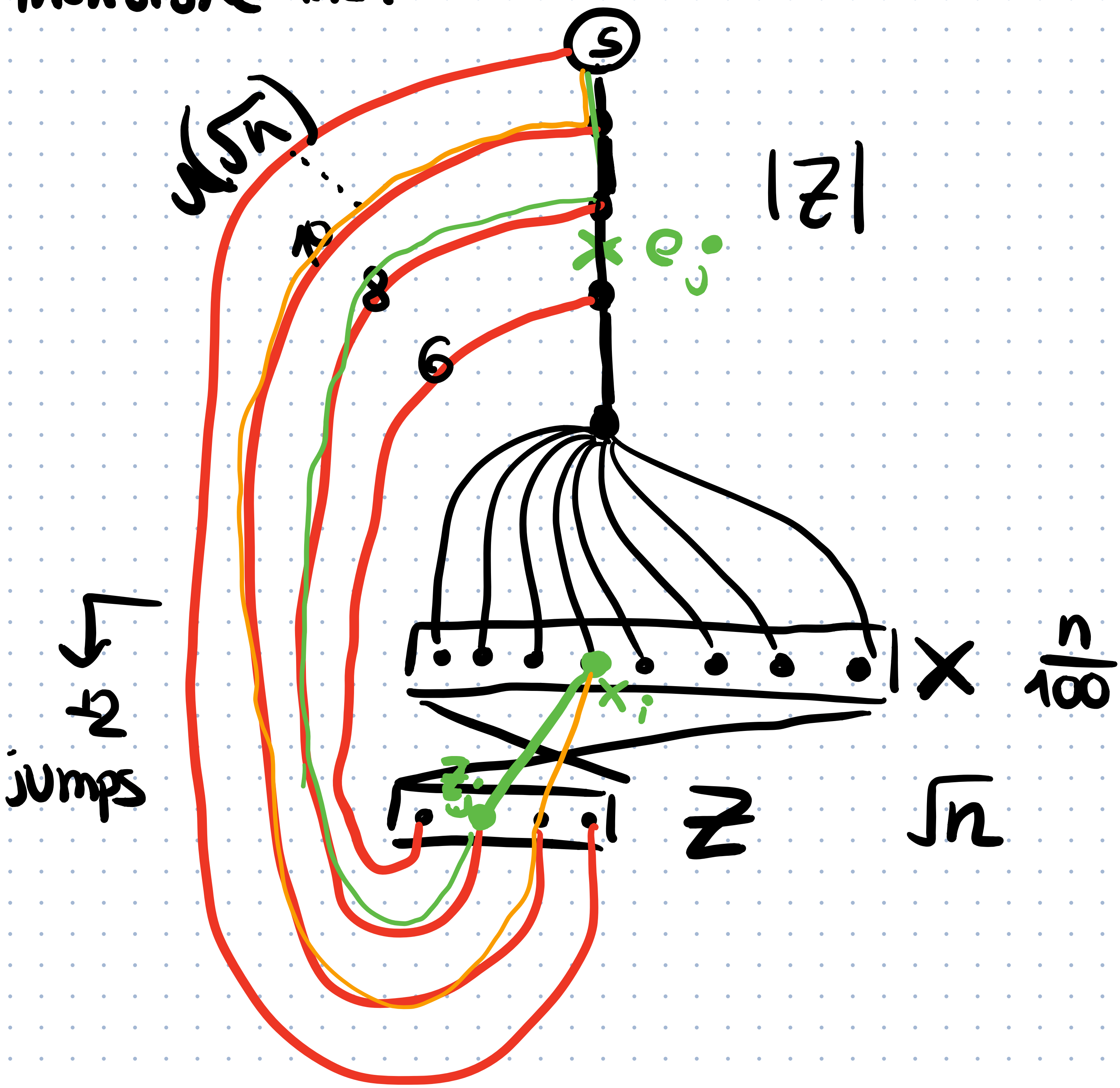
Theorem: For every n-vertex graph $G=(V,E)$, $s \in V$, one can compute an FT-BFS w.r.t s with $\boxed{O(n^{3/2})}$ edges. [upper bound]

Theorem [lower bound]:
$\forall n$ large enough,
$\exists$ n-vertex graph $G=(V,E)$ $s \in V$, s.t every FT-BFS $H \subseteq G$ w.r.t s must have $\boxed{\Omega(n^{3/2})}$ edges.

Lower Bound

G^*

vertex-disjoint
monotone-inc.



$V(G^*) = n$

$E(G^*) = n^{3/2}$

Assume (by contr.), $\exists$ FT-BFS H

s.t. $(x_i, z_j) \notin H$

$\mathrm{dist}(s, x_i, H \setminus e_j) = ?$

$\mathrm{dist}(s, x_i, G \setminus e_j)$

$\Rightarrow$ all edges of $X \times Z$

must be in any

FT-BFS w.r.t s

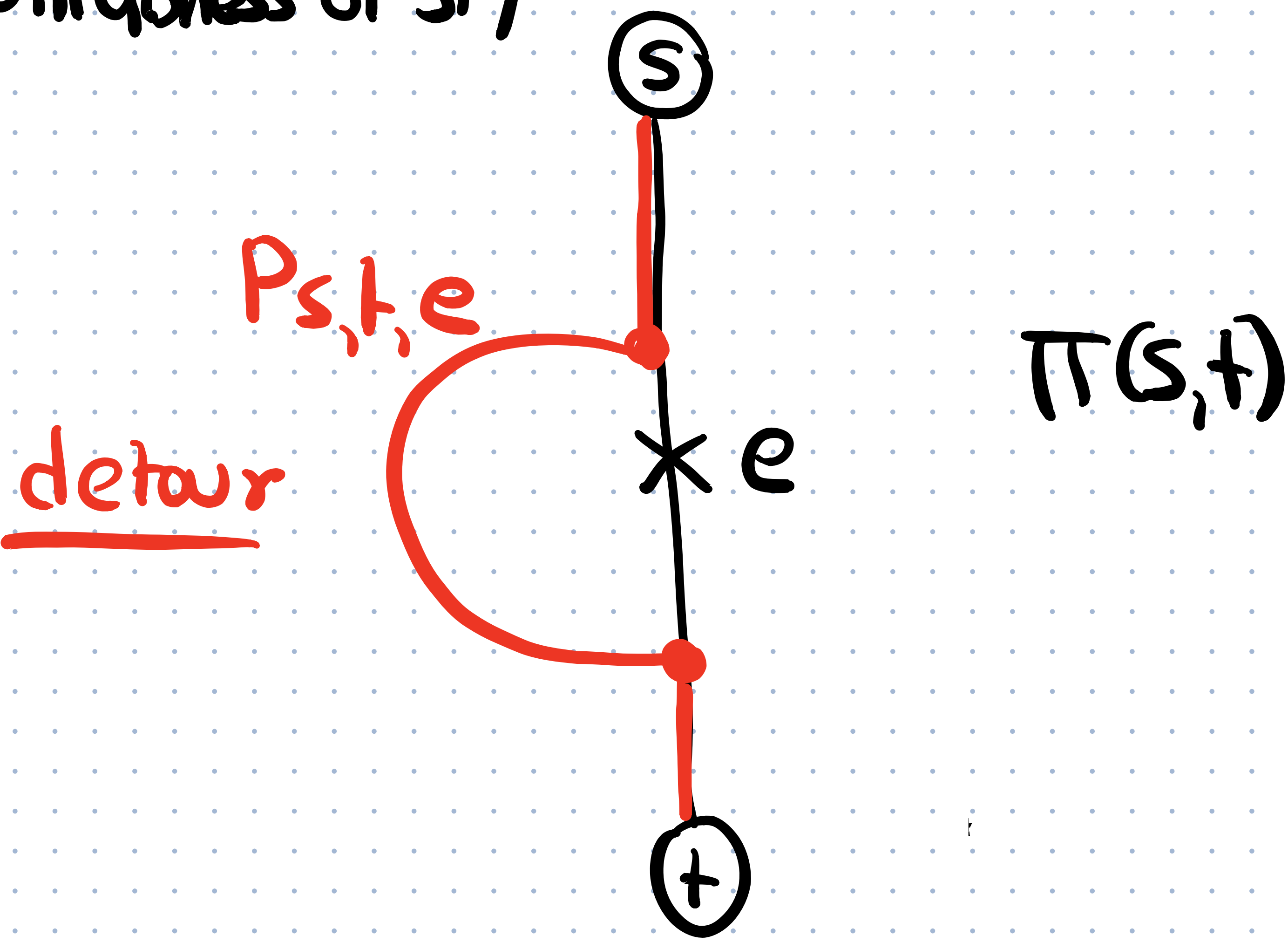
FT-BFS (what's known)

#faults	Upper bound	LB
1 ('13)	$O(n^{3/2})$	$\Omega(n^{3/2})$
2 ('15)	$O(n^{5/3})$	$\Omega(n^{5/3})$
f ('15)	?	$\boxed{\Omega(n^{2-\frac{1}{f+1}})}$
f ('17)	$\boxed{O(n^{2-\frac{1}{2^f}})}$	

Replacement Path

$P_{s,t,e}$:= s-t shortest-path in $G \setminus \{e\}$

(Uniqueness of SP)



$e \notin \pi(s,t)$: $P_{s,t,e} = \pi(s,t)$

RP problem: $s, t \in V$

Compute Fast $\bigcup_{e} P_{s,t,e}$

Now:

FT-BFS H

$$H = \bigcup_{t,e} P_{s,t,e}$$

$$|E(H)| = ?$$

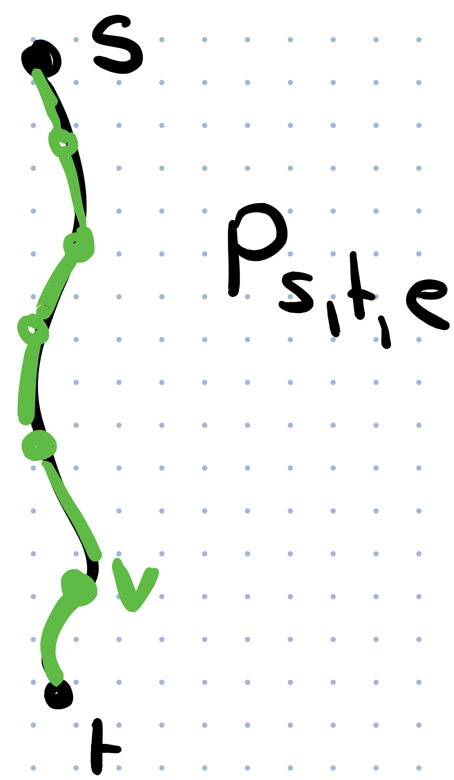
FT-BFS algorithm

Warm-up: $D = \mathrm{Diam}(G)$

Claim: H FT-BFS with $O(n \cdot D)$ edges.

Obs: $H = \bigcup_{t,e} \underline{\mathrm{LastE}}(P_{s,t,e})$

$P_{s,t,e} = P_{s,v,e} \circ (v,t)$



$$H = \bigcup_{t} \underbrace{\bigcup_{e} \mathrm{LastE}(P_{s,t,e})}_{\Downarrow}$$

at most D edsesl.

Proof! $\pi(s,t)$ with $\leq D$ edges

$$\left| \bigcup_{e \in \pi(s,t)} \mathrm{LastE}(P_{s,t,e}) \right| \leq D$$

$\tilde{O}(n^{3/2})$-size algorithm

Def: Edge $e=(u,v)$ is $\boxed{\text{near}}$ t if $d_G(u,t) \leq \sqrt{n}$

Otherwise e is $\boxed{\text{far}}$

$$\boxed{E_{near}(t) = \{ LastE(P_{s,t,e}) \mid e \in \pi(s,t) \text{ , } e \text{ near } t \}}$$

Note!

$|E_{near}(t)| = O(\sqrt{n})$

$$\boxed{H = \bigcup_t E_{near}(t) \cup \bigcup_{r \in R} BFS(r,G)}$$

s.t R is a set $O(\sqrt{n}\lg n)$ uni. sampled vertices.

Claim (size): $|E(H)| = \tilde{O}(n^{3/2})$

Correctness:

* Need to show $\mathrm{LastE}(P_{s,t,e}) \in H$

$\forall t, e.$

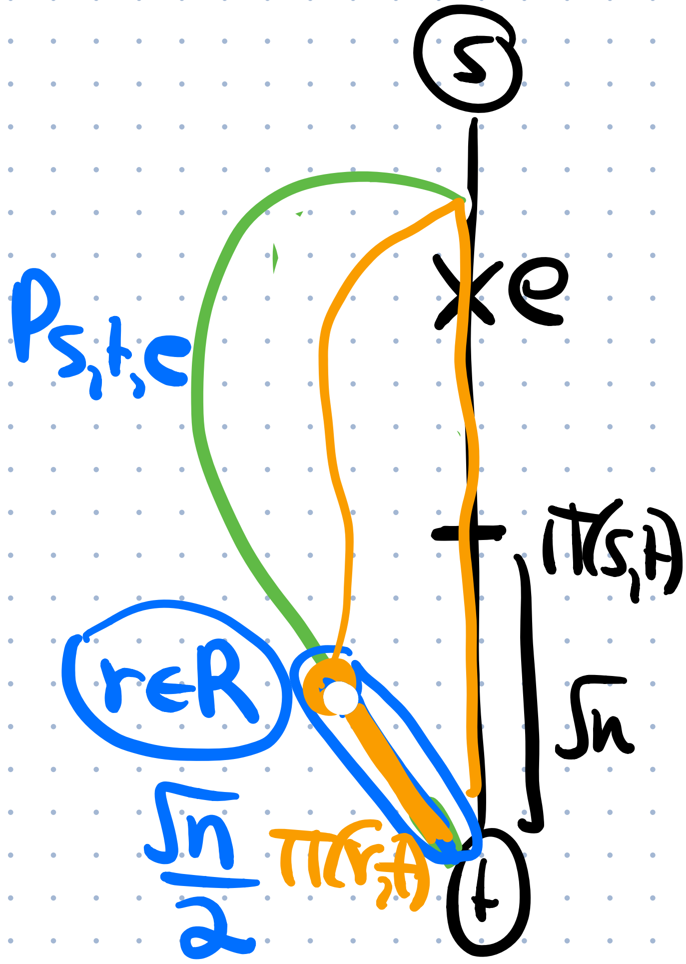
* Case 1: $e \in \pi(s,t)$ is near.

  $e \in E_{near}(t)$!

Case 2: $e \in \pi(s,t)$ is far

$|P_{s,t,e}| \geq \sqrt{n}$

$\boxed{P_{s,t,e}[r,t] = \pi(r,t)}$



Cl: $e \notin \pi(r,t)$

$P_{r,t,e} = \pi(r,t)$

$\mathrm{dist}_{G \setminus e}(r,t) \leq \frac{\sqrt{n}}{2}$

$\Rightarrow \mathrm{dist}_G(r,t) \leq \frac{\sqrt{n}}{2}$

Since $BFS(r) \subseteq H$

$\pi(r,t) \subseteq H$ and specifically

$LastE(\pi(r,t)) = LastE(P_{s,t,e}) \in H!$

Multi-Source FT-BFS

$S \subseteq V$, $H \subseteq G$ s.t

$d(s,t,H \setminus e) = dist(s,t,G \setminus e)$

$\forall s,t \in S \times V$

$H = \Theta(\sqrt{|S|} \cdot n^{3/2})$

$\boxed{S=V}$

$\boxed{\text{Approx. FT-BFS}}$

Stretch $\boxed{3}$: $2(n-1)$ edges

$\boxed{(1+\varepsilon)}$: $\tilde{O}\left(\frac{n}{\varepsilon}\right)$ edges

mult.

Additive stretch: $\Omega\left(n^{1+f(\beta)}\right)$ edges.

$+\beta$