Succinct Graph Structures and Their Applications **Spring 2022**

Home Exam: Aug 3

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Instructions: The exam consists of 5 problem sets, each worth 20 points. In case needed (and unless stated otherwise), you are allowed to use constructions that we showed in class as a black-box, but still required to add full proofs. Typing in LaTeX (and such) is highly recommended, but not a must (in case you do not have time). Please submit your solutions through email to merav.parter@weizmann.ac.il by midnight of Aug 4, 12am. Good luck!

Spanners

Exercise 1. We showed in class the construction of $(2k-1)$ spanners with $O(n^{1+1/k})$ edges. This size-stretch tradeoff is believed to be tight based on the Erdős girth conjecture. The given exercise demonstrates that unweighted graphs with large minimum degree admit considerably sparser spanners. (a) Show that every n-vertex graph G with minimum degree Δ has a 5-spanner with $\widetilde{O}((n/\Delta)^2)$ edges. (b) Generalize this result to show that it has an $(6k-1)$-spanner with $\widetilde{O}((n/\Delta)^{1+1/k})$ edges.

Exercise 2. Let $G=(A,B,E)$ be a (possibly weighted) bipartite graph where $|A|=\sqrt{n}$ and $|B|=n$. Show that one can construct a 3-spanner H with $O(n)$ edges.

Tree Embedding

Exercise 3. In this exercise we will consider a deterministic procedure for computing a low-diameter decomposition. The benefit of this procedure is that it also handles multi-graphs (where a given edge might have several copies in the graph). In addition, the clusters computed by this procedure will have small *strong-diameter*[1]. Let $c(e)$ be the number of copies of an edge $e \in G$ and for a subset of edges $F \subseteq E(G)$, let $C(F)=\sum_{e\in F} c(e)$. Let $E(u,r)=\{(x,y)\in E(G) \mid x,y \in B(u,r)\}$ be the G-edges connecting vertices in $B(u,r)$. The input to the decomposition algorithm $\mathsf{DetDecomp}$ (see Fig. 0.1) consists of (1) a multi-graph

Algorithm $\mathsf{DetDecomp}(G=(V,E,c),D,\alpha)$

1. Set $\ell \leftarrow 1$.
2. While G is nonempty do:
   (a) Pick a vertex v in G.
   (b) Let r_v be the smallest r satisfying that $C(E(v,r+1)) \le (1+\alpha)C(E(v,r))$.
   (c) $V_\ell \leftarrow B(v,r_v)$.
   (d) $\ell \leftarrow \ell+1$.
   (e) Remove all vertices of V_ℓ from G (along with their edges).
3. Return $V_1,\ldots,V_k$.

Figure 0.1: Deterministic low-diameter decomposition algorithm

$G=(V,E,c)$ (where each edge $e\in E$ has $c(e)$ copies in G), and (2) a desired diameter parameter D.

[1] The strong diameter of a subgraph $G' \subseteq G$ is $\max_{u,v\in G'} \mathtt{dist}_{G'}(u,v)$.

Consider an unweighted undirected graph $G = (V, E)$ with $C = C(E)$ and let $\alpha = 4\ln(C)/D$. Show that Alg. $\mathsf{DetDecomp}(G, D, \alpha)$ returns subsets $V_1, \ldots, V_k$ such that:

(Q1) The strong diameter of each subgraph $G[V_i]$ is at most D.

(Q2) There are at most $\alpha \cdot C(E)$ inter-cluster edges (i.e., edges connecting $u \in V_i$ and $v \in V_{j \neq i}$).

Cuts

Exercise 4. Let $G = (V, \mathcal{E})$ be a hypergraph where each edge $W \in \mathcal{E}$ is simply a subset of vertices. (In standard graphs, $|W| = 2$, but in hypergraph, the cardinality of an hyperedge can be arbitrarily large). For a subset of vertices S, the value of the cut $\delta_G(S)$ is the number of all hyperedges that contain both vertices in S and vertices not in S. That is, $\delta_G(S) = |\{W \in \mathcal{E} \ \mid W \setminus S \neq \emptyset \ \text{ and } \ W \cap S \neq \emptyset\}|$.

Consider the function $f : 2^V \to \mathbb{N}$ where $f(S) = \delta_G(S)$ for every $S \subseteq V$. Show that f is a symmetric and sub-modular function.

Fault Tolerant Graph Structures

Exercise 5. We consider an unweighted graph $G = (V, E)$ where shortest-path ties are decided in a consistent manner. For $s, t \in V(G)$ and $F \subseteq G$, let $P_{s,t,F}$ be the s-t shortest path in $G \setminus F$. For a fixed source s, and integer numbers $f, \ell \in [1, n]$, let

$$\mathcal{P}_{f,\ell} = \{P_{s,t,F} \ \mid \ t \in V, F \subseteq E, |F| \leq f, |P_{s,t,F}| \leq \ell\} \ .$$

I.e., $\mathcal{P}_{f,\ell}$ consists of all $\{s\} \times V$ replacement paths avoiding at most f edges, and of length at most ℓ. Let $H = \bigcup_{P \in \mathcal{P}_{f,\ell}} P$. Show that $|E(H)| = O(X \cdot n)$ where $X = f \log n \cdot \max\{\ell, f+1\}^{\min\{\ell, f+1\}}$. Hint: Compute a subgraph $H' \subseteq G$ that contains all edges of H using the FT-sampling approach (Class 11).